TILINGS

APERIODIC MEDIA

and their

NONCOMMUTATIVE GEOMETRY

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Content

Aperiodic Solids

1. *Perfect crystals* in d-dimensions:
   translation and crystal symmetries.
   Translation group $\mathcal{T} \simeq \mathbb{Z}^d$.

2. *Quasicrystals*: no translation symmetry, but
   icosahedral symmetry. Ex.:
   (a) $\mathbf{Al_{62.5}Cu_{25}Fe_{12.5}}$;
   (b) $\mathbf{Al_{70}Pd_{22}Mn_{8}}$;
   (c) $\mathbf{Al_{70}Pd_{22}Re_{8}}$;

3. *Amorphous media*: short range order
   (a) Glasses;
   (b) Silicium in amorphous phase;

4. *Disordered media*: random atomic positions
   (a) Normal metals (with defects or impurities);
   (b) Doped semiconductors (**Si**, **AsGa**, ...);

I - The Hull as a Dynamical System

J. Bellissard, D. Hermmann, M. Zarrouati, *Hull of Aperiodic Solids and Gap Labelling Theorems*
To appear in *Directions in Mathematical Quasicrystals,* M.B. Baake & R.V. Moody Eds, AMS, (2000).

I.1)- Point Sets

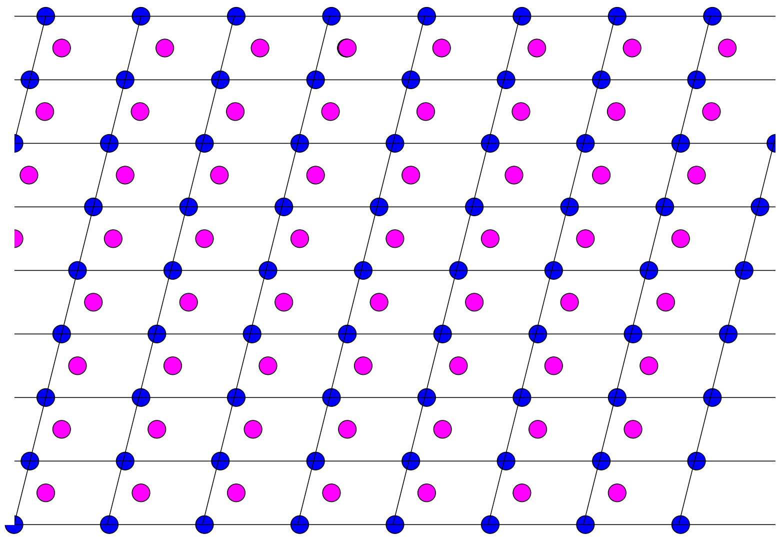
Equilibrium positions of atomic nuclei make up a point set $\mathcal{L} \subset \mathbb{R}^d$ *the set of lattice sites*. $\mathcal{L}$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.
4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.
5. *Finite type Delone* set: $\mathcal{L} - \mathcal{L}$ is discrete.
6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

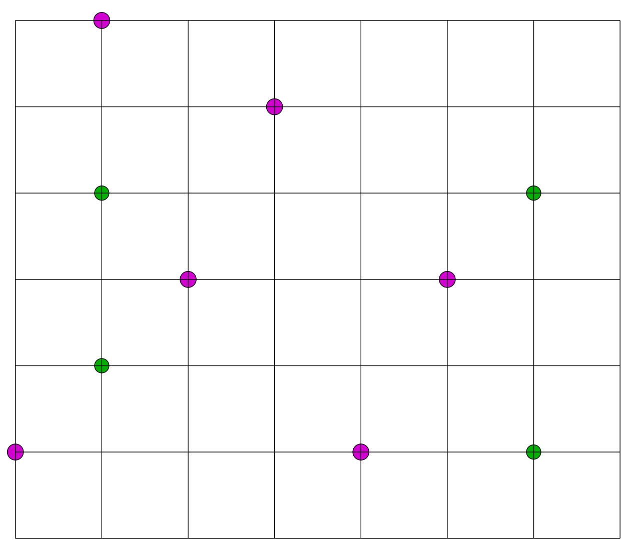
Examples:

1. A random Poissonian set in $\mathbb{R}^d$ is almost surely discrete but not uniformly discrete nor relatively dense.
2. Due to Coulomb repulsion and Quantum Mechanics, **lattices of atoms are always uniformly discrete**.
3. Impurities in semiconductors are not relatively dense.
4. In amorphous media $\mathcal{L}$ is Delone.
5. In a quasicrystal $\mathcal{L}$ is Meyer.

