The TRANSVERSE GEOMETRY of TILING SPACES

Sponsoring

Jean BELLISSARD

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics
e-mail: jeanbel@math.gatech.edu

Collaborations

J. Pearson, (*Gatech*, Atlanta, GA)

J. Savinien, (*U. Metz*, Metz, France)

A. Julien, (*U. Victoria*, Victoria, BC)

I. Palmer, (*NSA*, Washington DC)

R. Parada, (*Gatech*, Atlanta, GA)

Main References

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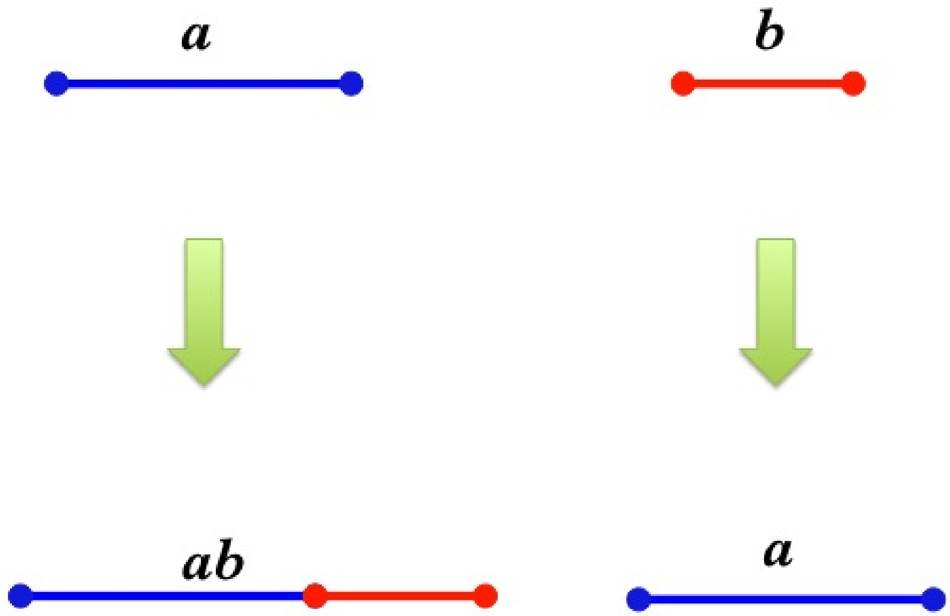
Content

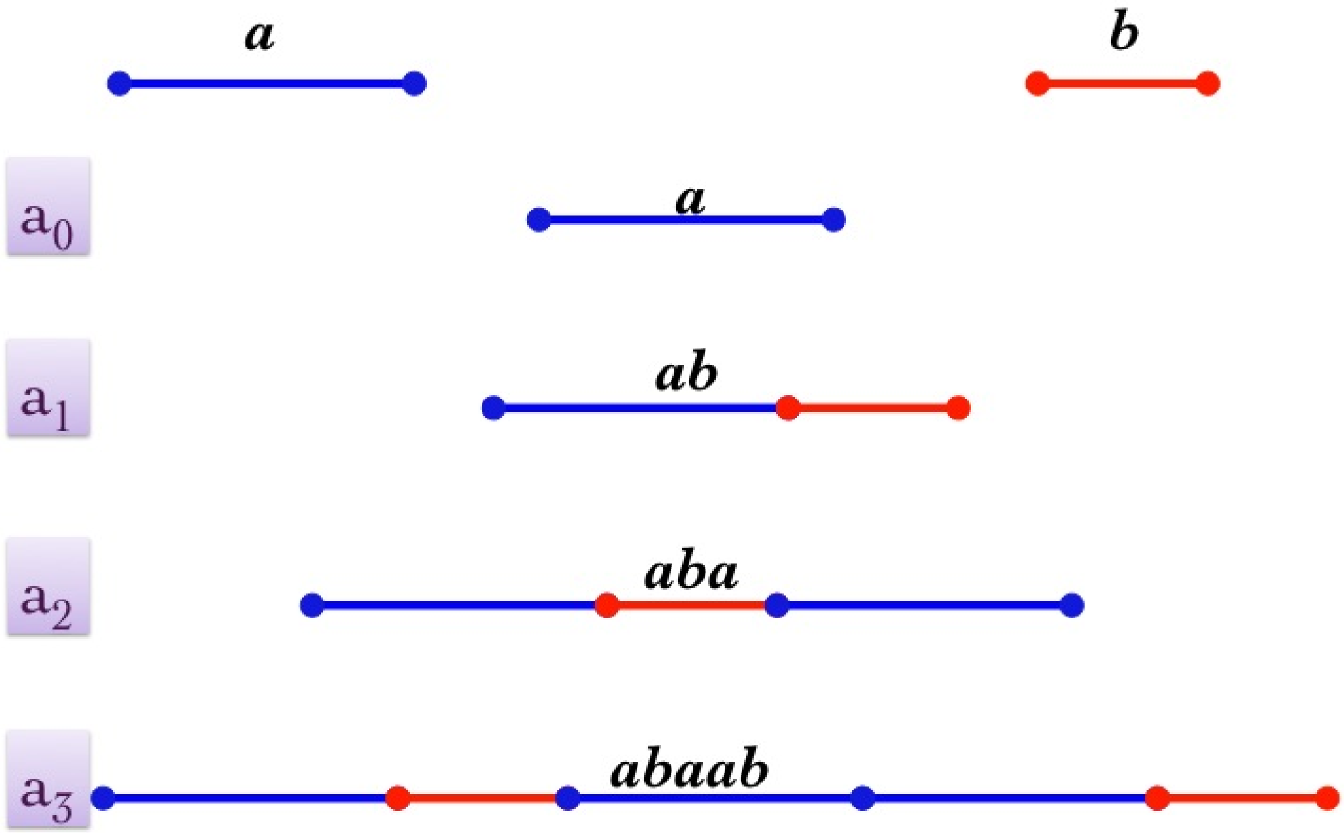
I - Tilings and their Transversal

The Fibonacci Tiling



The Fibonacci Substitution

The Fibonacci Tiling



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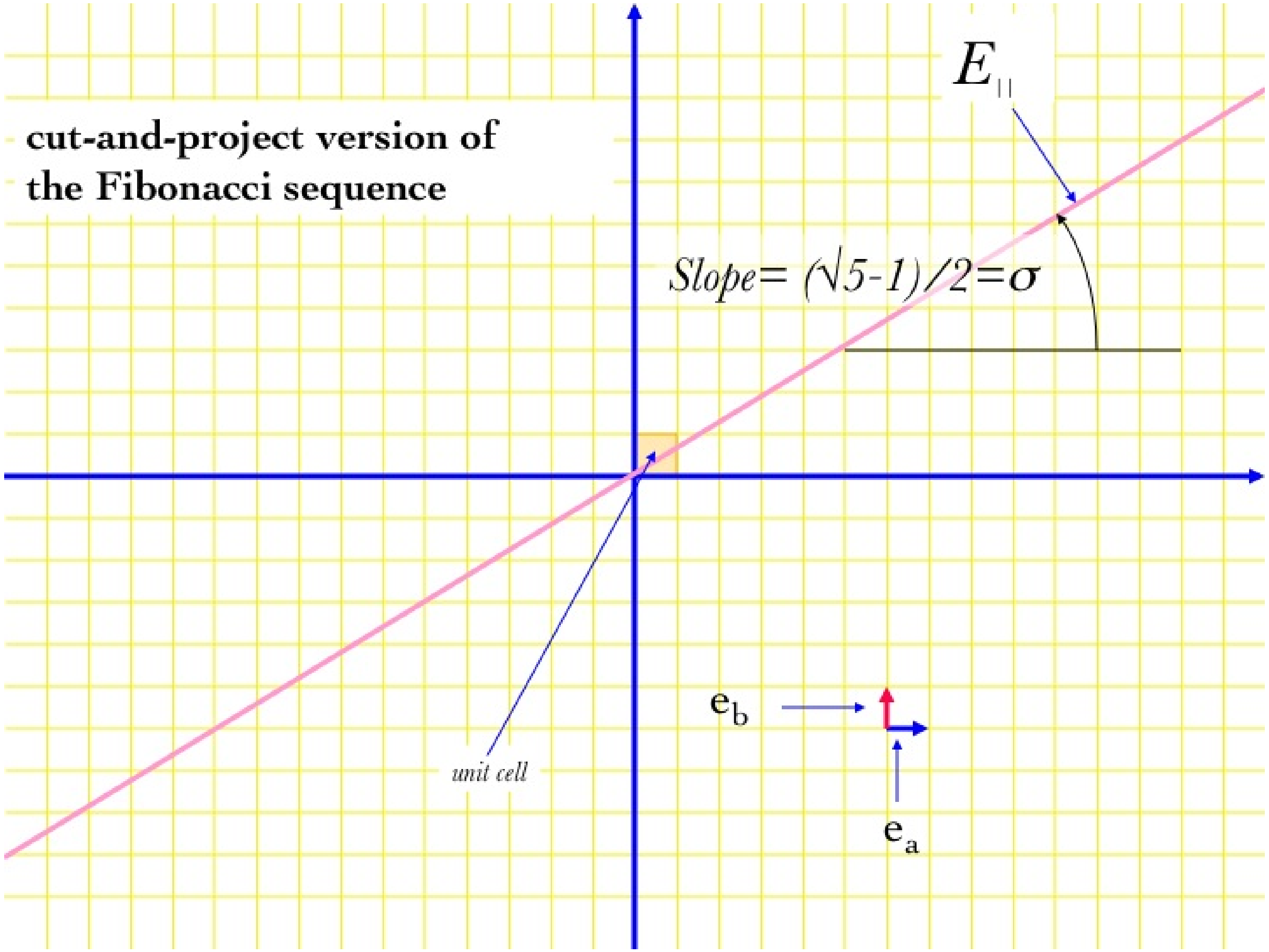

