RIEMANNIAN GEOMETRY

of

COMPACT METRIC SPACES

Jean BELLISSARD [1]

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics

Collaboration:

I. PALMER (Georgia Tech, Atlanta)

[1] e-mail: jeanbel@math.gatech.edu

Main References

J. Pearson, J. Bellissard,
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
Journal of Noncommutative Geometry, **3**, (2009), 447-480.

A. Connes,
Noncommutative Geometry,
Academic Press, 1994.

G. Michon,
Les Cantors réguliers,
C. R. Acad. Sci. Paris Sér. I Math., (19), **300**, (1985) 673-675.

K. Falconer,
Fractal Geometry: Mathematical Foundations and Applications,
John Wiley and Sons 1990.

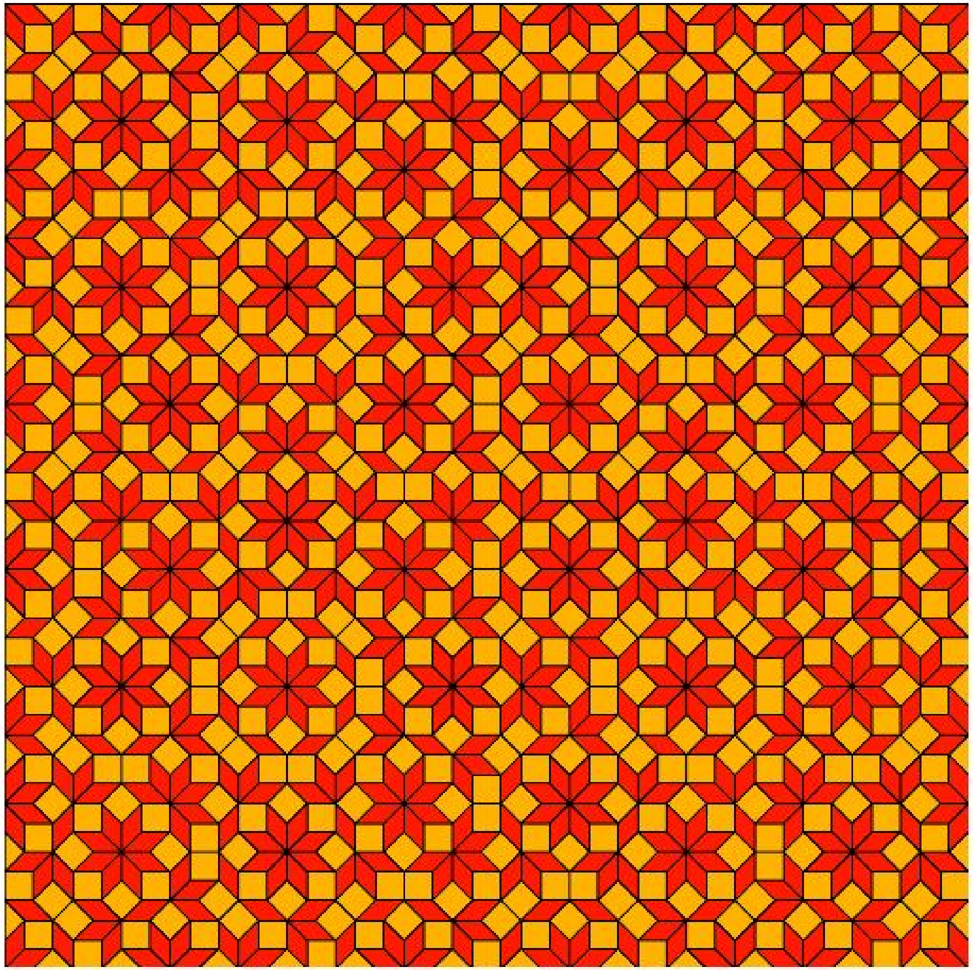
I. Palmer, *Noncommutative Geometry of compact metric spaces*, PhD Thesis, May 3rd, 2010.