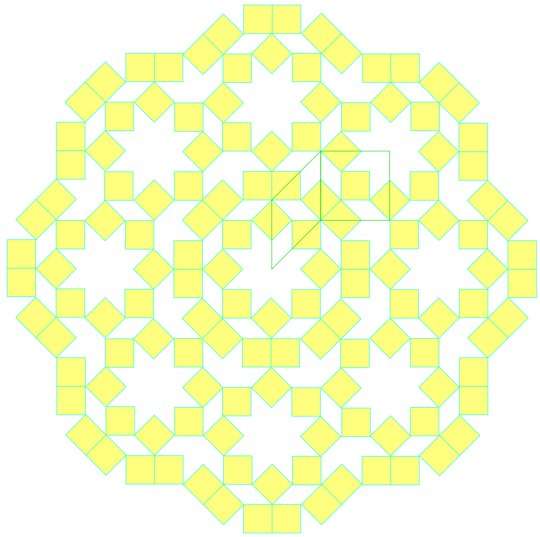
The ideal equilibrium positions of atomic nuclei sit on a discrete subset $\mathcal{L}$ of $\mathbb{R}^d$ ($d = 1, 2, 3$ in practice).



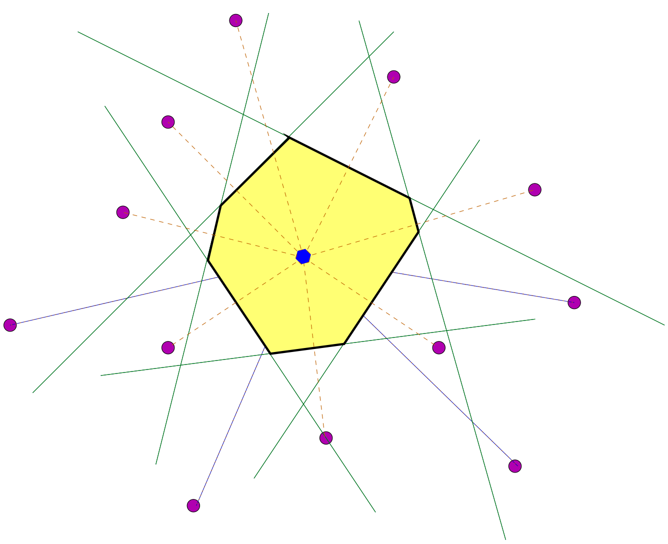
- A periodic array of atomic nuclei -



- A random array of atomic nuclei -



- A quasiperiodic array of atomic nuclei -



- Construction of VORONOI*'s tiling -*

I.2)- Point Measures

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $\mathcal{C}_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu \;:=\; \nu^{\mathcal{L}} = \sum_{y\in\mathcal{L}} \delta(x-y) \qquad \in \mathfrak{M}(\mathbb{R}^d)\;.$$

The *Hull* is the closure in $\mathfrak{M}(\mathbb{R}^d)$

$$\Omega = \overline{\left\{\mathrm{T}^a \nu^{\mathcal{L}}; a \in \mathbb{R}^d\right\}}\;,$$

where $\mathrm{T}^a\nu$ is the translated of ν by a.

Results:

1. Ω is compact and $\mathbb{R}^d$ acts by homeomorphisms.
2. If $\omega \in \Omega$, there is a uniformly discrete point set $\mathcal{L}_\omega$ in $\mathbb{R}^d$ such that ω coincides with $\nu_\omega = \nu^{\mathcal{L}_\omega}$.
3. If $\mathcal{L}$ is *Delone* (resp. *Meyer*) so are the $\mathcal{L}_\omega$'s.

I.3)- Properties

(a) Minimality

$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

Proposition 1 *$\mathbb{R}^d$ acts minimaly on Ω if and only if $\mathcal{L}$ is repetitive.*

(b) Transversal

The closed subset $X = \{\omega \in \Omega\ ;\ \nu_\omega(\{0\}) = 1\}$ is called the *canonical transversal*. Let G be the subgroupoid of $\Omega \rtimes \mathbb{R}^d$ induced by X.

A Delone set $\mathcal{L}$ has *finite type* if $\mathcal{L} - \mathcal{L}$ is closed and discrete.

(c) Cantorian Transversal

Proposition 2 *If $\mathcal{L}$ has finite type, then the transversal is completely discontinuous (Cantor).*