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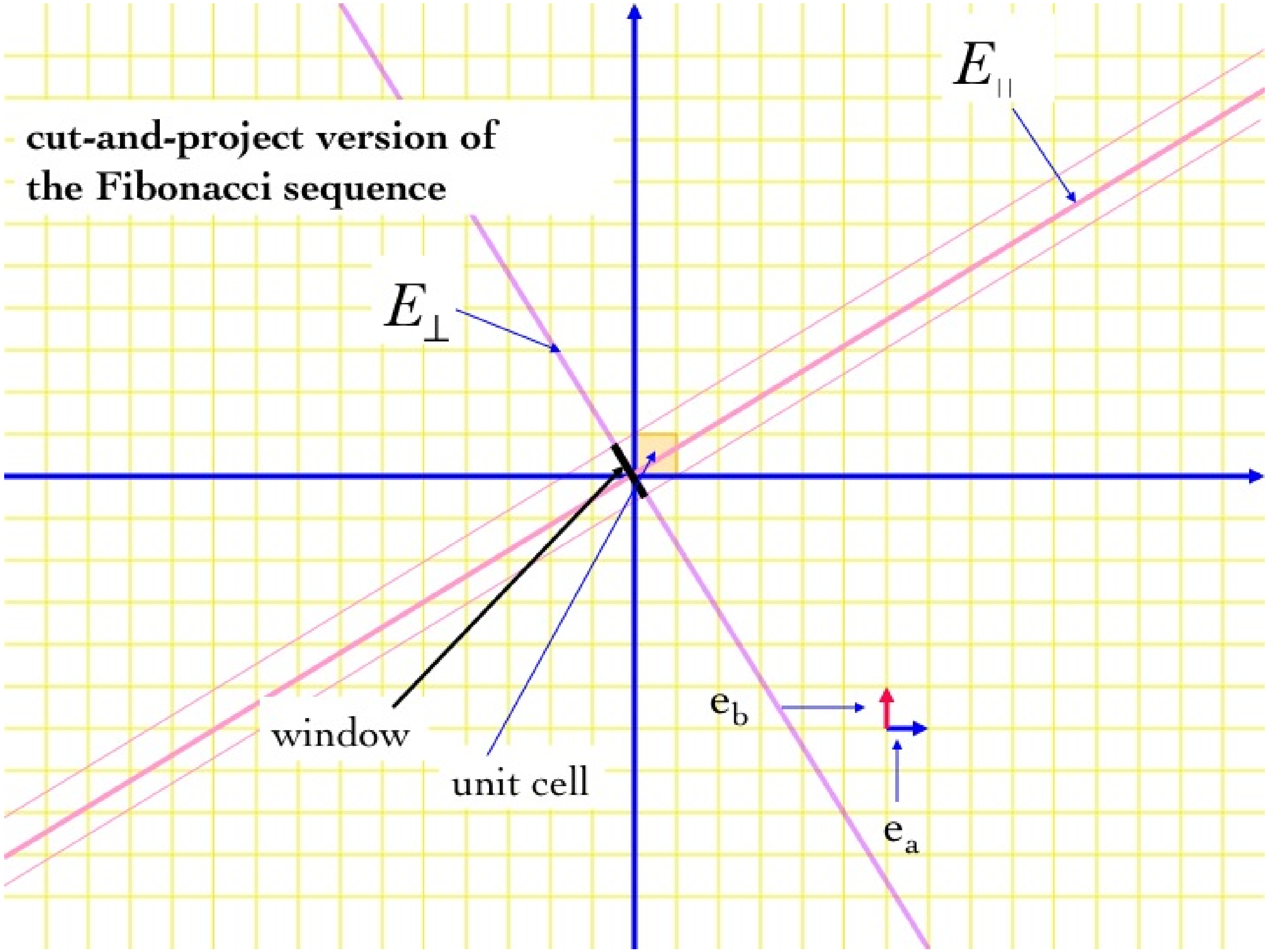

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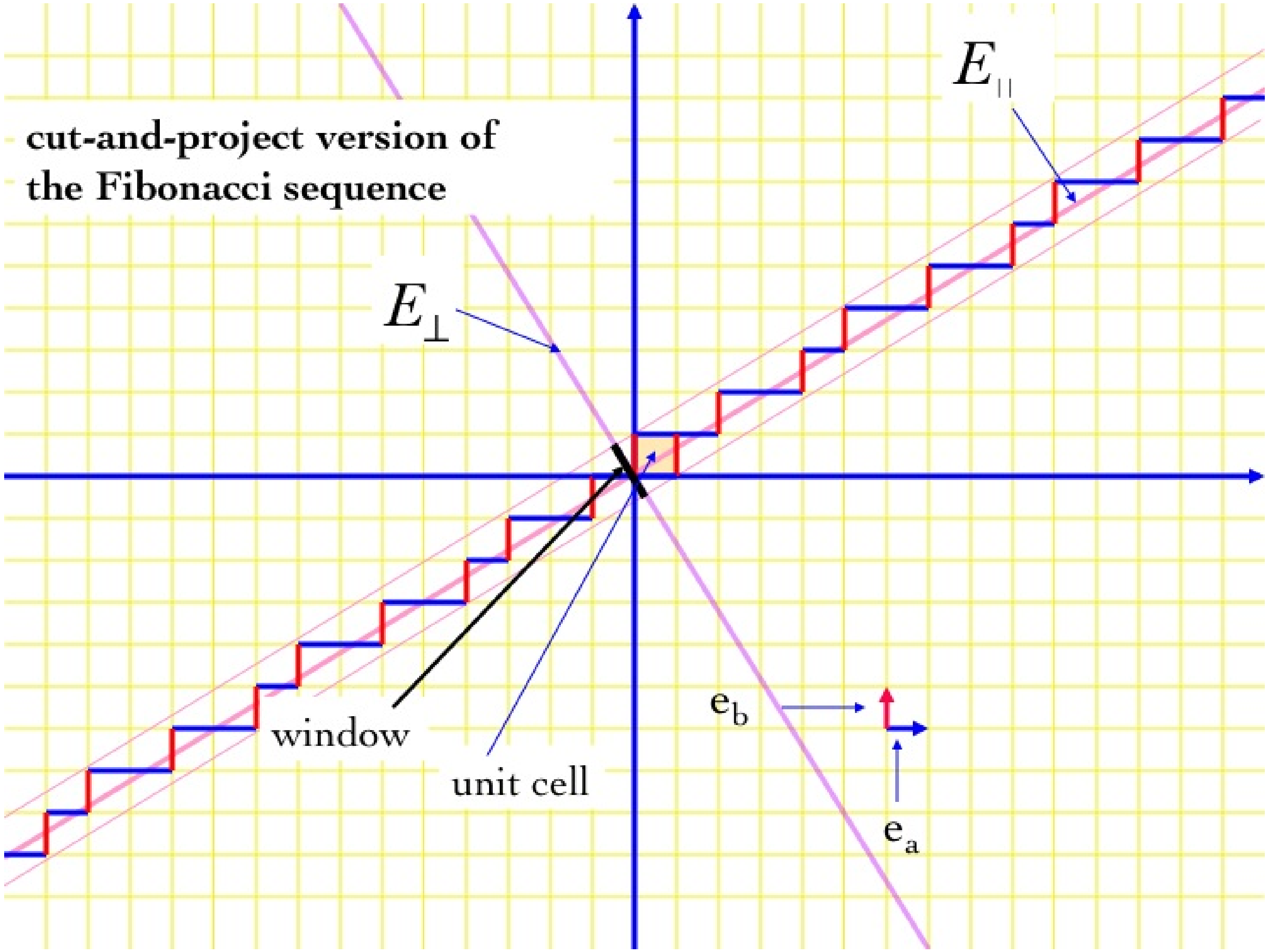