Motivation

A tiling of $\mathbb{R}^d$ or a Delone set describing the atomic positions in a solid defines a *tiling space*: a suitable closure of its translated. This space is compact. Various metrics may help describing the properties of the tiling itself such as

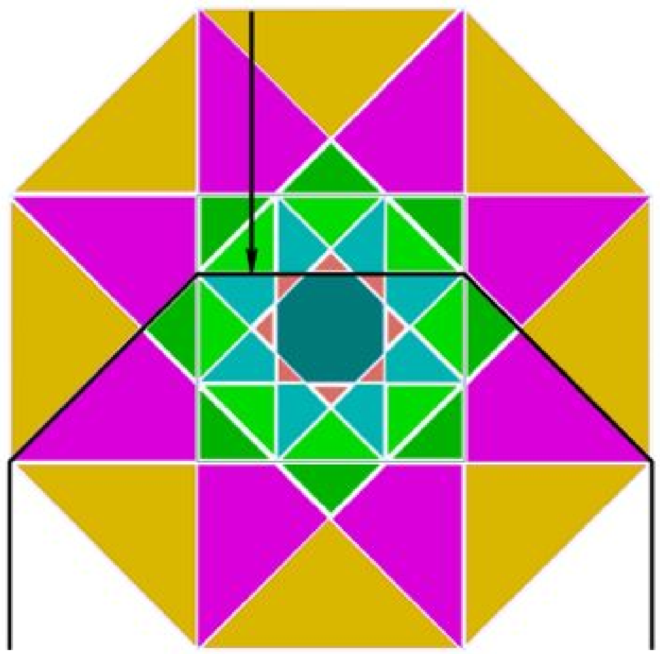
- Its algorithmic complexity or its configurational entropy.
- The atomic diffusion process
- Hopefully the mechanic of the solid (friction, fracture, ...)

Motivation



The octagonal tiling

Motivation



The tiling space of the octagonal tiling is a Cantor set

Content

1. Spectral Triples
2. ζ-function and Hausdorff Measure
3. The Laplace-Beltrami Operator
4. Ultrametric Cantor sets
5. To conclude

I - Spectral Triples

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

I.1)- Spectral Triples

A *spectral triple* is a family $(\mathcal{H}, \mathcal{A}, D)$, such that

- $\mathcal{H}$ is a Hilbert space
- D is a self-adjoint operator on $\mathcal{H}$ with *compact resolvent*
- $\mathcal{A}$ is a C^*-algebra with a representation π into $\mathcal{H}$ such that $\mathcal{A}_0 = \{a \in \mathcal{A}\,;\, \|[D, \pi(a)\| < \infty\}$ is *dense* in $\mathcal{A}$.

- $(\mathcal{H}, \mathcal{A}, D)$ is called *even* if there is $G \in \mathcal{B}(\mathcal{H})$ such that
  - $G = G^* = G^{-1}$
  - $[G, \pi(f)] = 0$ for $f \in \mathcal{A}$
  - $GD = -DG$

I.2)- Example of Spectral Triples

If $\mathbb{T}$ is the *1D-torus* then take $\mathcal{A} = C(\mathbb{T})$, $\mathcal{H} = L^2(\mathbb{T})$ and $D = -\imath d/dx$. $\mathcal{A}$ is represented by pointwise multiplication. This is a spectral triple such that

$$|x - y| = \sup\{|f(x) - f(y)| \,;\, f \in C(\mathbb{T}) \,,\, \|[D, \pi(f)]\| \leq 1\}$$

If M is *compact spin$_c$ Riemannian* manifold, then take $\mathcal{A} = C(M)$, $\mathcal{H}$ be the Hilbert space of L^2-sections of the *spinor bundle* and D the *Dirac* operator. $\mathcal{A}$ is represented by pointwise multiplication. This is a spectral triple such that the *geodesic distance* is given by

$$d(x, y) = \sup\{|f(x) - f(y)| \,;\, f \in C(\mathbb{T}) \,,\, \|[D, \pi(f)]\| \leq 1\}$$

I.3)- Properties of Spectral Triples

Definition *A spectral triple* $(\mathcal{H}, \mathcal{A}, D)$ *will be called regular whenever the following two properties hold*
(i) the commutant $\mathcal{A}' = \{a \in \mathcal{A} \,;\, [D, \pi(a)] = 0\}$ *is trivial*
(ii) the Lipshitz ball $B_{Lip} = \{a \in \mathcal{A} \,;\, \|[D, \pi(a)]\| \leq 1\}$ *is precompact in* $\mathcal{A}/\mathcal{A}'$

Theorem *A spectral triple* $(\mathcal{H}, \mathcal{A}, D)$ *is regular if and only if the Connes metric, defined on the state space of* $\mathcal{A}$ *by*

$$d_C(\omega, \omega') = \sup\{|\omega(a) - \omega'(a)| \,;\, \|[D, \pi(a)]\| \leq 1\}$$

is well defined and equivalent to the weak-topology*

I.4)- ζ-function and Spectral Dimension

Definition *A spectral triple* $(\mathcal{H}, \mathcal{A}, D)$ *is called summable is there is* $p > 0$ *such that* $\mathrm{Tr}\,(|D|^{-p}) < \infty$. *Then, the ζ-function is defined as*

