Random Matrix Theory and the 2D Anderson Model

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Main References

This work

J. Bellissard, J. Magnen, V. Rivasseau, *Supersymmetric Analysis of a Simplified Two Dimensional Anderson Model at Small Disorder*, `cond-mat/0210524`, to be published in *Markov Processes and Related Fields* (2003).

J. Bellissard, *Coherent and dissipative transport in aperiodic solids* Published in *Dynamics of Dissipation*, P. Garbaczewski, R. Olkiewicz (Eds.), Lecture Notes in Physics, **597**, Springer (2002), pp. 413-486.

Motivated by

M. Disertori, H. Pinson, T. Spencer, *Density of states for Random Band Matrix* `math-ph/0111047`, *Commun. Math. Phys.*, **232**, (2002), 83-124.

The Anderson Model:

For $\mathcal{H} = \ell^2(\mathbb{Z}^D)$

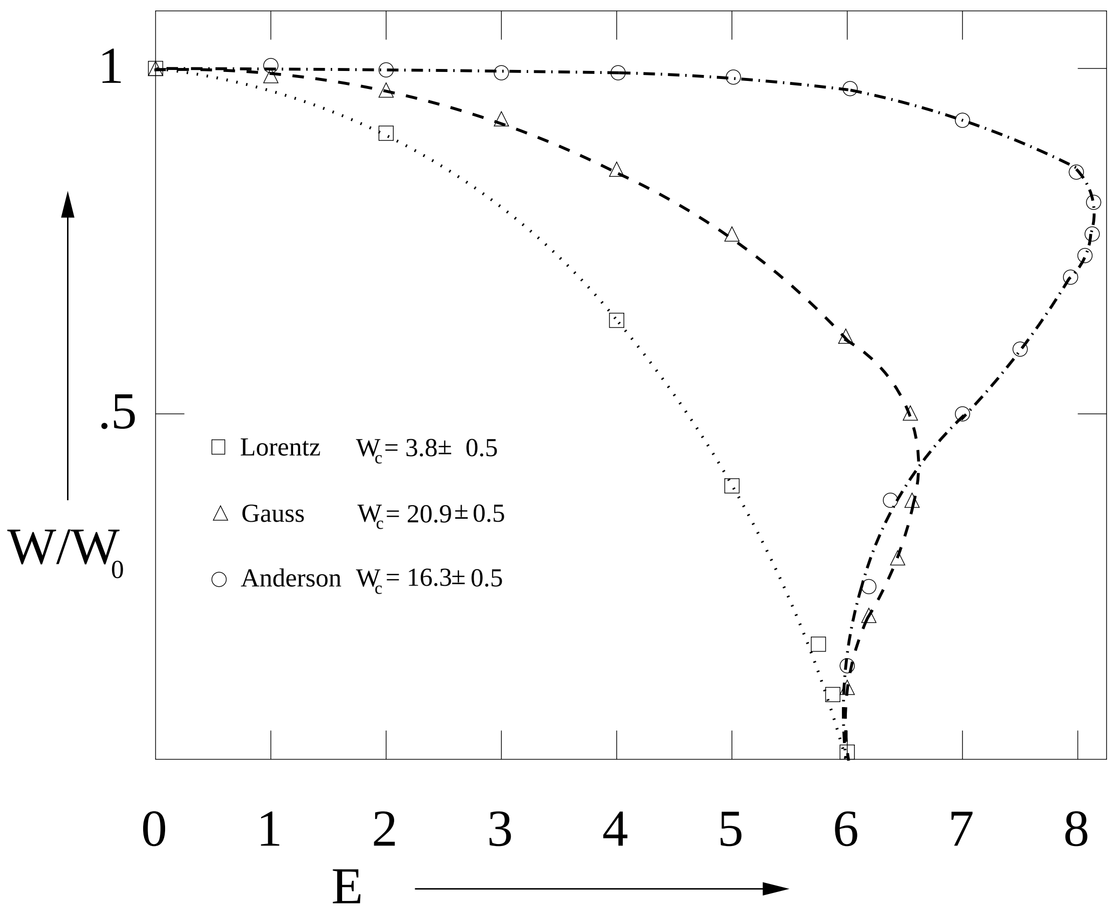
$$H_\omega \psi\,(x) \;=\; \sum_{y;|x-y|=1} \psi(y) + V(x)\psi(x)$$