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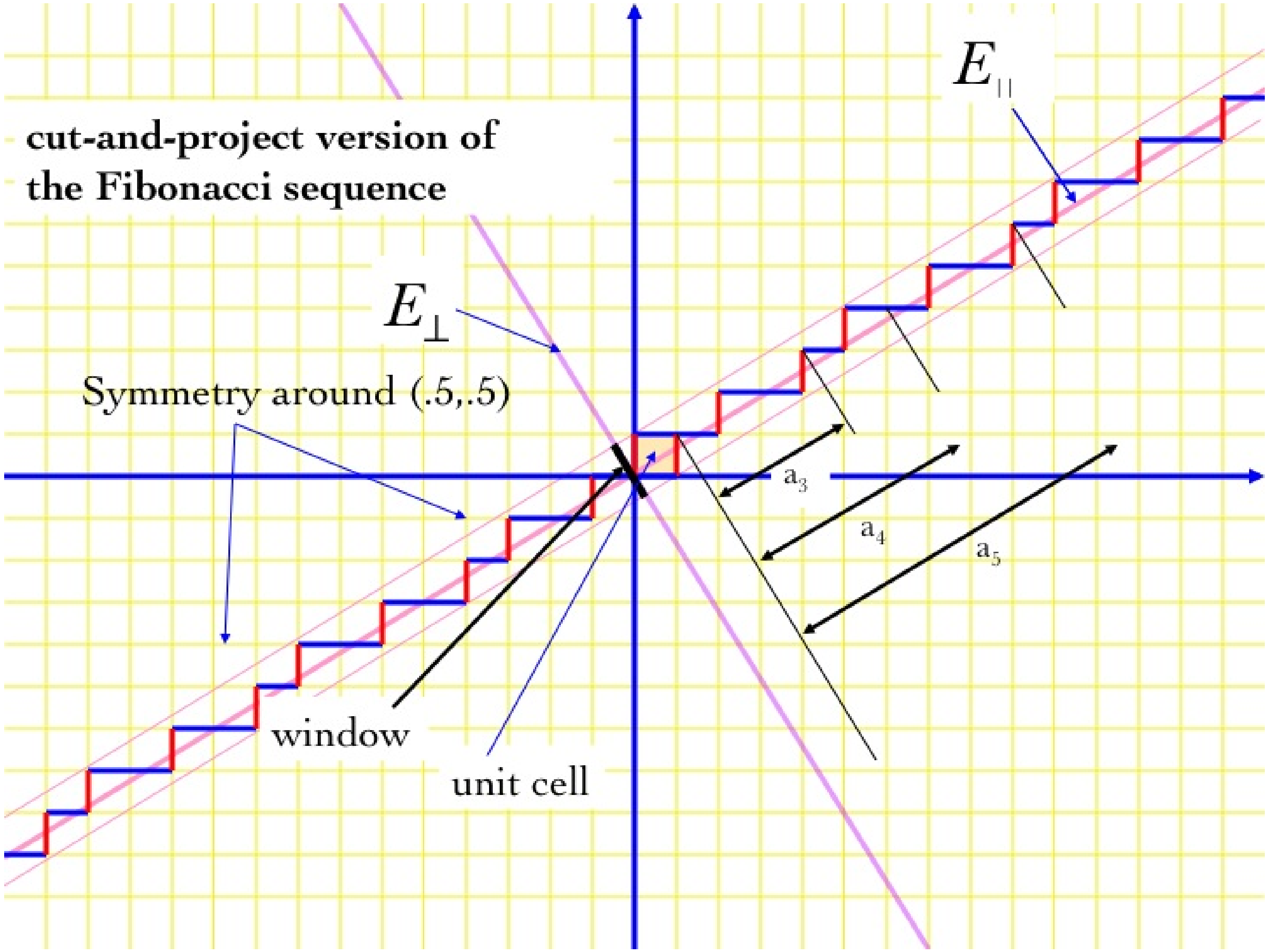
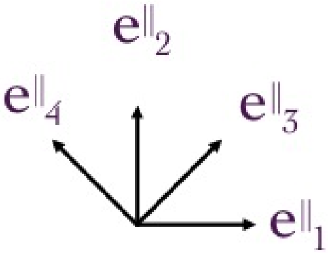
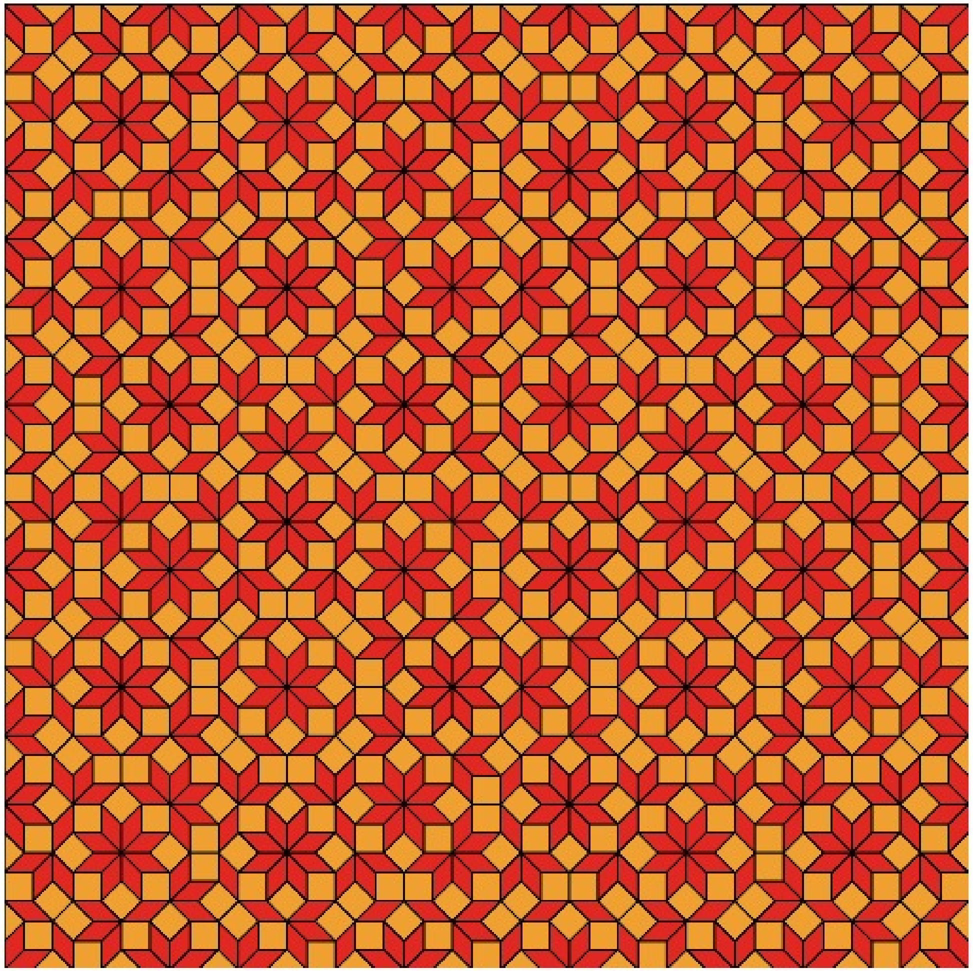