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

The *spectral dimension* is

$$s_D = \inf\left\{s > 0\,;\, \mathrm{Tr}\left(\frac{1}{|D|^s}\right) < \infty\right\}$$

Then ζ is *holomorphic* in $\Re(s) > s_D$

Remark *For a Riemannian manifolds* $s_D = \dim(M)$

I.5)- Connes state & Volume Form

The spectral triple is *spectrally regular* if the following limit is unique

$$\omega_D(a) = \lim_{s \downarrow s_D} \frac{1}{\zeta(s)} \operatorname{Tr}\left(\frac{1}{|D|^s}\,\pi(a)\right) \qquad a \in \mathcal{A}$$

Then ω_D is called the *Connes state*.

Remark

(i) By compactness, limit states always exist, but the limit may not be unique.

(ii) Even if unique this state might be trivial.

(iii) In the example of compact Riemannian manifold the Connes state exists and defines the volume form.

I.6)- Hilbert Space

If the Connes state is well defined, it induces a *GNS-representation* as follows

- The Hilbert space $L^2(\mathcal{A}, \omega_D)$ is defined from $\mathcal{A}$ through the inner product

$$\langle a|b\rangle = \omega_D(a^*b)$$

- The algebra $\mathcal{A}$ acts by *left multiplication*.
- If the quadratic form

$$Q(a,b) = \lim_{s\downarrow s_D} \frac{1}{\zeta(s)} \operatorname{Tr}\left(\frac{1}{|D|^s}[D,\pi(a)]^*[D,\pi(b)]\right)$$

extends to $L^2(\mathcal{A}, \omega_D)$ as a *closable quadratic form*, then, it defines a positive operator which generates a *Markov semi-group* and is a candidate for being the analog of the *Laplace-Beltrami operator*.

II - Compact Metric Spaces

I. Palmer, *Noncommutative Geometry of compact metric spaces*, PhD Thesis, May 3rd, 2010.

II.1)- Open Covers

Let (X, d) be a *compact metric space* with an infinite number of points. Let $\mathcal{A} = C(X)$.

- An *open cover* $\mathcal{U}$ is a family of open sets of X with union equal to X. Then $\mathrm{diam}\mathcal{U} = sup\{\mathrm{diam}(U)\,;\, U \in \mathcal{U}\}$. All open covers used here will be at most *countable*
- A *resolving sequence* is a family $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \mathrm{diam}(\mathcal{U}_n) = 0$$

- A resolving sequence is *strict* if all $\mathcal{U}_n$'s are finite and if

$$\mathrm{diam}(\mathcal{U}_n) < \inf\{\mathrm{diam}(U)\,;\, U \in \mathcal{U}_{n-1}\} \quad \forall n$$

II.2)- Choice Functions

Given a resolving sequence $\xi = (\mathcal{U}_n)_{n\in\mathbb{N}}$ a *choice function* is a map $\tau : \mathcal{U}(\xi) = \coprod_n \mathcal{U}_n \mapsto X \times X$ such that

- $\tau(U) = (\tau_+(U), \tau_-(U)) \in U \times U$
- there is $C > 0$ such that

$$\mathrm{diam}(U) \geq d(\tau_+(U), \tau_-(U)) \geq \frac{\mathrm{diam}(U)}{1 + C\,\mathrm{diam}(U)}, \qquad \forall U \in \mathcal{U}(\xi)$$

The *set* of such choice functions is denoted by $\Upsilon(\xi)$.

II.3)- A Family of Spectral Triples

- Given a *resolving sequence* ξ, let $\mathcal{H}_\xi = \ell^2(\mathcal{U}(\xi)) \otimes \mathbb{C}^2$
- For τ a *choice* let $D_{\xi,\tau}$ be the *Dirac operator* defined by

$$D_{\xi,\tau}\psi\,(U) = \frac{1}{d(\tau_+(U),\tau_-(U))}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\psi(U) \qquad \psi \in \mathcal{H}$$

- For $f \in C(X)$ let $\pi_{\xi,\tau}$ be the *representation* of $\mathcal{A} = C(X)$ given by

$$\pi_{\xi,\tau}(f)\psi\,(U) = \begin{bmatrix} f(\tau_+(U)) & 0 \\ 0 & f(\tau_-(U)) \end{bmatrix}\psi(U) \qquad \psi \in \mathcal{H}$$