where $\omega = \{V(x)\}_{x\in\mathbb{Z}^D} \in \Omega$ is a family of *i. i. d.*'s with

$$\langle V(x)\rangle = 0 \qquad \langle V(x)^2\rangle = W^2$$

The Anderson conjecture:

1. $D \leq 2$ *pure point spectrum*, with finite *localization length* for $W > 0$.
2. $D \geq 3$ there is a region of the phase space (E, W) in which the spectrum is absolutely continuous with positive *residual conductivity*.



- The 3D phase diagram of localization -

(*after B. Kramer & A. MacKinnon ('81-'85)*)

Results:

1. Anderson (1958): *localization*. Gang of 4 (1979): Anderson's transition for $D \geq 3$.
2. Wegner (1979)P: the *n-orbital model*; Wegner & Schaeffer (1980): Goldstone's boson.
3. Numerics: Pichard-Sarma (1981-84) Kramer & MacKinnon (1981-86).
4. Rigorous 1D: Pastur-Molchanov (1978), Kunz-Souillard (1979).
5. Rigorous $D \geq 2$: Fröhlich-Spencer (1983), Fröhlich-Spencer-Martinelli-Scoppola (1984), $\cdots$, Aizenman-Molchanov (1993). $\cdots$, Klein-Germinet (2002).
6. Supersymmetry: Wegner, Efetov (1983).
7. Random Matrices: Altshuler-Shklovskii (1986), mesoscopic systems.
8. Universality: Quasicrystals Berger, Mayou *et al.* (1987-89)

Noncommutative Calculus:

A *covariant operator* is a family $A = \{A_\omega\}$ of operators on $\mathcal{H}$ such that

1. $\omega \mapsto A_\omega$ is measurable,
2. $T(a) A_\omega T(a)^{-1} \;=\; A_{\mathrm{T}^a \omega}$

($T(a)$ *= translation by* $a \in \mathbb{Z}^D$, $\mathrm{T}^a \omega = \{V(x-a)\}_{x \in \mathbb{Z}^D}$*).*

Trace per unit volume:

$$\mathcal{T}_{\mathbb{P}}(A) \;=\; \int_\Omega \mathbb{P}(d\omega) \langle 0 | A_\omega | 0 \rangle \;=\; \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{1}{|\Lambda|} \mathrm{Tr}_\Lambda (A_\omega)$$

$\mathbb{P}$-almost surely *(*$\mathbb{P}$ *= probability distribution of* ω*).*

Derivatives:

$$(\partial_\mu A)_\omega \;=\; \imath [X_\mu \,,\, A_\omega] \qquad \vec{\nabla} \;=\; (\partial_1, \cdots, \partial_D)$$

$X = (X_1, \cdots, X_D)$ *position operator.*

IDS (Shubin's formula): **IDS (Shubin's formula):**

IDS = Integrated Density of States

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{1}{|\Lambda|} \#\{eigen.\ \ H_\omega \restriction_\Lambda \leq E\}$$

$$= \mathcal{T}_{\mathbb{P}}(\chi(H \leq E)) \qquad a.e.\ \ \omega$$

Current-Current correlation:

Current operator: $\vec{J} = e^2/\hbar\, \vec{\nabla} H.$

$$\mathcal{T}_{\mathbb{P}}(f(H)\ \partial_\nu H\ \ g(H)\ \partial_{\nu'} H)$$

$$= \int_{\mathbb{R}\times\mathbb{R}} m_{\nu,\nu'}(dE, dE')\ f(E)\, g(E')$$

Transport

Diffusion exponent:

$$(L_\Delta(t))^2 = \int_{-t}^{+t} \frac{ds}{2t} \int_X d\mathbb{P}_{tr}(\omega)$$
$$\cdots \langle 0|\Pi_{\omega,\Delta}\, |\vec{X}_\omega(s) - \vec{X}|^2\, \Pi_{\omega,\Delta}|0\rangle$$
$$\stackrel{t\uparrow\infty}{\sim} \; t^{2\,\beta_2(\Delta)} .$$

where $\Pi_{\omega,\Delta} = \chi(H_\omega \in \Delta)$. Equivalently *(J. B. & H. Schulz-Baldes ('98))*, if

$$m(dE, dE') = \sum_{\nu=1}^{D} m_{\nu,\nu}(dE, dE')$$

then

$$m\{(E, E' \in \Delta \times \mathbb{R}\,;\, |E - E'| \leq \epsilon\} \stackrel{\epsilon\downarrow 0}{\sim} \epsilon^{2(1-\beta_2(\Delta))} .$$

Kubo's formula:

In the *Relaxation Time Approximation*

$$\sigma_{\nu,\nu'} = \frac{e^2}{\hbar} \int_{\mathbb{R}^2} m_{\nu,\nu'}(dE, dE') \\ \cdots \frac{f_{T,\mu}(E) - f_{T,\mu}(E')}{E' - E} \frac{1}{\hbar/\tau_{coll} - \imath(E' - E)},$$

T = *temperature*,
μ = *chemical potential*
τ_{coll} = *average collision time*
k_B = *Boltzmann constant*

$$f_{T,\mu}(E) = \frac{1}{1 + e^{(E-\mu)/k_B T}}$$

Fermi level:

$$\mathcal{N}(E_F) = n_{el}$$

where n_{el} is the charge carrier density.

Theorem 1 *If* $m_{\nu,\nu'} = \rho^{(2)}_{\nu,\nu'} dE dE'$ *with* $\rho^{(2)}_{\nu,\nu'}(E,E')$ *continuous near* $E = E' = E_F$ *then, for any Borel set* $\Delta \subset \mathbb{R}$ *small enough containing* E_F

1. $\beta_2(\Delta) = 1/2$
2. *The diffusion constant* $D(\Delta) = \lim_{t\uparrow\infty} L_\Delta(t)^2/t$ *is finite and*

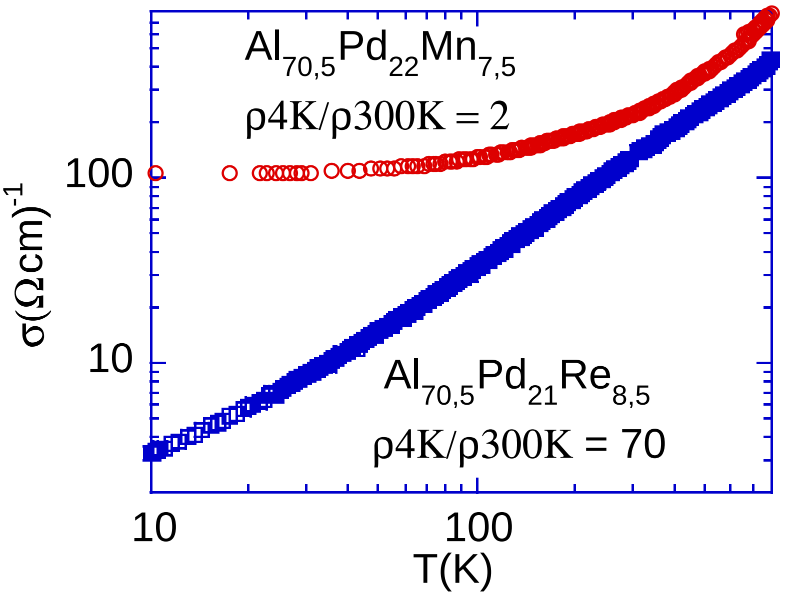
$$D(\Delta) \;=\; \pi \int_\Delta dE \,\sum_\nu \rho^{(2)}_{\nu,\nu}(E,E)$$