The Octagonal Tiling

Octagonal
Lattice

$Z^4 \to R^2$





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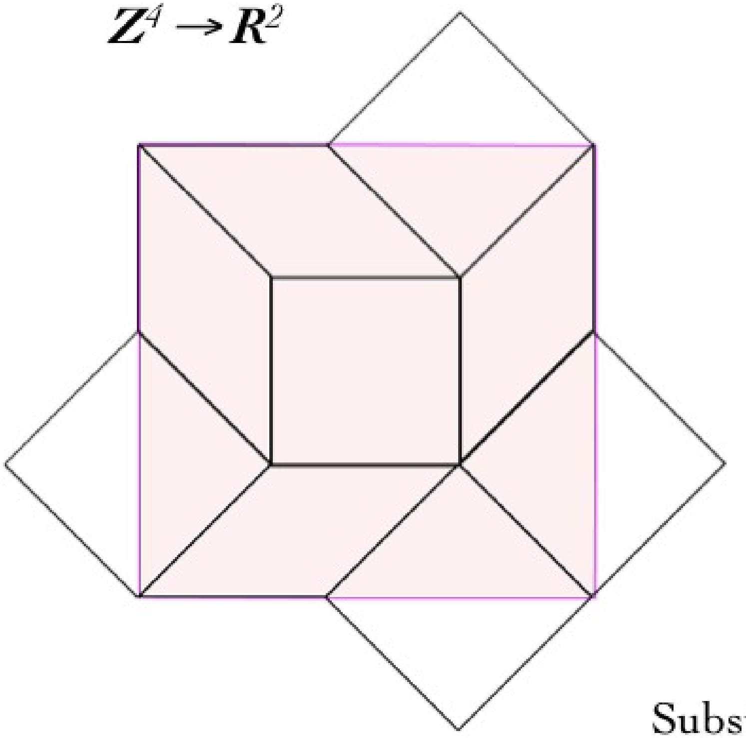

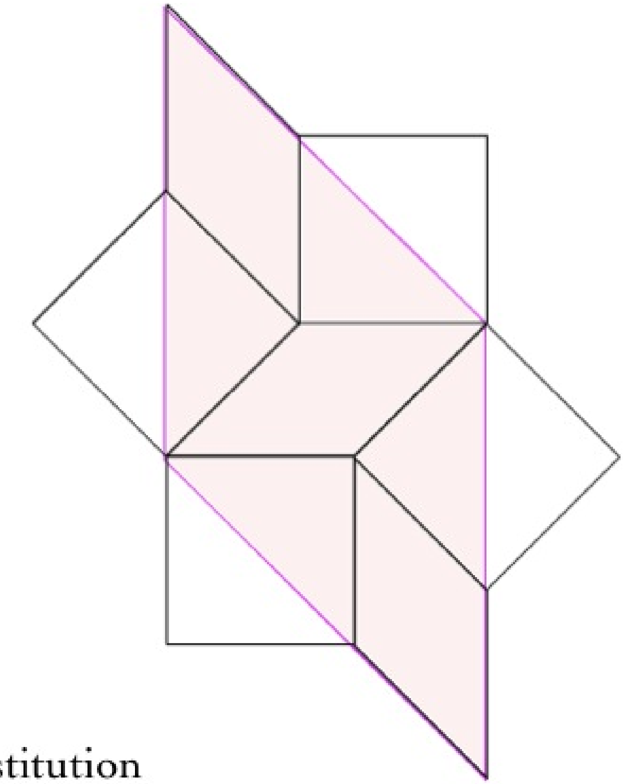

Substitution

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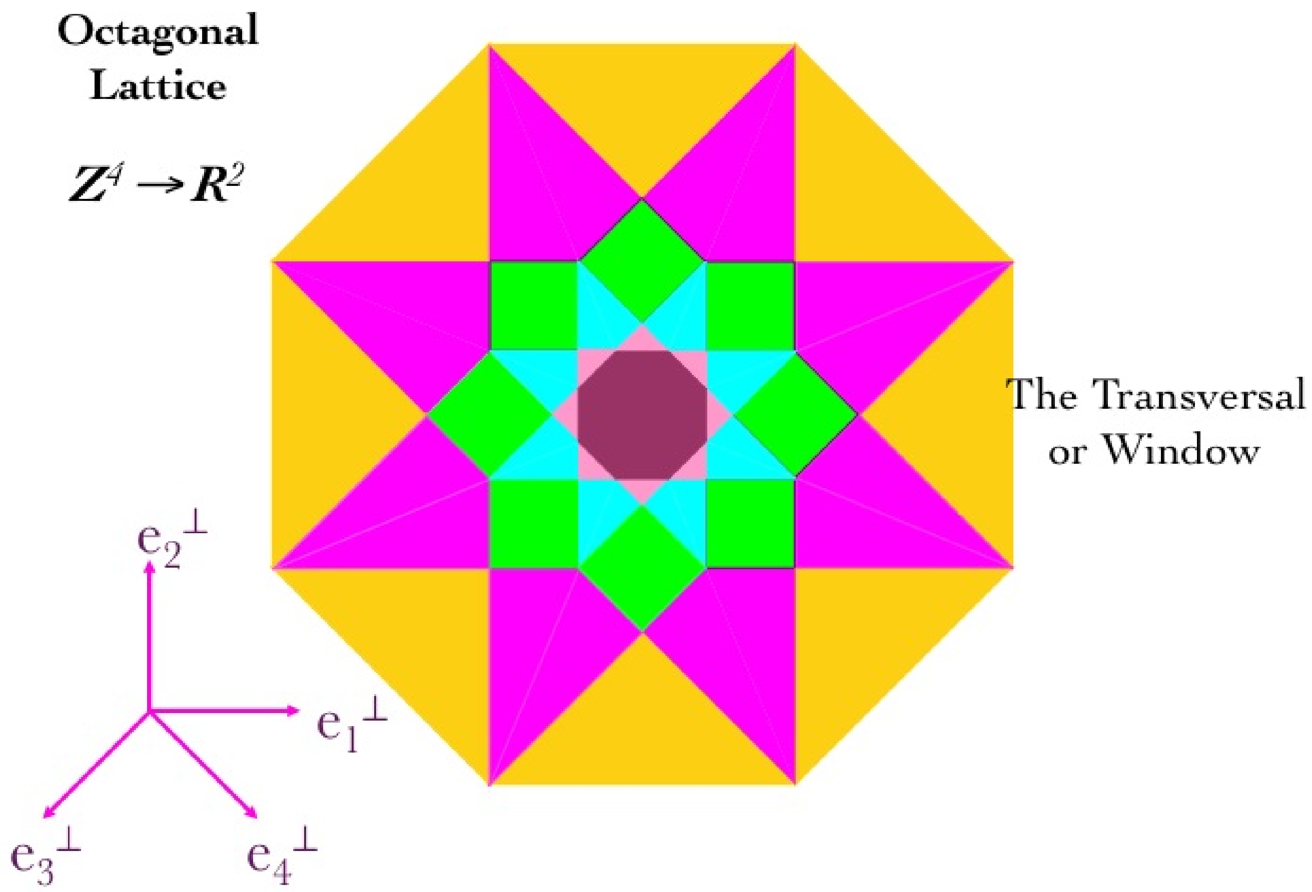

