COHOMOLOGY

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Content

1. Cohomology of The Hull
2. K-theory
3. Physical Interpretation

I - Cohomology for the Hull

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Integer Čech Cohomology of Icosahedral Projection Tilings

Čech Cohomology

- Let X be a *compact metrizable* space. Given an *open cover* $\mathcal{U}$, let $C_k(\mathcal{U})$ be the free group generated by the families $(U_0, \cdots, U_k)$ of elements of $\mathcal{U}$, such that
  - $U_0 \cap \cdots \cap U_k \neq \emptyset$
  - for any permutation σ, $(U_{\sigma 0}, \cdots, U_{\sigma k}) = (-1)^{\sigma}(U_0, \cdots, U_k)$
- A boundary map $\partial : C_k(\mathcal{U}) \to C_{k-1}(\mathcal{U})$ is defined by

$$\partial(U_0, \cdots, U_k) = \sum_{j=0}^{k} (-1)^j (U_0, \cdots, \overset{j}{\vee}, \cdots U_k)$$

- The homology and the cohomology of this chain complex are denoted by $H_k(\mathcal{U})$ and $H^k(\mathcal{U})$

Čech Cohomology

- If $\mathcal{V} \leq \mathcal{U}$ is a *refinement* of $\mathcal{U}$, a *restriction* map π is a choice, for any $V \in \mathcal{V}$ of an open set $U \in \mathcal{U}$ such that $V \subset U$. Any such map induces a map $\pi_k : C_k(\mathcal{V}) \to C_k(\mathcal{U})$, commuting with the boundary. Hence it defines a map $\pi^k_{\mathcal{U}\to\mathcal{V}} : H^k(\mathcal{U}) \to H^k(\mathcal{V})$.

- **Theorem:** *the maps* $(\pi^k_{\mathcal{U}\to\mathcal{V}})^{\infty}_{k=0}$ *are independent of the choice* π *for the restriction map*

- The *Čech cohomology* groups are the direct limits

$$\check{H}^k(X) = \varinjlim (H^k(\mathcal{U}), \pi^k_{\mathcal{U}\to\mathcal{V}})$$

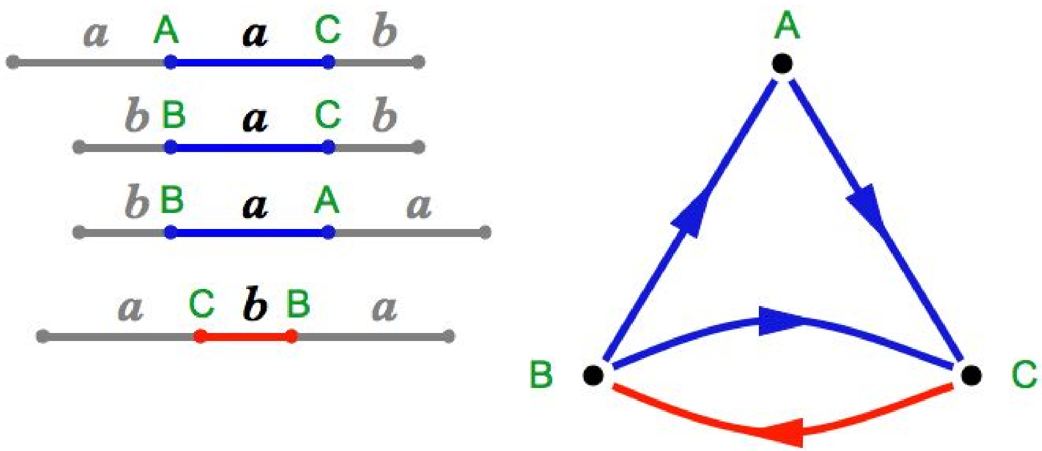
Čech Cohomology

- **Theorem:**
  If X, Y are compact metrizable spaces and $f : X \to Y$ is continuous, it induces a group homomorphism $f^* : \check{H}^k(Y) \to \check{H}^k(X)$.