3. *The DC conductivity at zero temperature is finite and given by*

$$\sigma_{\nu,\nu} \;=\; \frac{\pi e^2}{\hbar}\; \rho^{(2)}_{\nu,\nu}(E_F, E_F)$$

Universality:

1. Other systems like *quasicrystals* exhibit diffusion exponents smaller than 1/2
2. At low temperature *quasicrystals* show a weak localization regime as for the Anderson model (*Mayou et al. '98*) suggesting $\beta_2 = 1/2$.
3. Numerics show a Wigner-Dyson spectral statistics (*Schreiber et al. '99*) for the octagonal tiling.



Conductivities of $AlPdMn$ and $AlPdRe$

C. Berger et al.

How can these observations be reconciled ?

1. Spectral statistics requires *finite size sample*. For a sample of size L,
2. *Heisenberg' time:* it takes a time $t_H = O(L^D)$ to see the discretization of the spectrum.
3. *Thouless time:* it takes a time $t_{Th} = O(L^{1/\beta_2})$ for the wave packet to reach the boundary.
4. If $t_{Th} \ll t_H$ the wave packet feels only the finite size.

$$\beta_2 \geq 1/D \quad \Rightarrow \quad \textit{weak localization}$$

$$\beta_2 < 1/D \quad \Rightarrow \quad \textit{scaling law}$$

5. *Guarneri's bound* shows that $\beta_2 \geq 1/D$ is compatible with an absolute continuous spectrum. Whereas if $\beta_2 < 1/D$ the spectrum near the Fermi level is *singular*.

Voiculescu's Free Calculus:

Let $\mathcal{A}$ be *unital algebra* $\mathcal{A}$.

1. A *distribution* is a linear map $\phi : \mathcal{A} \mapsto \mathbb{C}$ such that $\phi(1) = 1$,
2. A *random variable* is an element X of $\mathcal{A}$. Its *distribution* is the map $\phi_X : p \in \mathbb{C}[X] \mapsto \phi(p(X)) \in \mathbb{C}$.
3. $X_1, \cdots, X_n$ are *free* if for any $(i_1, \cdots, i_l) \in [1, n]^l$ such that $i_k \neq i_{k+1}$ and any polynomials $p_1, \cdots, p_l$

$$\phi(p_k(X_{i_k})) = 0 \ \forall k \Rightarrow \phi\left(p_1(X_{i_1}) \cdots p_l(X_{i_l})\right) = 0$$

4. *free convolution:* if X, Y are free, ϕ_{X+Y} depends only upon ϕ_X and ϕ_Y and is denoted $\phi_X \boxplus \phi_Y$.
5. *R-transform:* if $G_X(z) = \phi((z - X)^{-1})$

$$G_X = \frac{1}{z - R_X \circ G_X(z)}$$

Then X, Y free $\Rightarrow$ $R_{X+Y} = R_X + R_Y$.

Examples:

1. Let $X_1, \cdots, X_n$ be a family of $N \times N$ *independent random matrices* and $\phi = \mathbb{E}(1/N\mathrm{Tr}(.))$, then as $N \to \infty$ this family becomes free.

2. If, in the Anderson model, H and $\vec{\nabla} H$ are free with respect to $\mathcal{T}_{\mathbb{P}}$, then use

$$\phi(XYZY) = \phi(X)\phi(Y^2)\phi(Z)$$

if $\phi(Y) = 0$.Using this gives

$$\mathcal{T}_{\mathbb{P}}(f(H)\vec{\nabla}Hg(H)\vec{\nabla}H) =$$
$$\mathcal{T}_{\mathbb{P}}(f(H))\mathcal{T}_{\mathbb{P}}((gH))\mathcal{T}_{\mathbb{P}}(\vec{\nabla}H^2)$$

so that

$$m(dE, dE') = \mathcal{T}_{\mathbb{P}}(\vec{\nabla}H^2)\, \mathcal{N}(dE)\mathcal{N}(dE')$$