II - Building Hulls

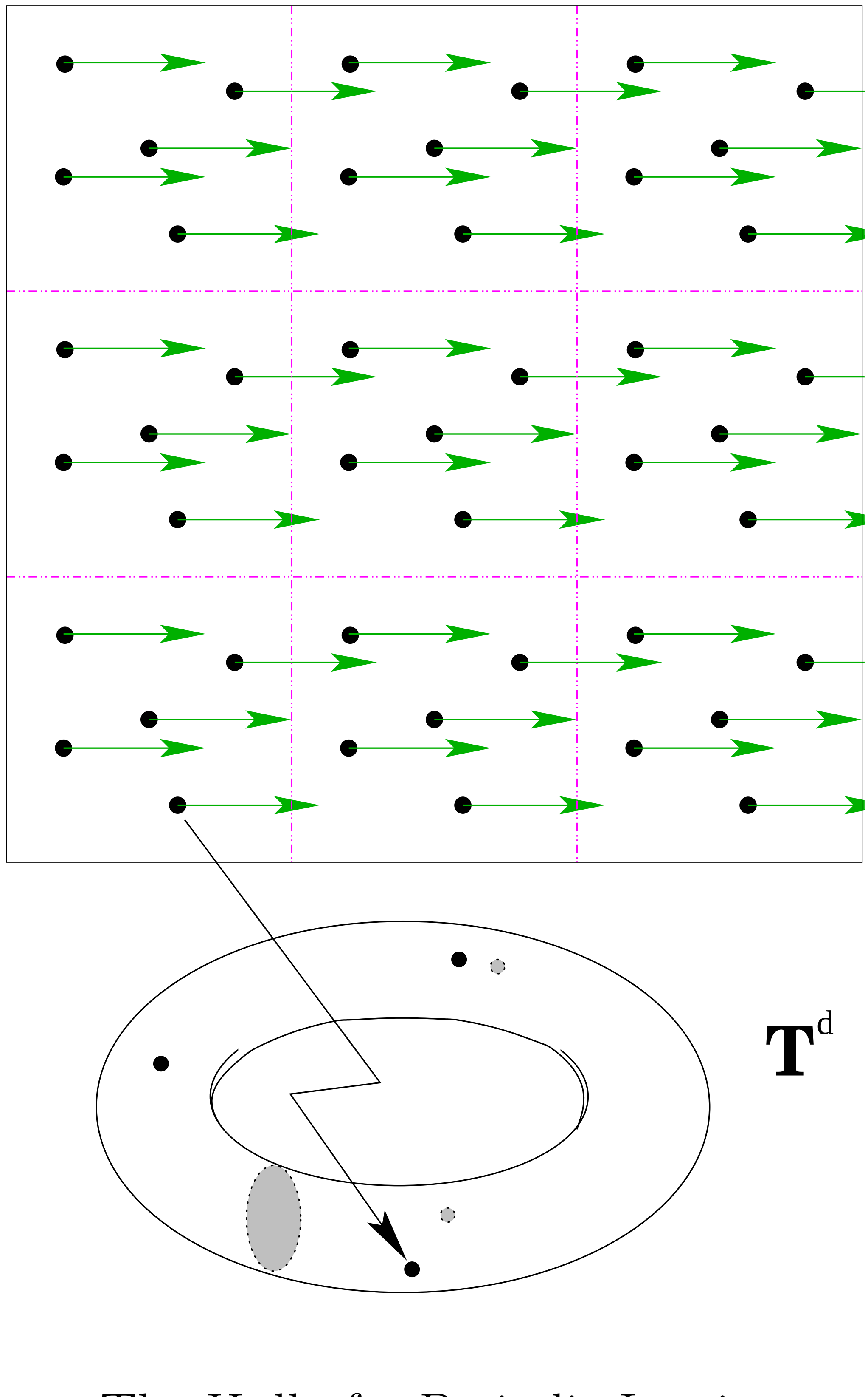
J. Bellissard, R. Benedetti, J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling,* preprint August (2001).

II.1)- Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Quasicrystals :* $\Omega \simeq \mathbb{T}^n,\ n > d$ with an irrational action of $\mathbb{R}^d$ and a completely discontinuous topology in the transverse direction to the $\mathbb{R}^d$-orbits. The transversal is a Cantor set.

3. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for Si atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