- **Theorem:**
  If Ω is the inverse limit of a sequence $X_n \xrightarrow{f_n} X_{n-1}$, with f_n continuous, then
  $$\check{H}^k(\Omega) = \varinjlim (H^k(X_n), f_n^*)$$

- In particular, the Čech cohomology of the *Hull* can be computed from the Čech cohomology of the *Anderson-Putnam complexes*

Example: the Fibonacci chain

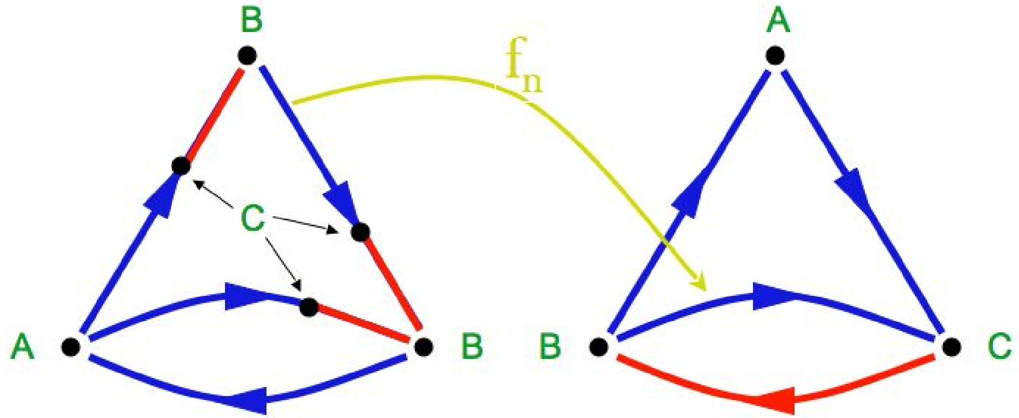


$$\partial(aab) = C - A$$
$$\partial(bab) = C - B$$
$$\partial(baa) = A - B$$
$$\partial(aba) = B - C$$

$$0 \to C_1 \simeq \mathbb{Z}^4 \xrightarrow{\partial} C_0 \simeq \mathbb{Z}^3 \to 0$$

$$H_1(X) = \mathrm{Ker}(\partial) \simeq \mathbb{Z}^2 \qquad H_0(X) = \mathrm{Im}(\partial) \simeq \mathbb{Z}$$

Example: the Fibonacci chain



Action of the substitution on the Anderson-Putnam complex

$$X_{n+1} \xrightarrow{f_n} X_n$$

$$f_n \restriction_{H^0} : \mathbb{Z} \xrightarrow{1} \mathbb{Z} \qquad f_n \restriction_{H^1} : \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2$$

$$\check{H}^0(\text{Hull}) \simeq \mathbb{Z} \qquad \check{H}^1(\text{Hull}) \simeq \mathbb{Z}^2$$

A Table of Results

Tiling	$\check{H}^0$	$\check{H}^1$	$\check{H}^2$	$\check{H}^3$
Fibonacci 1D	$\mathbb{Z}$	$\mathbb{Z}^2$		
Thue-Morse 1D	$\mathbb{Z}$	$\mathbb{Z}[1/2] \oplus \mathbb{Z}$		
Penrose 2D	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	
Chair 2D	$\mathbb{Z}$	$\mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$	$\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$	
A.-B. 2D plain	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^9$	
A.-B. 2D decorated	$\mathbb{Z}$	$\mathbb{Z}^9$	$\mathbb{Z}^{23}$	
Tübingen triangle	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{24} \oplus \mathbb{Z}_5^2$	
Canonical 3D D_6	$\mathbb{Z}$	$\mathbb{Z}^7$	$\mathbb{Z}^{72}$	$\mathbb{Z}^{208} \oplus \mathbb{Z}_2^2$

Cohomology groups for some tilings in one, two and three dimensions.

"A.-B." stands for Ammann-Beenker *(octagonal symmetry)*

II - *K*-Theory

A. H. Forrest, J. R. Hunton, Erg. Th. Dyn. Syst., **19** (1999), 611-625.

J. Bellissard, J. Savinien, Erg. Th. Dyn. Syst., **29**, (2009), 997-1031.

The Thom-Connes Isomorphism

- The *Thom-Connes* isomorphism gives

$$K_i(C(\text{Hull}) \rtimes \mathbb{R}^d) \simeq K_{i+d}(C(\text{Hull}))$$

- It follows that it is sufficient to compute the topological K-theory of the Hull.

- An analog of the *Atiyah-Hirzebruch spectral sequence* for CW-complexes, will permit to compute it from the computation of the *Čech cohomology*.