$d\mathcal{N}$ continuous $\Rightarrow$ finite conductivity

Wegner's n-orbital Model:

$$H_\omega \psi(x) = \sum_{y;|x-y|=1} \psi(y) + V(x)\psi(x)$$

where $\omega = \{V(x)\}_{x\in\mathbb{Z}^D}$ is a family of *identically distributed random $n \times n$ matrices* with

$$\langle V(x) \rangle = 0 \qquad \langle (V(x))^2_{i,j} \rangle = \frac{W^2}{n}$$

Theorem 2 *1. In the limit $n \to \infty$ the $V(x)$'s become free variables.*

2. In the limit $n \to \infty$ the density of states $d\mathcal{N}$ and the current-current correlation of the Wegner model are continuous.

(Wegner '81, Pastur-Khorunzhy '93, Neu-Speicher '94)

The DPS model:

$H = (H_{ij})$ is a random gaussian matrix with zero average and covariance

$$\langle H_{ij} H_{kl} \rangle = \delta_{ij} \delta_{kl} J_{ij}$$

with

$$J_{ij} = \left(\frac{1}{-W^2 \Delta + 1} \right)_{ij}$$

where i, j vary in $\Lambda \cap \mathbb{Z}^3$, Λ a set of cubes of size W and $W > 0$ large but fixed. Δ is the discrete Laplacian with periodic $b.c.$.

The DOS is defined by

$$\rho_\Lambda(E) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \Im \langle \left(\frac{1}{E + \imath\epsilon - H} \right)_{00} \rangle$$

This model should behave like the 3D Anderson one.

The derivative of the DOS is the imaginary part of $\sum_x R(E+\imath 0^+;0,x)/\pi$ where $R(E+\imath\epsilon;0,x)$ is defined by

$$\langle \left(\frac{1}{E+\imath\epsilon-H}\right)_{0x}\left(\frac{1}{E+\imath\epsilon-H}\right)_{x0}\rangle$$

Theorem 3 *For W large enough the DOS of this model is smooth and coincides, in $[-2,2]$, with the Wigner semicircle distribution modulo a correction of order W^{-2}. Moreover $R(E+\imath\epsilon;0,x)$ decays exponentially fast in x uniformly as $\epsilon\downarrow 0$ and as $\Lambda\uparrow\mathbb{Z}^3$.*

Reference:

M. Disertori, H. Pinson, T. Spencer, *Density of states for Random Band Matrix* math-ph/0111047, *Commun. Math. Phys.*, **232**, (2002), 83-124.

RMT & Anderson:

The Anderson's model at weak coupling:

(Poirot, Magnen, Rivasseau '98)