II.2)- Quasicrystals

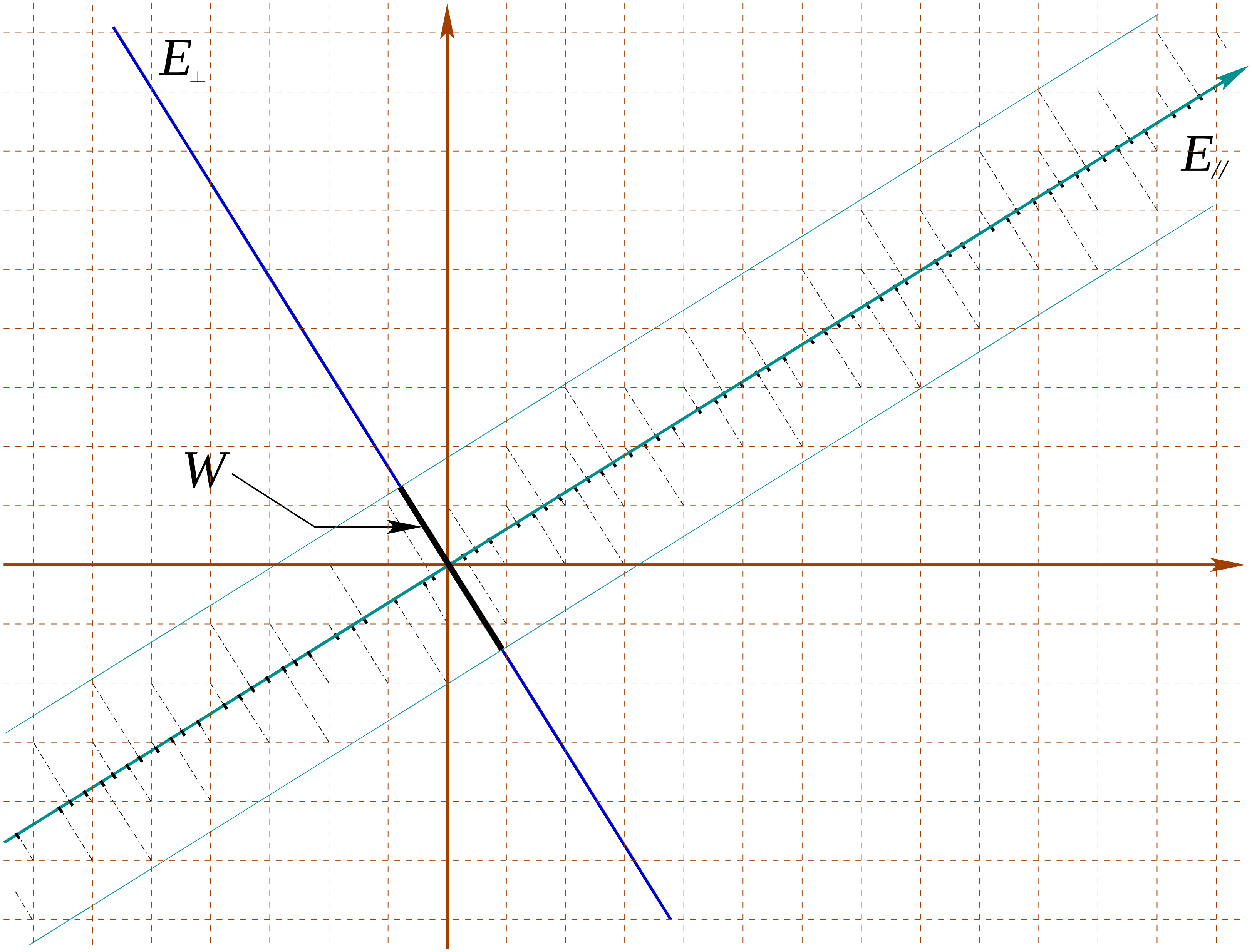
Use the *cut-and-project* construction:

$$\mathbb{R}^d \simeq \mathcal{E}_{\|} \xleftarrow{\pi_{\|}} \mathbb{R}^n \xrightarrow{\pi_{\perp}} \mathcal{E}_{\perp} \simeq \mathbb{R}^{n-d}$$