II.4)- Regularity

Theorem *Each* $\mathfrak{T}_{\xi,\tau} = (\mathcal{H}_\xi, \mathcal{A}, D_{\xi,\tau}, \pi_{\xi,\tau})$ *defines a spectral triple such that* $\mathcal{A}_0 = C_{Lip}(X, d)$ *is the space of Lipshitz continuous functions on X. Such a triple is even when endowed with the grading operator*

$$G\psi(U) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

In addition, the family $\{\mathfrak{T}_{\xi,\tau}\,;\, \tau \in \Upsilon(\xi)\}$ *is regular in that*

$$d(x,y) = \sup\{|f(x - f(y)|\,;\, \sup_{\tau\in\Upsilon(\xi)} \|[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]\| \leq 1\}$$

II.5)- Summability

Theorem *There is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral triples if and only if the Hausdorff dimension of X is finite.*

If so, the spectral dimension s_D *satisfies* $s_D \geq \dim_H(X)$.

If $\dim_H(X) < \infty$ *there is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral triples with spectral dimension* $s_D = \dim_H(X)$.

II.6)- Hausdorff Measure

Theorem *There exist a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of spectrally regular spectral triples if and only if the Hausdorff measure of* X *is positive and finite.*

In such a case the Connes state coincides with the normalized Hausdorff measure on X*.*

Then the Connes state is given by the following limit *independently* of the choice τ

$$\frac{\int_X f(x)\mathcal{H}^{s_D}(dx)}{\mathcal{H}^{s_D}(X)} = \lim_{s\downarrow s_D} \frac{1}{\zeta_{\xi,\tau}(s)} \operatorname{Tr}\left(\frac{1}{|D_{\xi,\tau}|^s}\,\pi_{\xi,\tau}(f)\right) \qquad f \in C(X)$$

III - The Laplace-Beltrami Operator

M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland (1980).

J. Pearson, J. Bellissard,
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
Journal of Noncommutative Geometry, **3**, (2009), 447-480.