The Pimsner-Voiculescu Exact Sequence

Let $\mathcal{A}$ be a C*-algebra endowed with an action α of $\mathbb{Z}$. Then there is a *6-terms exact sequence* in K-theory

$$\begin{array}{ccccc}
K_0(\mathcal{A}) & \stackrel{\alpha_*-\mathbf{1}}{\longrightarrow} & K_0(\mathcal{A}) & \stackrel{i_*}{\longrightarrow} & K_0(\mathcal{A} \rtimes \mathbb{Z}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(\mathcal{A} \rtimes \mathbb{Z}) & \stackrel{i_*}{\longleftarrow} & K_1(\mathcal{A}) & \stackrel{\alpha_*-\mathbf{1}}{\longleftarrow} & K_1(\mathcal{A})
\end{array}$$

which allows to compute $K_*(\mathcal{A} \rtimes \mathbb{Z})$

The PV-Cohomology

- Let $\mathcal{A}$ be a C*-algebra endowed with an action $\alpha = (\alpha_1, \cdots, \alpha_d)$ of $\mathbb{Z}^d$. If $\{e_1, \cdots, e_d\}$ are the generators of $\mathbb{Z}^d$, the *PV-complex* is given by

$$K_*(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d \xrightarrow{d_{PV}} K_*(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d \qquad d_{PV} = \sum_{i=1}^{d} (\alpha_{i*} - \mathbf{1}) \otimes e_i \wedge$$

- **Theorem:** *There is a spectral sequence starting with the cohomology of the PV-complex and converging to the K-theory of* $\mathcal{A} \rtimes \mathbb{Z}^d$

- The proof uses the mapping torus $M_\alpha(\mathcal{A})$.

The *PV*-Cohomology

The keys to lift the previous result to the C*-algebra of a tiling are

$$\textit{Mapping Torus} \longleftrightarrow \mathcal{C}(\text{Hull})$$

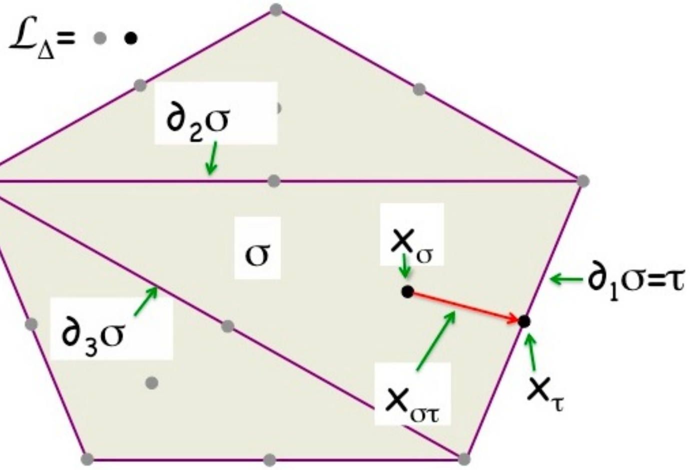
$$\textit{Torus} \longleftrightarrow \textit{Anderson-Putnam complex}$$

$$\mathcal{A} \longleftrightarrow \mathcal{C}(\text{Trans})$$

$$\mathbb{Z}^d - \textit{action} \longleftrightarrow \textit{Groupoid of the Transversal}$$

The *PV*-Cohomology

- Let T be an aperiodic, repetitive, FLC tiling with polyhedral tiles. Let $\Omega = \mathrm{Hull}(T)$.



Replace each tile by a *simplicial decomposition* Δ. Each simplex σ is endowed with a *base point* x_σ(its barycenter) compatible with the equivalence by translation making a Delone set $\mathcal{L}_\Delta$.

- Let Ξ_Δ be the *transversal* for $\mathcal{L}_\Delta$ (it is a Cantor set). Let Ξ_Δ^n be the part of the transversal corresponding to base points of n-simplices ($0 \leq n \leq d$). The Ξ_Δ^n make up a *partition* of Ξ_Δ.

The PV-Cohomology

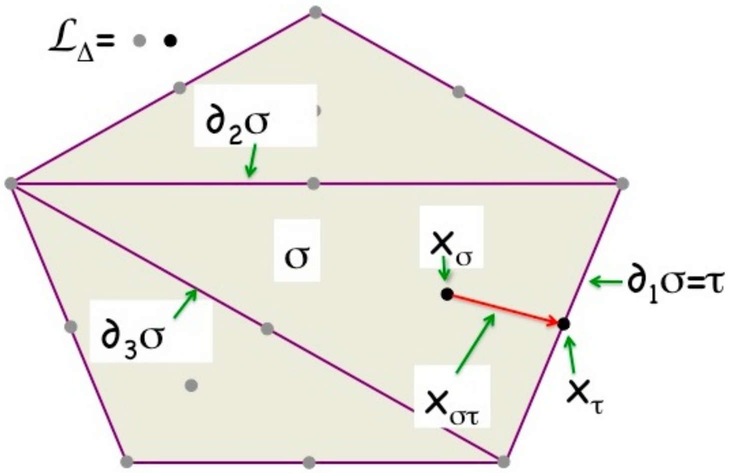
- Hence $C(\Xi_\Delta, \mathbb{Z}) = \bigoplus_{n=0}^{d} C(\Xi_\Delta^n, \mathbb{Z})$

- The construction of the Anderson-Putnam complex X_0 associated with the *collared prototiles* leads also to a simplicial decomposition. Let S_0^n be the set of n-simplices in X_0.