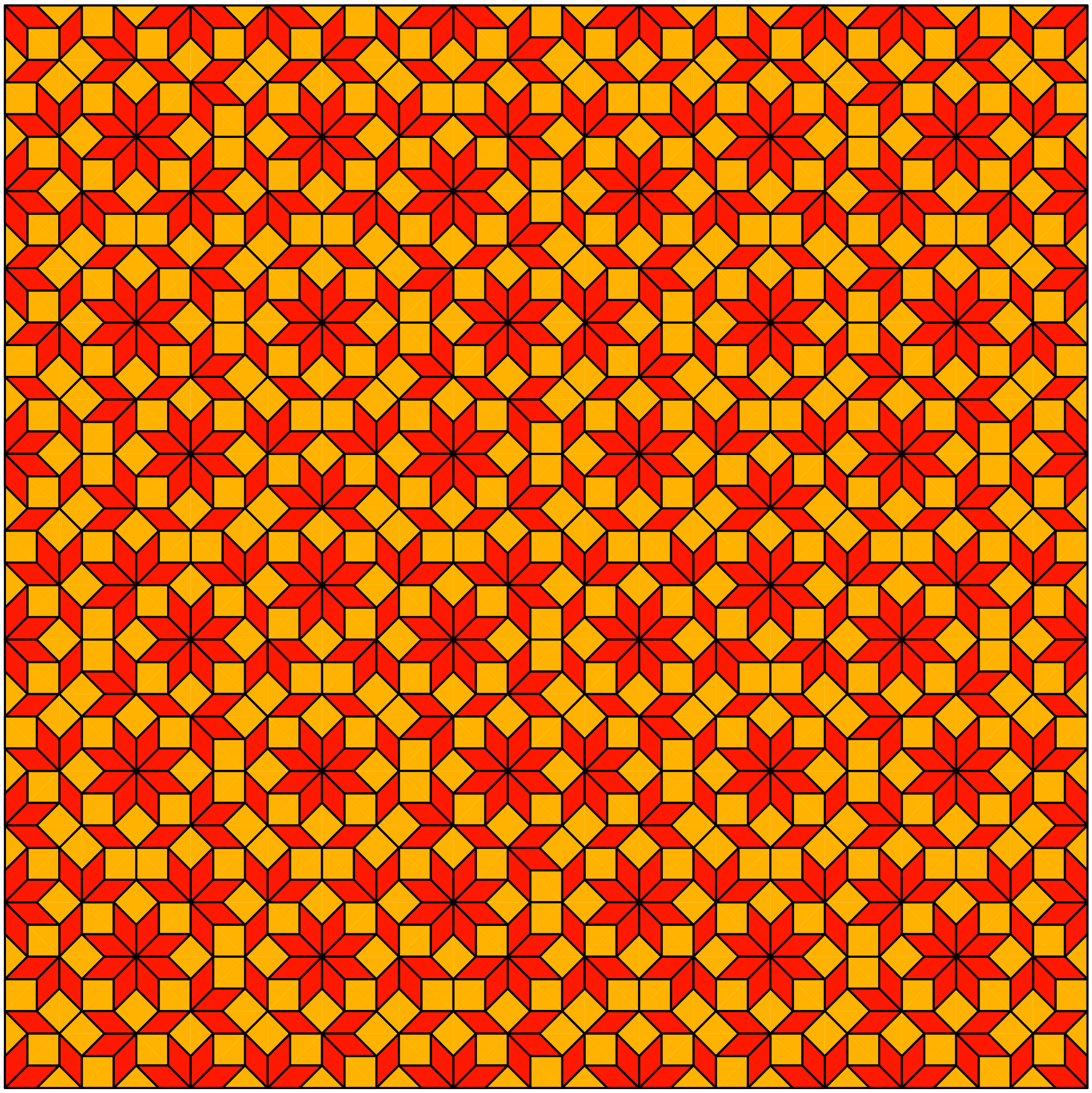
$$\mathcal{L} \xleftarrow{\pi_{\|}} \tilde{\mathcal{L}} \xrightarrow{\pi_{\perp}} W \quad ,$$

Here

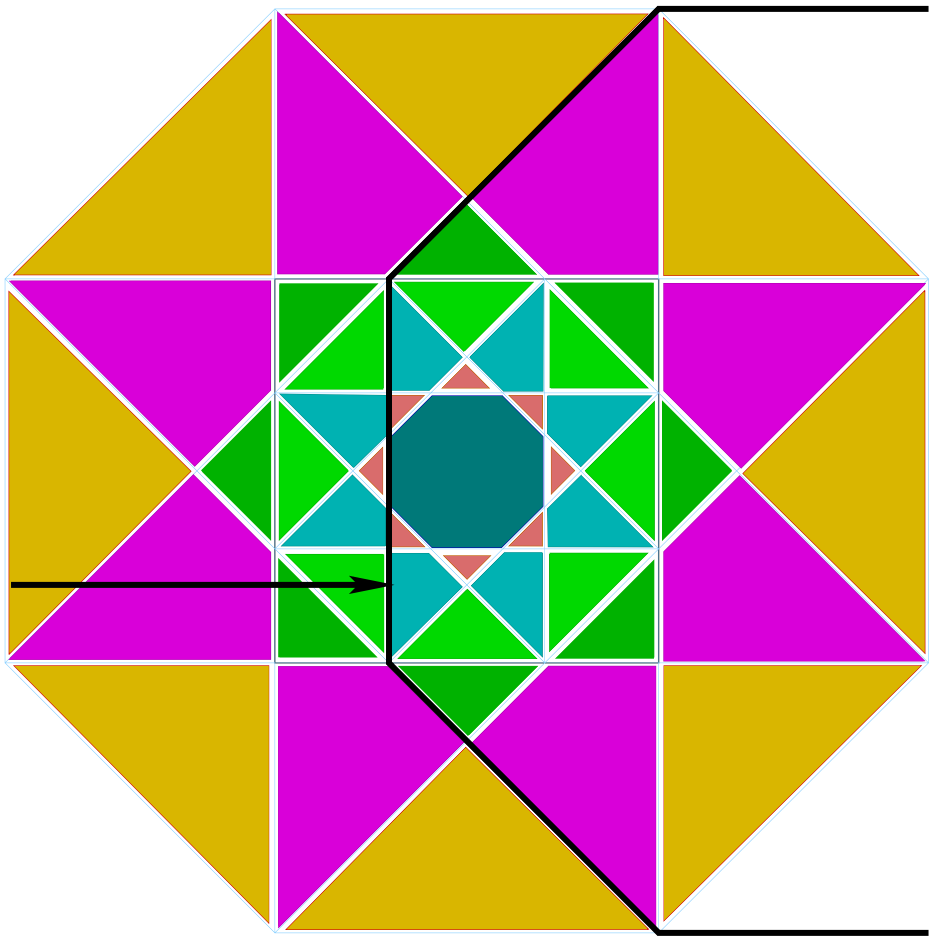
1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{L}$ is the induced *quasilattice* in $\mathcal{E}_{\|}$.