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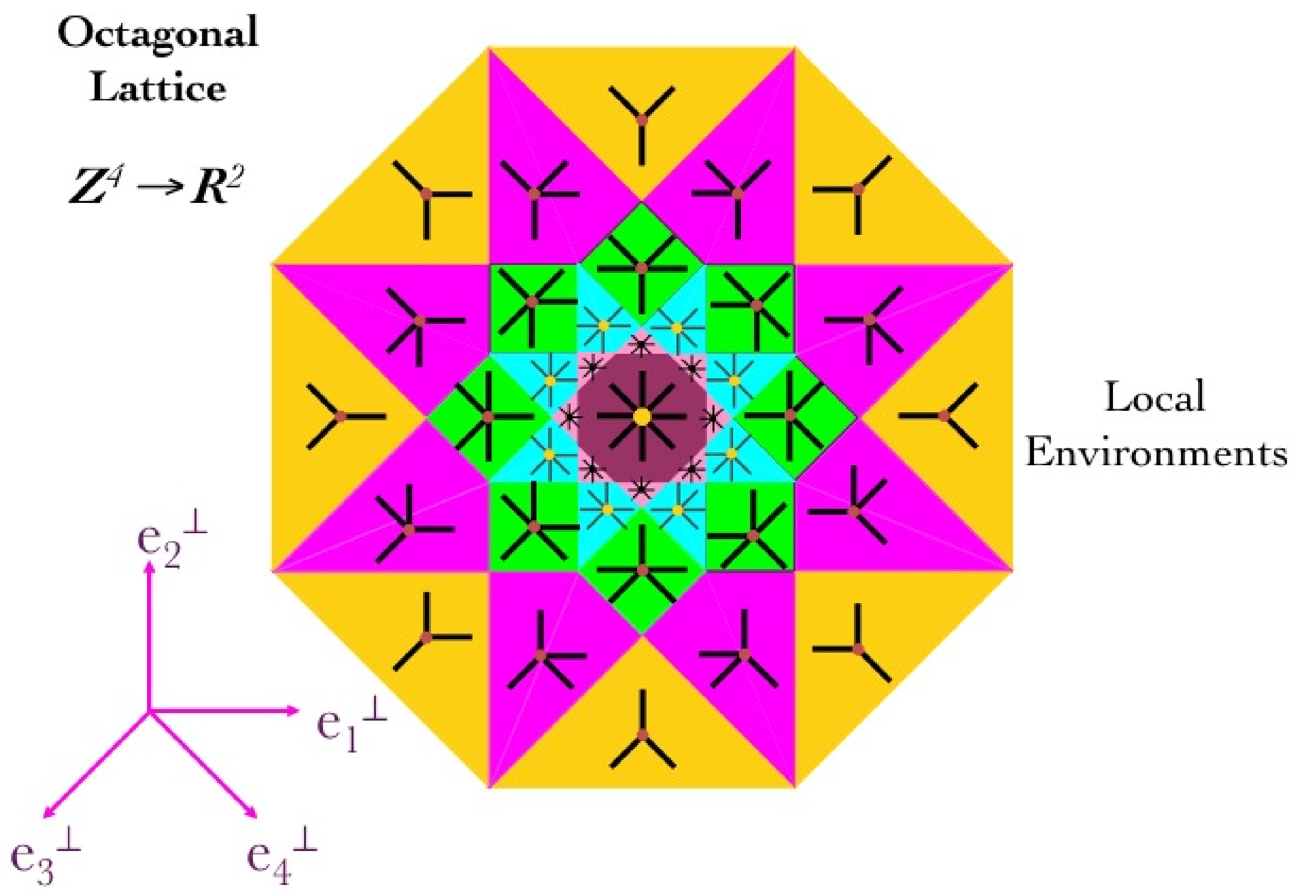
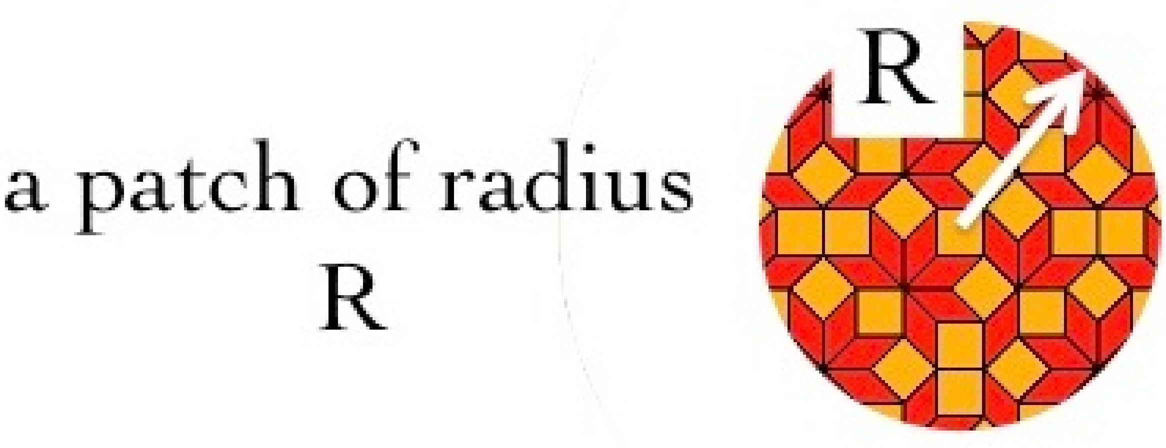


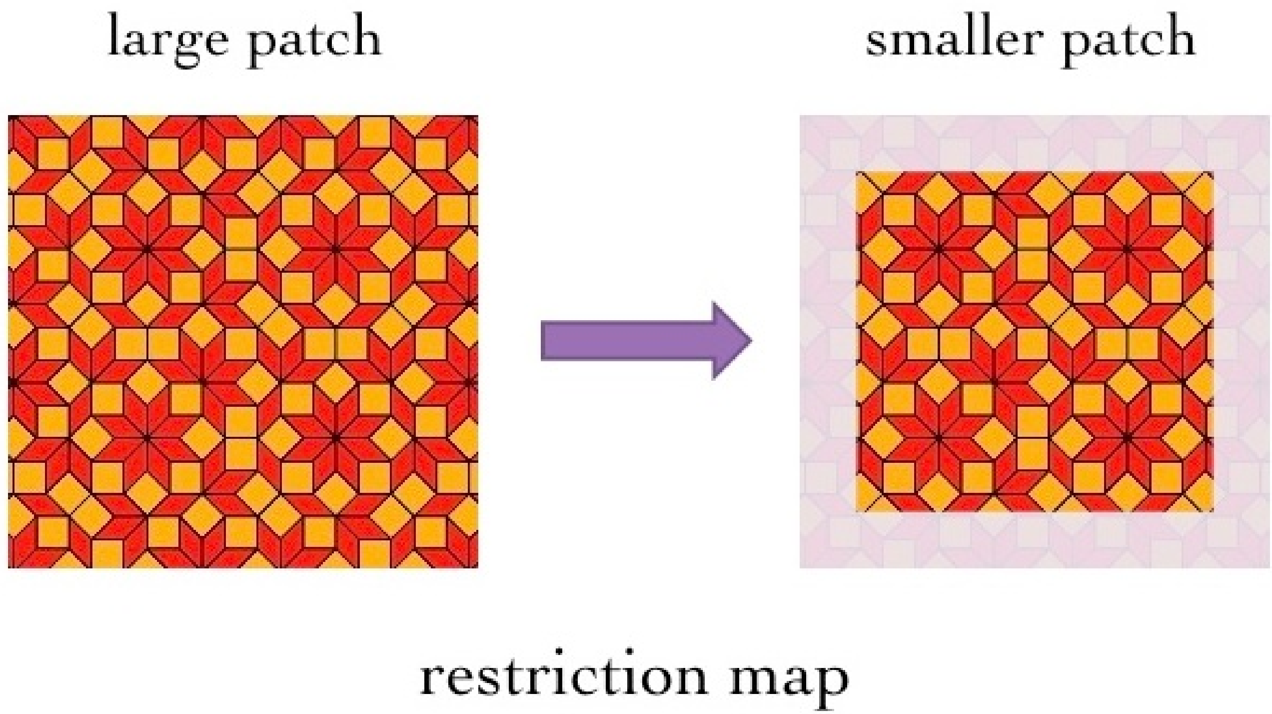
Inverse Limit



Let $\mathcal{P}_R$ be the set of patches of radius R, modulo translation.

The tiling has *finite local complexity* (FLC), if and only if $\mathcal{P}_R$ is a *finite* set for all R. In particular $R \to \mathcal{P}_R$ is *locally constant* and *non-decreasing*. Thus there is a sequence $R_0 = 0 < R_1 < \cdots < R_n < \cdots$ with $R_n \to \infty$ such that $\mathcal{P}_R = \mathcal{P}_n$ for $R_n \leq R < R_{n+1}$.