J. Bellissard, I. Palmer, *in progress*.

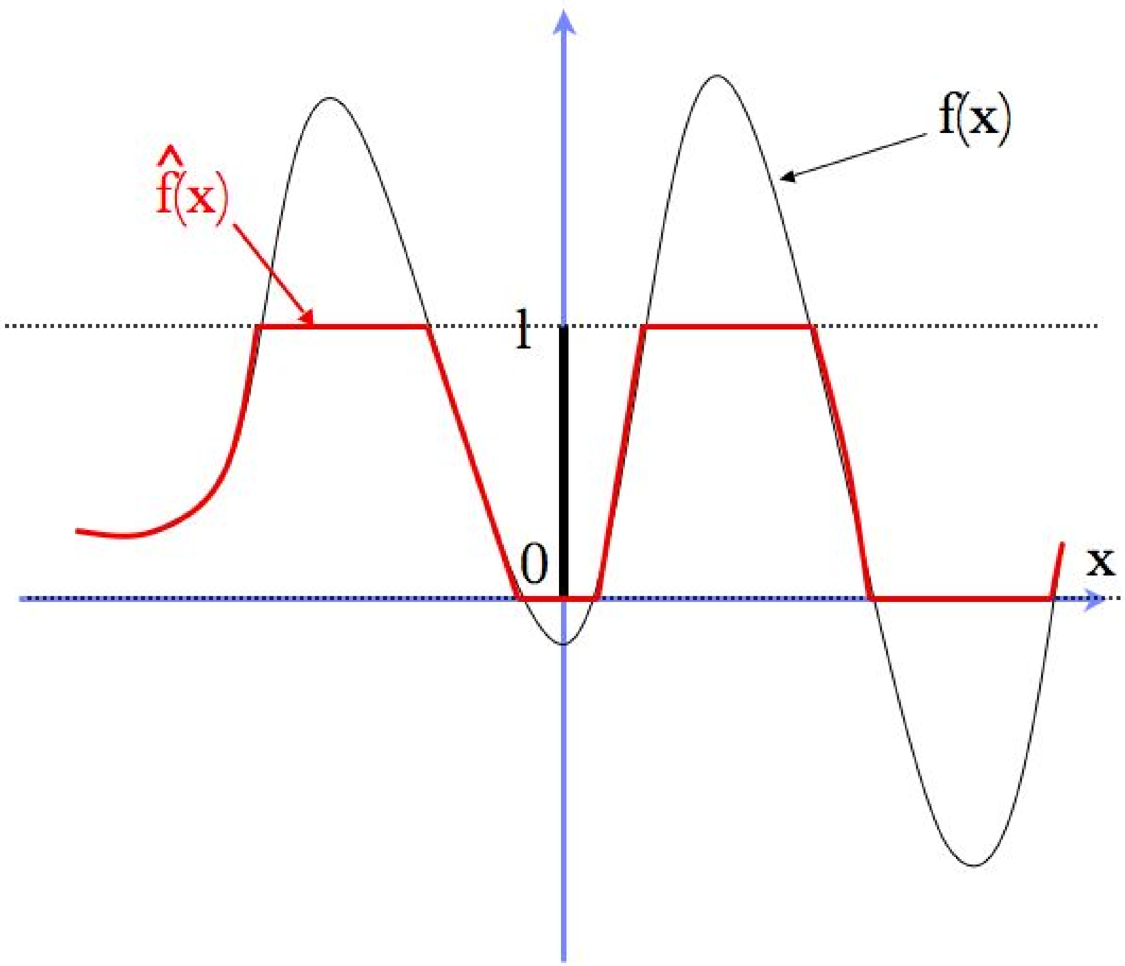
III.1)- Dirichlet Forms

Let (X, μ) be a probability space space. For f a *real valued* measurable function on X, let $\hat{f}$ be the function obtained as

$$\hat{f}(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq 1 \\ f(x) & \text{if} \quad 0 \leq f(x) \leq 1 \\ 0 & \text{if} \quad f(x) \leq 0 \end{cases}$$

A Dirichlet form Q on X is a *positive definite sesquilinear form* $Q : L^2(X, \mu) \times L^2(X, \mu) \mapsto \mathbb{C}$ such that

- Q is densely defined with domain $\mathcal{D} \subset L^2(X, \mu)$
- Q is closed
- Q is *Markovian*, namely if $f \in \mathcal{D}$, then $Q(\hat{f}, \hat{f}) \leq Q(f, f)$



Markovian cut-off of a real valued function

The simplest typical example of Dirichlet form is related to the Laplacian Δ_{Ω} on a bounded domain $\Omega \subset \mathbb{R}^D$

$$Q_{\Omega}(f,g) = \int_{\Omega} d^{D}x \; \overline{\nabla f(x)} \cdot \nabla g(x)$$

with domain $\mathcal{D} = C_0^1(\Omega)$ the space of continuously differentiable functions on Ω vanishing on the boundary.

This form is closable in $L^2(\Omega)$ and its closure defines a Dirichlet form.

Any closed positive sesquilinear form Q on a Hilbert space, defines canonically a *positive self-adjoint operator* $-\Delta_Q$ satisfying

$$\langle f| - \Delta_Q \, g\rangle = Q(f,g)$$

In particular $\Phi_t = \exp(t\Delta_Q)$ (defined for $t \in \mathbb{R}_+$) is a strongly continuous *contraction* semigroup.

If Q is a Dirichlet form on X, then the contraction semigroup $\Phi = (\Phi_t)_{t\geq 0}$ is a *Markov semigroup*.

A *Markov semi-group* Φ on $L^2(X, \mu)$ is a family $(\Phi_t)_{t\in[0,+\infty)}$ where

- For each $t \geq 0$, Φ_t is a *contraction* from $L^2(X, \mu)$ into itself
- *(Markov property)* $\Phi_t \circ \Phi_s = \Phi_{t+s}$
- *(Strong continuity)* the map $t \in [0, +\infty) \mapsto \Phi_t$ is strongly continuous
- $\forall t \geq 0$, Φ_t is *positivity preserving* : $f \geq 0 \ \Rightarrow \ \Phi_t(f) \geq 0$
- Φ_t is *normalized*, namely $\Phi_t(1) = 1$.

Theorem (Fukushima) *A contraction semi-group on $L^2(X, \mu)$ is a Markov semi-group if and only if its generator is defined by a Dirichlet form.*

III.2)- The Laplace-Beltrami Form

Let M be a *Riemannian manifold* of dimension D. The *Laplace-Beltrami operator* is associated with the Dirichlet form

$$Q_{\mathrm{M}}(f,g) = \sum_{i,j=1}^{D} \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))} \, g_{ij}(x) \, \overline{\partial_i f(x)} \, \partial_j g(x)$$

where g is the metric. Equivalently (in local coordinates)

$$Q_{\mathrm{M}}(f,g) = \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))} \int_{S(x)} d\nu_x(u) \, \overline{u \cdot \nabla f(x)} \, u \cdot \nabla g(x)$$

where $S(x)$ represent the *unit sphere* in the tangent space whereas ν_x is the *normalized Haar measure* on $S(x)$.