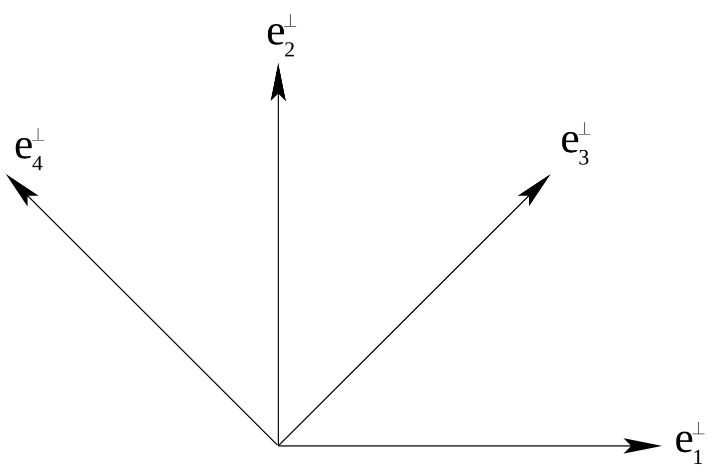


– The cut–and–project construction –



- The octagonal tiling -





- The transversal of the Octagonal Tiling -

- is completely disconnected -

II.3)- Hull of Quasicrystals

1. Let $\mathcal{F}$ be the family of affine hyperplanes in $\mathbb{R}^n$ with projections on $\mathcal{E}_\perp$ containing the maximal faces of W. Endow $\mathbb{R}^n$ with the topology such that for any $F \in \mathcal{F}$ and $a \in \tilde{\mathcal{L}}$, the two half spaces separated by the affine hyperplane $F + a$ are both closed and open. Let $\mathbb{R}^n_{\mathcal{F}}$ be the completion of $\mathbb{R}^n$ with this topology.
2. By construction, for $a \in \tilde{\mathcal{L}}$, the map $\tilde{x} \in \mathbb{R}^n \mapsto \tilde{x} + a \in \mathbb{R}^n$ extends to $\mathbb{R}^n_{\mathcal{F}}$ by continuity. Let then $\mathbf{T}^n_{\mathcal{F}} = \mathbb{R}^n_{\mathcal{F}}/\tilde{\mathcal{L}}$ be the corresponding *pseudo torus*.
3. By construction, for $x \in \mathcal{E}_\parallel$, the map $\tilde{x} \in \mathbb{R}^n \mapsto \tilde{x} + x \in \mathbb{R}^n$ extends also by continuity to $\mathbb{R}^n_{\mathcal{F}}$ and commutes with $\tilde{\mathcal{L}}$. Thus it defines an $\mathbb{R}^d$ action on $\mathbf{T}^n_{\mathcal{F}}$.

Theorem 1 *$\mathcal{L}$ is a Meyer set. Its Hull is conjugate to $\mathbf{T}^n_{\mathcal{F}}$ endowed with its canonical $\mathbb{R}^d$-action. This Hull is uniquely ergodic.*