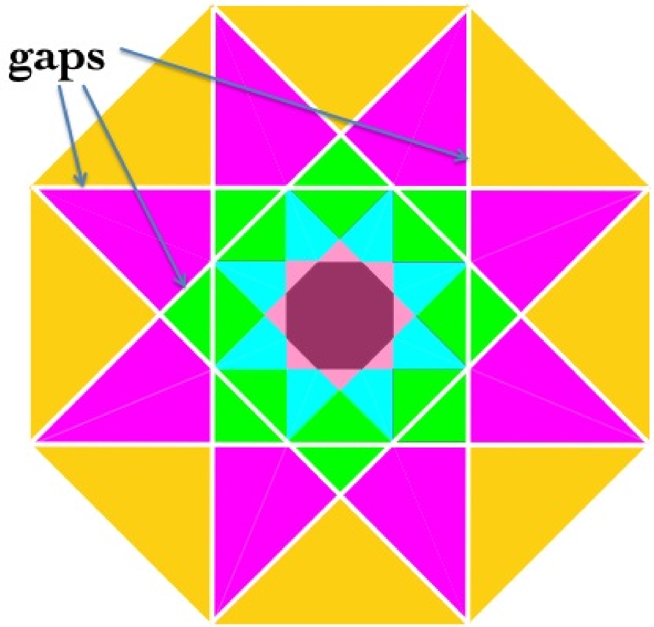
Inverse Limit



There is a restriction map $\pi : \mathcal{P}_{n+1} \to \mathcal{P}_n$. Then the *transversal* is defined by the inverse limit

$$\Xi = \lim_{\leftarrow \pi} \mathcal{P}_n$$

Inverse Limit



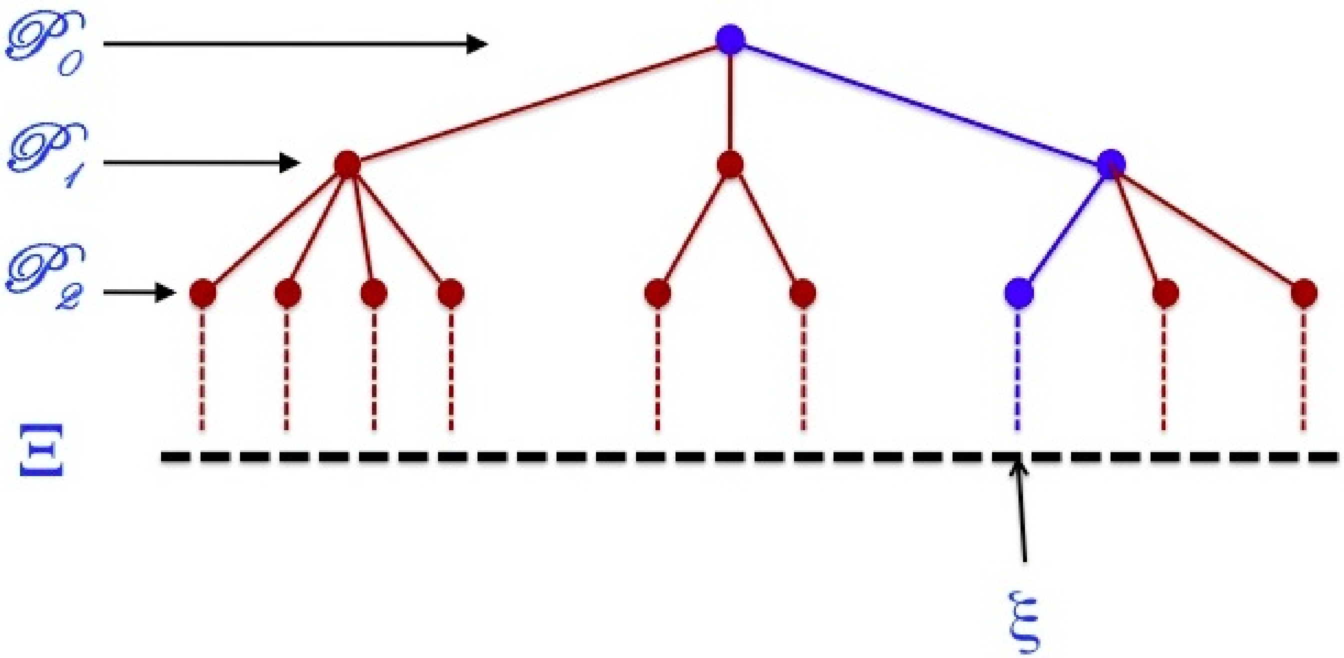
For The *Fibonacci* and *Octagonal* Tilings, as for all *cut-and-project tilings*, the transversal coincides with the window provided the window is endowed with a topology that makes all acceptance domains *closed and open*

Rooted Tree

Since all the $\mathcal{P}_n$'s are finite set, Ξ is a *Cantor set*.

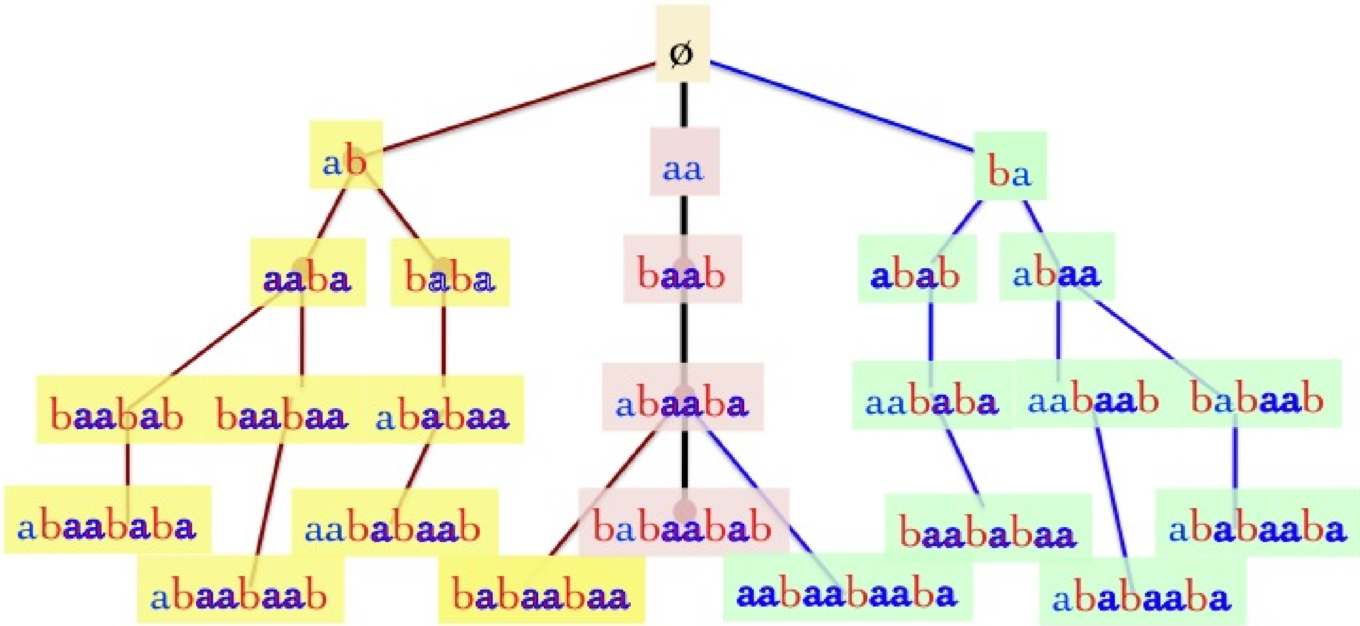
A point of Ξ is an infinite sequence $\xi = (p_n)_{n=0}^{\infty}$ of compatible patches, so it defines a unique *tiling*.

This inverse limit can be represented by a *rooted tree*



Rooted Tree

For the *Fibonacci sequence* this gives



The Fibonacci Tree

II - Spectral Triples

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

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Proposition: *Then $C^1(X)$ is a dense $*$-subalgebra of $\mathcal{A}$, invariant under the holomorphic functional calculus.*

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

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Hence the algebra $\mathcal{A}$ encodes the *space*, the Dirac operator D encodes the *metric*. $\mathcal{H}$ is needed to define D.