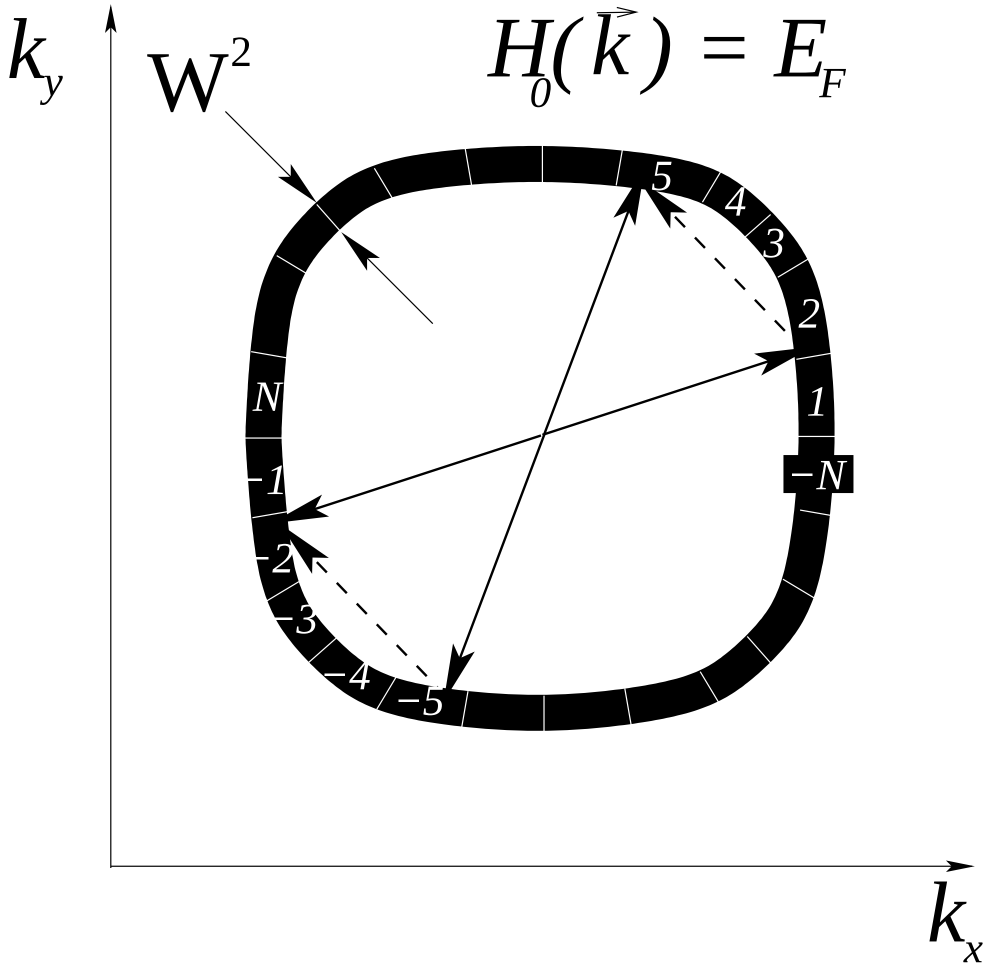
1. The unperturbed Hamiltonian is the discrete Laplacian H_0. Set $\chi_W = \chi(|H_0 - E_F| \leq c \cdot W^2))$.

2. The part of the Hamiltonian with energies away from E_F by $O(W^2)$ can be treated as a perturbation. Let then $\chi_W = \chi(|H_0 - E_F| \leq O(W^2))$ and $H_{eff} = \chi_W H \chi_W$.

3. Divide $\mathbb{Z}^D$ into boxes of size $O(W^{-2})$. In each such box Λ, let V_Λ be the restriction of H_{eff} to Λ.

4. The V_Λ's play a role similar to the potential in the Wegner n-orbital model with $n = O(W^{-2})$. The connecting operators between boxes play the role of the discrete Laplacian.

5. For $D = 2$ and *gaussian potential*, the V_Λ's are gaussian random matrices of the type GOE with an extra discrete symmetry: *flip matrices*



- Fermi Surface for the Flip Model -

$$\textit{here } V_{5,2} = V_{-2,-5} = \overline{V_{2,5}} = \overline{V_{-5,-2}}$$

The Flip Matrix model for $D = 2$:

(Bellissard, Magnen, Rivasseau '02)

1. Since Λ is a finite box the quasimomentum space is discrete. The thicken Fermi surface defined by χ_W contains only $n = 2N = O(W^{-2})$ quasimomenta denotes by $\alpha, \beta \in \{1, \cdots, N\} \cup \{-N, \cdots, -1\}$
2. V_Λ is a $2N \times 2N$ selfadjoint gaussian random matrix, indexed by quasimomenta
$$V_{\alpha,\beta} = \overline{V_{\beta,\alpha}}$$
3. Momentum conservation leads to
$$V_{\alpha,\beta} = V_{-\beta,-\alpha} \qquad V_{\alpha,\alpha} = V_0$$
4. Modulo these constraints the matrix elements are independent and
$$\langle V_{\alpha,\beta} \rangle = 0 \qquad \langle |V_{\alpha,\beta}|^2 \rangle = W^2 = O(1/2N)$$
5. **Main result:** *The DOS od the flip model is a semicircular distribution.*