III - Tilings & Point Sets

C. Radin, *Miles of Tiles*, Student Mathematical Library, Vol 1, Amer. Math. Soc., Providence, (1999).

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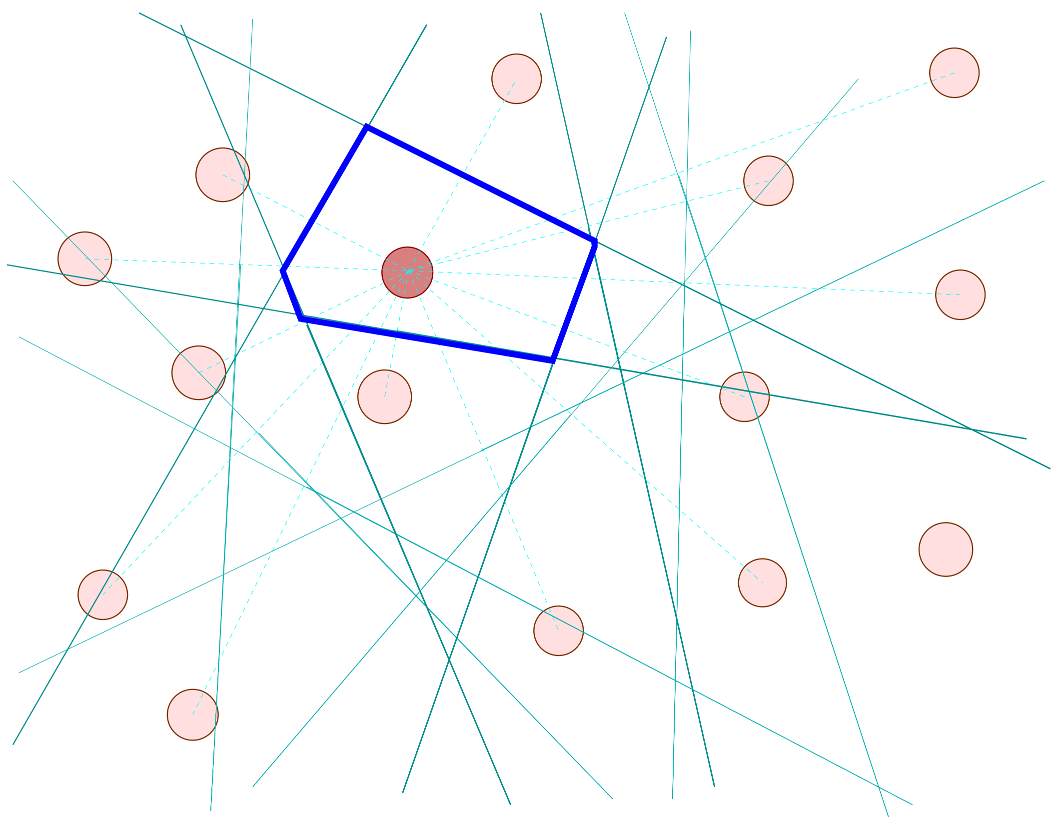
J. Bellissard, R. Benedetti, J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling*, preprint August (2001).

III.1)- Voronoi Cells

For $\mathcal{L}$ *Delone* and $x \in \mathcal{L}$, the *Voronoi cell of x* is

$$V_x = \{y \in \mathbb{R}^d ;\ |y - x| < |y - x'|,\ \forall x' \in \mathcal{L} \setminus \{x\}\}$$



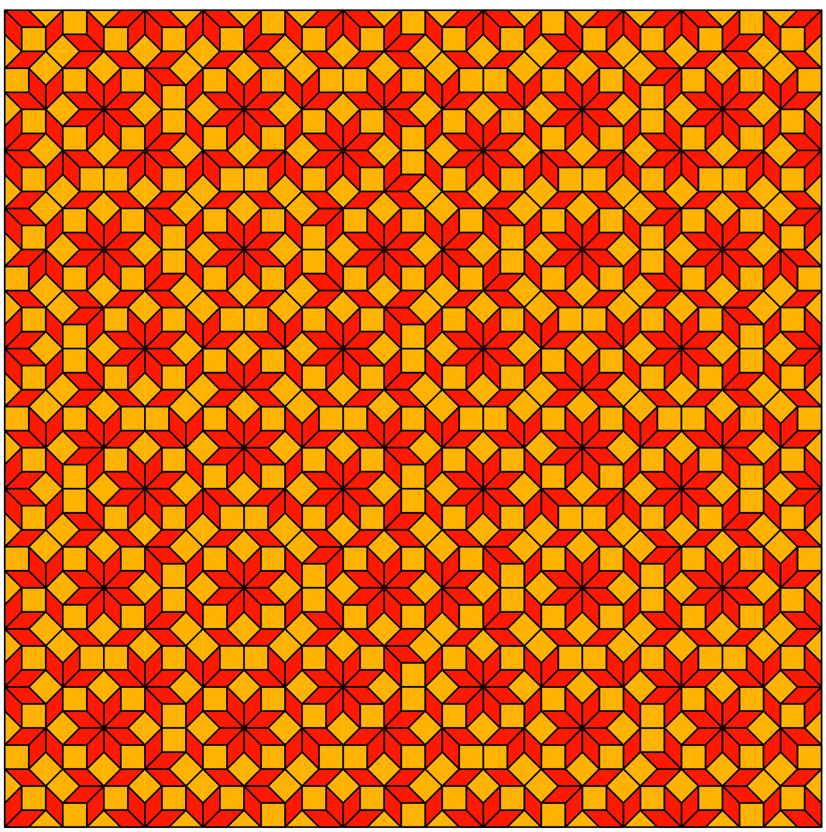
- Building a Voronoi Cell -

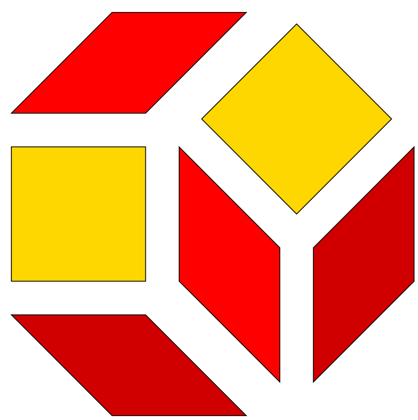
The V_x's are open polyhedrons with uniformly bounded diameter. They are mutually disjoint and their closure cover $\mathbb{R}^d$: it is a *tiling of* $\mathbb{R}^d$