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Theorem, (Michon '84) *If $\xi, \eta \in \Xi$ let $\xi \wedge \eta$ be the least common ancestor of the path ξ and η. Then $d_\kappa(\xi, \eta) = \kappa(\xi \wedge \eta)$ defines an ultrametric on Ξ.*

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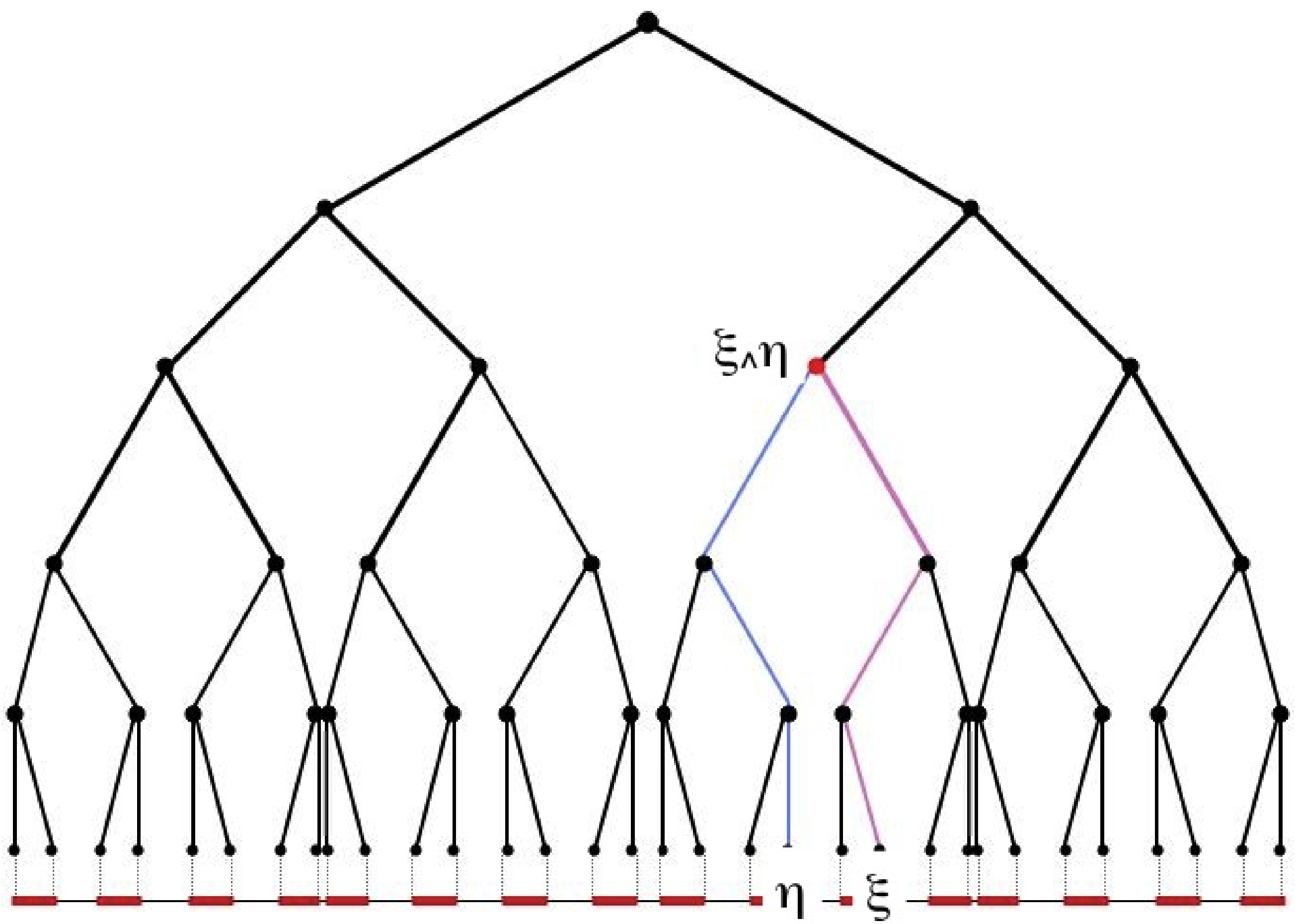
Each ultrametric on Ξ can be obtained in such a way through a rooted tree defined from the metric.

Ultrametric on Ξ



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- If p is a patch, take $\kappa(p)$ to be the *maximum potential energy difference* at the origin, produced by atoms outside p on all tilings of Ξ compatible with p.

The Pearson-Palmer Spectral Triple

Given p a patch, let $\Xi(p)$ be the set of all tilings in Ξ compatible with p at the origin. The family $(\Xi(p))_{p\in\mathcal{P}}$ is a basis of clopen set for the topology of Ξ.

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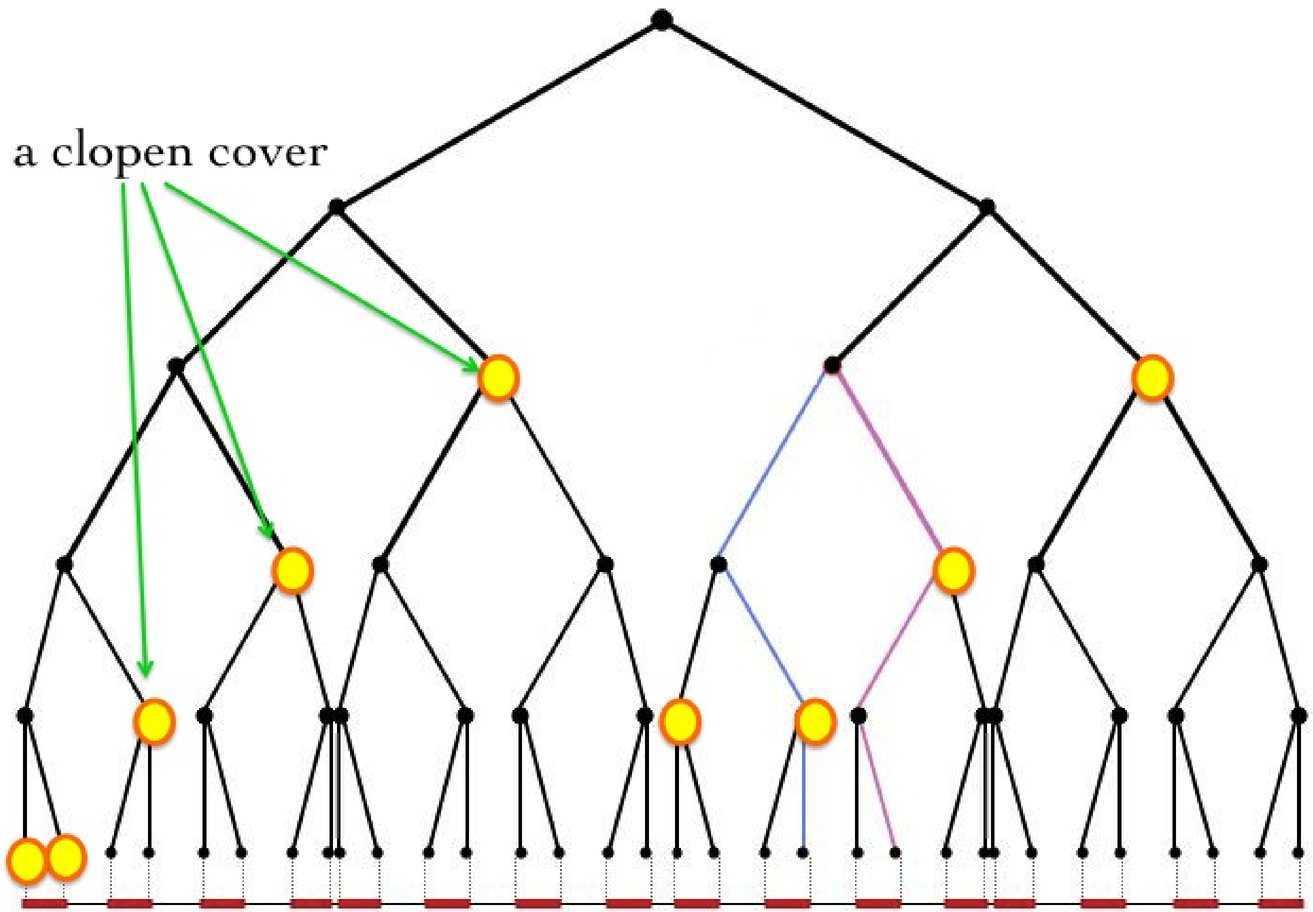

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$$\operatorname{diam}\mathcal{P} = \max\{\kappa(p)\,;\, p \in \mathcal{P}\}$$

An infinite sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of clopen cover is called *resolving* if $\lim_{n\to\infty} \operatorname{diam}\mathcal{P}_n = 0$.

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- **Representation:** for each choice τ and $f \in C(\Xi)$

$$(\pi_\tau(f)\psi)(p) = \begin{bmatrix} f(\xi_p) & 0 \\ 0 & f(\eta_p) \end{bmatrix} \psi(p)\,.$$

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There exists a (non unique) resolving sequence of clopen covers $(\mathcal{P}_n)_{n\in\mathbb{N}}$*, called a Hausdorff sequence, such that* $s_0 = \dim_H(\Xi)$.