III.3)- Choices and Tangent Space

The main remark is that, if $\tau(U) = (x, y)$ then

$$[D, \pi(f)]_\tau\ \psi\ (U) = \frac{f(x) - f(y)}{d(x, y)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(U)$$

The commutator with the Dirac operator is a coarse graining version of a *directional derivative*. Therefore

- it could be written as $\nabla_\tau f$
- $\tau(U)$ can be interpreted as a coarse grained version of a *normalized tangent vector* at U.
- the set $\Upsilon(\xi)$ can be seen as the set of *sections of the tangent sphere bundle*.

III.4)- Choice Averaging

To mimic the previous formula, a *probability* over the set $\Upsilon(\xi)$ is required.

For each open set $U \in \mathcal{U}(\xi)$, the set of choices is given by the set of pairs $(x, y) \in U \times U$ such that $d(x, y) > \mathrm{diam}(U)\,(1 + C\,\mathrm{diam}(U))^{-1}$. This is an *open set*.

Thus the probability measure ν_U defined as the *normalized measure* obtained from *restricting* $\mathcal{H}^{s_D} \otimes \mathcal{H}^{s_D}$ to this set is the right one.

This leads to the probability

$$\nu = \bigotimes_{U \in \mathcal{U}(\xi)} \nu_U$$

III.5)- The Quadratic Form

This leads to the quadratic form (omitting the indices ξ, τ)

$$Q(f,g) = \lim_{s \to s_D} \int_{\Upsilon(\xi)} d\nu(\tau) \, \frac{1}{\zeta(s)} \, \mathrm{Tr}\left(\frac{1}{|D|^s} [D, \pi(f)]^* \, [D, \pi(g)] \right)$$

Claim (unproved yet) *This quadratic form is closable and Markovian.*

Claim *If X is a Riemannian manifold equipped with the geodesic distance this quadratic form coincides with the Laplace-Beltrami one.*

Theorem *If (X,d) is an ultrametric Cantor set, this quadratic form vanishes identically.*

III.6)- Cantor sets

If (X, d) is an ultrametric Cantor set, the characteristic functions of clopen sets are continuous. For such a function $[D, \pi(f)]$ is a finite rank operator. To replace the previous form simply set, for any real $s \in \mathbb{R}$

$$Q_s(f, g) = \int_{\Upsilon(\xi)} d\nu(\tau) \ \mathrm{Tr}\left(\frac{1}{|D|^s}[D, \pi(f)]^* \ [D, \pi(g)]\right)$$

Theorem *If (X, d) is an ultrametric Cantor set, the quadratic forms Q_s are closable in $L^2(X, \mathcal{H}^{s_D})$ and Markovian. The corresponding Laplacean have pure point spectrum. They are bounded if and only if $s > s_D + 2$ and have compact resolvent otherwise. The eigenspaces are common to all s's and can be explicitly computed.*

IV - Conclusion & Prospect

IV.1)- Results

- A compact metric space can be described as *Riemannian manifolds*, through Noncommutative Geometry.
- An analog of the *tangent unit sphere* is given by *choices*
- The *Hausdorff dimension* plays the role of the *dimension*.
- A *Hausdorff measure* is the analog of the *volume form*
- A *Laplace-Beltrami operator* can be defined which coincided with the usual definition if X is a Riemannian manifold.
- It generates a *stochastic process* playing the role of the *Brownian motion*.

IV.2)- Cantor Sets

If the space is an *ultrametric Cantor set* more is known

- The set of ultrametric can be described and characterized
- The Laplace-Beltrami operator *vanishes* but can be replaced by a *one parameter family of Dirichlet forms*, defined by Pearson in his PhD thesis
- The *Pearson operators* have point spectrum and for the right domain of the parameter, they have compact resolvent.
- A *Weyl asymptotics* for the eigenvalues can be shown to hold.
- The corresponding stochastic process is a *jump process*
- This process exhibits *anomalous diffusion*.

IV.3)- Open Problems

- Prove that the Laplace-Beltrami operator is *well defined* at least for a compact metric space with *nonzero finite Hausdorff measure*.
- Prove that the Laplace-Beltrami operator has *compact resolvent*
- Prove that the Laplace-Beltrami operator coincides with the generator of *diffusion on fractal sets* such as the *Sierpinski gasket*.