- For $\sigma \in S_0^n$ there are
  - a *Delone* subset $\mathcal{L}_\sigma \subset \mathcal{L}_\Delta$ of base point of the simplices in the tiling, translation equivalent to σ.
  - the corresponding *transversal* $\Xi(\sigma) \subset \Xi_\Delta$ (acceptance domain)
  - the characteristic function $\chi_\sigma \in C(\Xi_\Delta, \mathbb{Z})$ of $\Xi(\sigma)$

The PV-Cohomology

- Given any pair (σ, τ) of simplices in X_0 such that $\tau \subset \partial\sigma$, there is a well defined *translation vector* $x_{\sigma\tau} = x_\tau - x_\sigma \in \mathbb{R}^d$ translating the base point of a representative of σ in the tiling, to the corresponding one for τ



The translation vector $x_{\sigma\tau}$

The *PV*-Cohomology

- Given any pair (σ,τ) of simplices in X_0 such that $\tau \subset \partial\sigma$, there is a well defined *translation vector* $x_{\sigma\tau} = x_\tau - x_\sigma \in \mathbb{R}^d$ translating the base point of a representative of σ in the tiling, to the corresponding one for τ

- Then define the *translation operator* acting on $C(\Xi_\Delta, \mathbb{Z})$ by

$$\theta_{\sigma\tau} = \begin{cases} \chi_\sigma \mathrm{T}^{x_{\sigma\tau}} \chi_\tau & \text{if} \quad \tau \subset \partial\sigma \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_{\sigma\tau}\theta^*_{\sigma\tau} = \chi_\sigma \qquad\qquad \sum_{\tau\subset\partial\sigma} \theta^*_{\sigma\tau}\theta_{\sigma;\tau\subset\partial\sigma} = \chi_\tau$$

The PV-Cohomology

- The following operator acting on $C(\Xi_\Delta, \mathbb{Z}) = \bigoplus_{n=0}^{d} C(\Xi_\Delta^n, \mathbb{Z})$ defines a cochain complex

$$d_{PV} = \sum_{\sigma \in S_0^n} \sum_{j=1}^{n} (-1)^j \, \theta_{\sigma \partial_j \sigma} \qquad\qquad d_{PV}^2 = 0$$

$$C(\Xi_\Delta^0) \xrightarrow{d_{PV}} \cdots \longrightarrow C(\Xi_\Delta^{n-1}) \xrightarrow{d_{PV}} C(\Xi_\Delta^n) \longrightarrow \cdots \xrightarrow{d_{PV}} C(\Xi_\Delta^d)$$

- It leads to the *PV-cohomology* groups

$$H_{PV}^n(X_0; C(\Xi_\Delta, \mathbb{Z})) = \frac{\mathrm{Ker}\left\{C(\Xi_\Delta^n) \xrightarrow{d_{PV}} C(\Xi_\Delta^{n+1})\right\}}{\mathrm{Im}\left\{C(\Xi_\Delta^{n-1}) \xrightarrow{d_{PV}} C(\Xi_\Delta^n)\right\}}$$

The PV-Cohomology

The main results are the following *(Savinien-Bellissard '08)*

- **Theorem:** *the PV-cohomology of this complex coincides with the Čech cohomology of the Hull*

- **Theorem:** *There is a spectral sequence starting with the PV-cohomology and converging to the K-theory of the Hull*

- **Theorem:** *If $d = 1, 2, 3$, the K-theory of the Hull coincides with the PV-cohomology*

Other Cohomologies

- **Longitudinal cohomology:** defined for foliated space
  The de Rham complex of differential form along the leaves with continuous coefficient on the space leads to this cohomology
  (Connes '79, Moore-Schochet '88)
  The coefficient group is $\mathbb{R}$ or $\mathbb{C}$.