About the Proof:

1. Using supersymmetry the DOS can be written as

$$\frac{d\mathcal{N}}{dE} = \lim_{\Im m E \downarrow 0} \frac{1}{\pi} \int S^{+}_{+\alpha} S_{+\alpha}\, e^{L}\, \mathcal{D}\Psi^{\dagger}\mathcal{D}\Psi$$

   where $\Psi_{\pm\alpha} = (S_{\pm\alpha}, \chi_{\pm\alpha})$ is a superfield
   L is a sum of quartic terms of the form
   $\Psi^{\dagger}_{\pm\alpha}\Psi_{\pm\beta}\Psi^{\dagger}_{\pm\beta}\Psi_{\pm\alpha}$

2. Using commutation rules it can be written as a sum of terms of the form

$$\Psi^{\dagger}_{\pm\alpha}\Psi_{\pm\alpha}\Psi^{\dagger}_{\pm\beta}\Psi_{\pm\beta}$$

   separating the α's from the β's.

3. Using a gaussian integral

$$e^{L} = \int \mathcal{D}R\, e^{\imath W \sum_{\alpha} \Psi^{\dagger}_{\alpha} R \Psi_{\alpha}}$$

   where R varies in a set of 4×4 supermatrices.

4. The gaussian integral can be performed and gives rise to a 5-dimensional integral of the form

$$\int_{\mathbb{R}^5} d^5x \ (F(x))^N$$

   Since $1 \ll N = O(W^{-2})$, this can be analyzed through a saddle point method.

5. Among the 4 saddle points only one contributes in the limit $N \to \infty$, giving rise to a semi-circle distribution.

Remark:

Such a Supermatrix R is attached to each box of size $O(W^{-2})$. R is the order parameter. There is an effective lattice Hamiltonian connecting them similar to a *spin system*. The low energy excitations are given by *spin waves*: this is the famous *Goldstone mode (Wegner '80)*

Conclusion

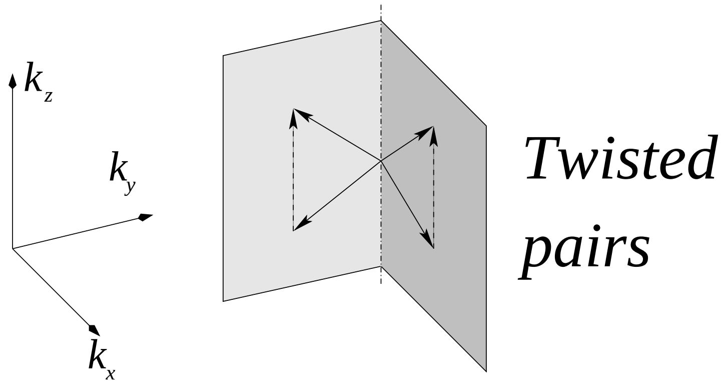
1. The Anderson model in $2D$ at low coupling W can be approximated by a Random Matrix of size $O(W^{-2})$ for energies $|E - E_F| \leq O(W^2)$.

2. The rest of the Hamiltonian can be treated perturbatively: *prove it.*

3. Within this approximation the DOS is smooth: *control the finite size correction.*

4. For all $W > 0$ *localization* is expected. This is because the Goldstone mode gets a mass $e^{-O(W^{-2})}$. (*a non perturbative result: hard to prove*)

5. For $D \geq 3$, the previous model has a bigger degeneracy. A power counting of Feynmann graphs for the SUSY theory shows that the dominant contribution comes from a flip matrix similar to the $D = 2$ case: *prove it.*



6. The limit $W \downarrow 0$ looks similar to a Wegner model in any dimension: *Prove it.*

7. For $D \geq 3$ the Goldstone mode remains massless: *an infra-red inequality, based upon Osterwalder-Schrader positivity, is possibly a strategy for a rigorous proof.*