The Connes State

The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s} \pi_\tau(f) \right), \qquad f \in C(\Xi)$$

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III - The Pearson Laplacian

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\,\psi\,(p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

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- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at p.
- the set Υ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*.

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$$[D, \pi_\tau(f)]\,\psi\,(p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at p.
- the set Υ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*.
- $[D, \pi_\tau(f)]$ could be written as $\nabla_\tau f$.

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Hence ν_p can be interpreted as the average over the tangent unit sphere at p.

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

$$Q_s(f, g) = \int_\Upsilon d\nu(\tau)\ \mathrm{Tr}\left(\frac{1}{|D|^s}[D, \pi_\tau(f)]^*\ [D, \pi_\tau(g)]\right)$$

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The eigenspaces are common to all s's and can be explicitly computed.

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Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in Ξ.

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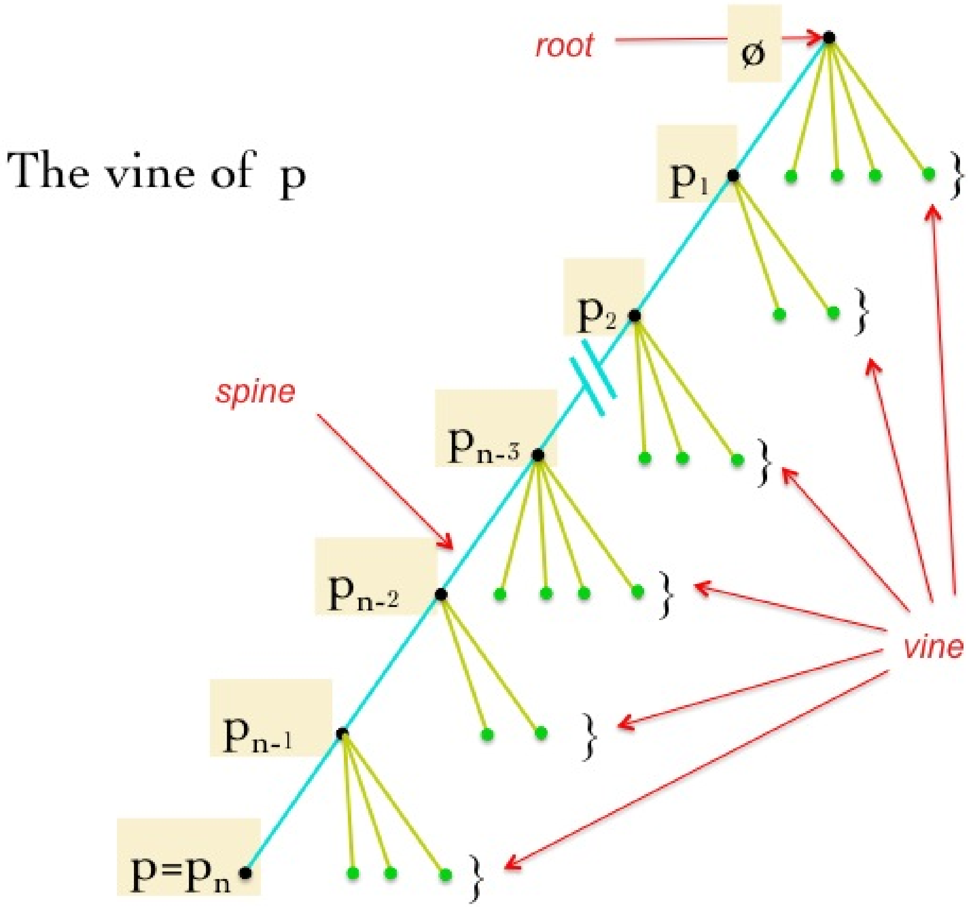
Given a patch p, its *spine* is the set of vertices located along the finite path joining the root to p. The *vine* $\mathcal{V}(p)$ *of* p is the set of patches, not in the spine, which are children of one vertex of the spine.



The vine of a vertex v

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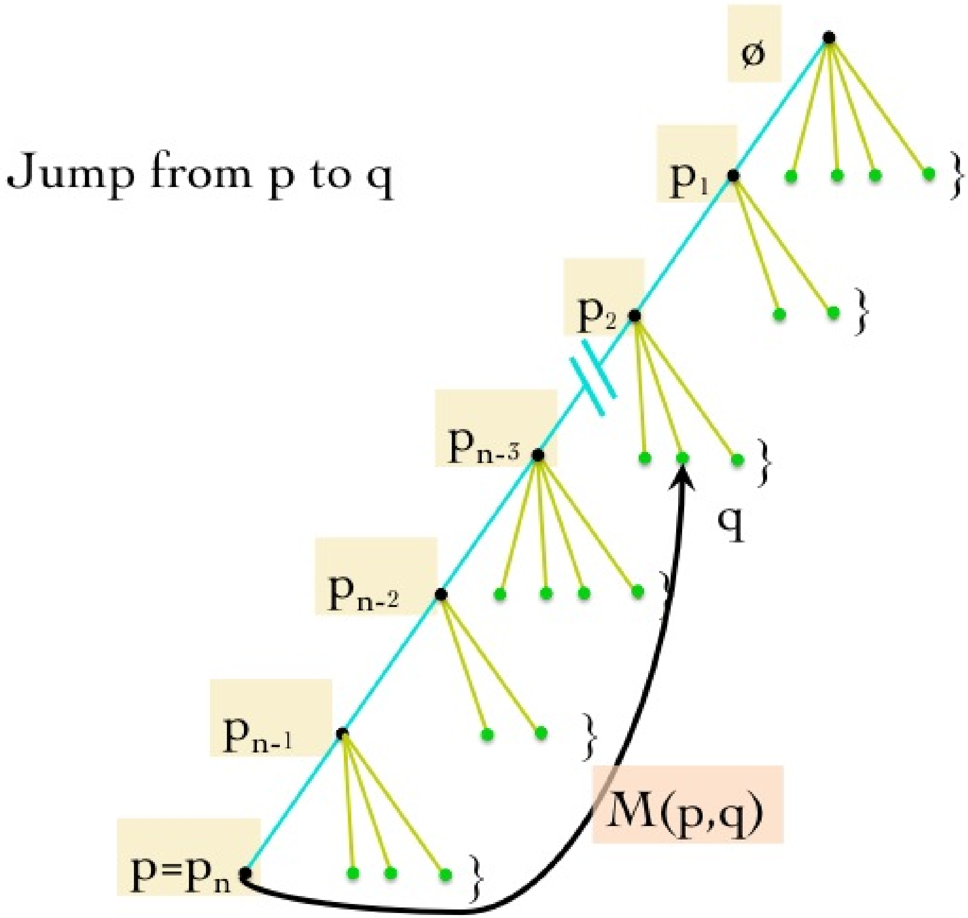
where $M(p,q) > 0$ represents the
probability rate (per unit time) for X_t *to jump from* $\Xi(p)$ *to* $\Xi(q)$.



Jump process from v to w

Jump Process

Concretely, if $\hat{q}$ denotes the *father* of q (which belongs to the spine)

$$M(p,q) \;=\; 2\kappa(\hat{q})^{s-2}\;\frac{\mu_p}{Z_{\hat{q}}} \qquad\qquad \mu_p = \mu_H(\Xi(p))$$

where $Z_{\hat{q}}$ is the *normalization constant* for the measure $\nu_{\hat{q}}$ on the set of choices at $\hat{q}$, namely

$$Z_{\hat{q}} = \sum_{q' \neq q'' \in \mathrm{Ch}(\hat{q})} \mu_{q'}\,\mu_{q''}$$

where $\mathrm{Ch}(\hat{q})$ denotes the set of children of $\hat{q}$.

Jump Process

The Markov semigroup $e^{-t\Delta_s}$ plays the role of a Brownian motion on the Cantor set. Thus $d_\kappa(X_t, X_{t+\tau})$ denotes the distance between the process at times t and $t+\tau$.

However this process is slightly *overdiffusive*, namely, *in most examples computed* the following holds

$$\mathbb{E}\left(d_\kappa(X_t, X_{t+\tau})^2\right) \overset{\tau\downarrow 0}{=} C\,\tau \ln \tau\,(1+o(1))$$

if

$$s = \dim_H(\Xi)$$

Thanks for Listening !