- **Pattern Equivariant cohomology:** *(Kellendonk '03, Sadun '06)*:
  - If $\mathcal{L}$ is a Delone set, a continuous function $f : \mathbb{R}^d \to \mathbb{C}$ is *strongly $\mathcal{L}$-pattern equivariant* whenever there is $R > 0$ such that $f(x) = f(y)$ every time $\overline{\mathrm{B}}(0; R) \cap (\mathcal{L} - x) = \overline{\mathrm{B}}(0; R) \cap (\mathcal{L} - y)$
  - The *de Rham complex* of differential forms on $\mathbb{R}^d$ with strongly pattern equivariant coefficient defines the *PE-cohomology*
  - The *PE*-cohomology with coefficient in $\mathbb{Z}$ can also be defined

Other Cohomologies

- **Theorem:** *If $\mathcal{L}$ is aperiodic, repetitive and FLC, the PV-, the longitudinal and the PE-cohomologies coincide with the Čech cohomology of the Hull*

- **Remark:** this result may not apply if the FLC condition is relaxed *(Bellissard-Ulgen-Weinberger '12, in preparation)*

III - Cohomology and Physics

Is Cohomology Measurable ?

- Since the atomic positions in a solid define the Hull it should be possible to *"measure"* the Hull in some way. Such measurements could help physicists to identify the properties of the atomic arrangements.
- **Diffraction Experiments:**
  - X-ray diffraction (transmission or absorption), transmission electron microscopy (TEM), neutron diffraction, surface electron...
  - It intends to give the *Fourier transform* of the measure $\nu^{\mathcal{L}}$. Actually it gives only some amplitude and the phases are not accessible. Still, some part of the cohomology of the Hull should be *"visible"*

Is Cohomology Measurable ?

- **Electron Transport:**
  - the electron dynamics can be used, also since the Hull of the one-particle Hamiltonian coincides with the Hull of $\mathcal{L}$.
  - This provides some information about the electronic energy spectrum, measurable in various ways, using for instance X-rays absorption or emission, or junction experiments
  - As already seen, the *Gap Labeling Theorem* is a possible way to get some information about the cohomology of the Hull.
  - Transport experiments can also provide some information. The *Quantum Hall Effect* is a spectacular example.

Is Cohomology Measurable ?

- **Other Degrees of Freedom:** phonons, electron spin, atomic diffusion
  - The *phonon vibrational modes* are obtained through a bounded selfadjoint operator belonging toin the C^*-algebra of the transversal. It also contains the same information as the groupoid of the transversal. Hence the cohomology of the Hull should have some *measurable traces*. Here also the *Gap Labeling Theorem* applies.
  - Not much is known about the spin degrees of freedom, even though the recent topic of *Topological Insulators* suggests that some topological invariants might be available through them.

Lattice Deformation

A. Clark, L. Sadun, Ergodic Theory Dynam. Systems, **26**, (2006), 69-86.

- Let T be a polyhedral tiling. An *edge* is a $1D$-face of a prototile and can be seen as a subset of the Anderson-Putnam complex X_0. The free abelian group generated by edges is $C_1(X_0)$. A *displacement* is a map $f : edges \to \mathbb{R}^d$, such that
  - the displacements of the edges in any *k-cell* lie in a kD-subspace of $\mathbb{R}^d$
  - the *sum* of the displacements along the edge of a *2-cell* is zero
- A displacement can be seen as an element of $C^1(X_0, \mathbb{R}^d) = \mathrm{Hom}(C_1(X_0), \mathbb{R}^d)$.
  Moreover, $df(t) = f(\partial t) = 0$ implies that f defines an element $I(f) \in H^1(X_0, \mathbb{R}^d)$ hence there is $\check{I}(f) \in \check{H}^1(X_0, \mathbb{R}^d)$

Lattice Deformation

- Given a displacement function f, a new tiling T_f is built,

- **Theorem:**
  Hull(T) *and* $\mathrm{Hull}(T_f)$ *are homeomorphic*

Lattice Deformation

- A tilings T' is *locally derivable* from T if there is $R > 0$ such that for all $x, y \in \mathbb{R}^d$ the condition $(T - x) \cap \overline{\mathrm{B}}(0; R) = (T - y) \cap \overline{\mathrm{B}}(0; R)$ implies $(T' - x) \cap \overline{\mathrm{B}}(0; 1) = (T' - y) \cap \overline{\mathrm{B}}(0; 1)$.
- If T and T' are locally derivable from each other that they are called *mutually locally derivable* (MLD).

- **Theorem:**
  If f, g are displacements with $\check{I}(f) = \check{I}(g)$ then T_f and T_g are MLD

Lattice Deformation

- There is a notion of an element $\eta \in \check{H}^1(\mathrm{Hull}(T), \mathbb{R}^d)$ being *asymptotically negligible* leading to

- **Theorem:**
  If f, g are displacements with Ǐ$(f) -$ Ǐ(g) asymptotically negligible, then $\mathrm{Hull}(T_f)$ and $\mathrm{Hull}(T_g)$ are topologically conjugate

It is time for coffee !