III.2)- The Finite Pattern Condition

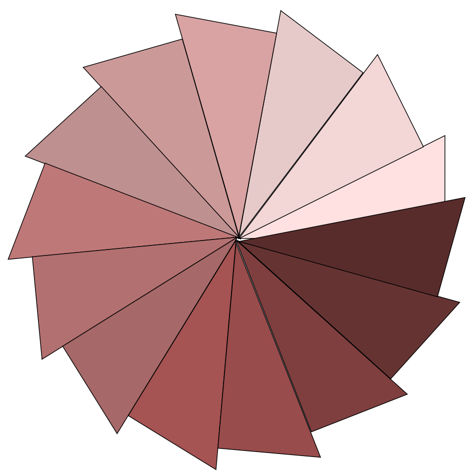
A *patch* is the set of tiles of $\mathcal{T}$ contained in some ball. A tiling $\mathcal{T}$ fulfills the *finite pattern condition* (FPC) if the number of patches of radius smaller than R *modulo translations* is finite for all R's.

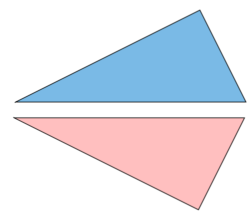
Then the transversal is a Cantor set.

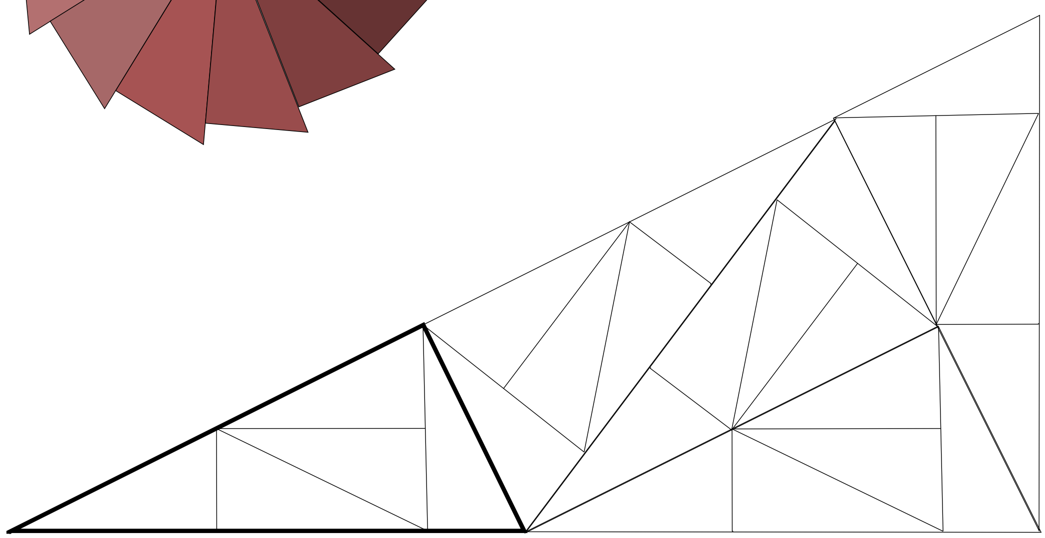




- The octagonal tiling is FPC -





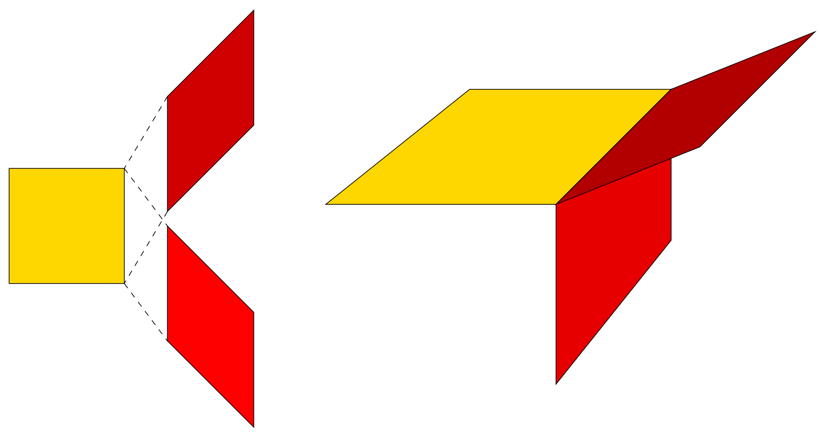


- The pinwheel tiling is NOT FPC ! -

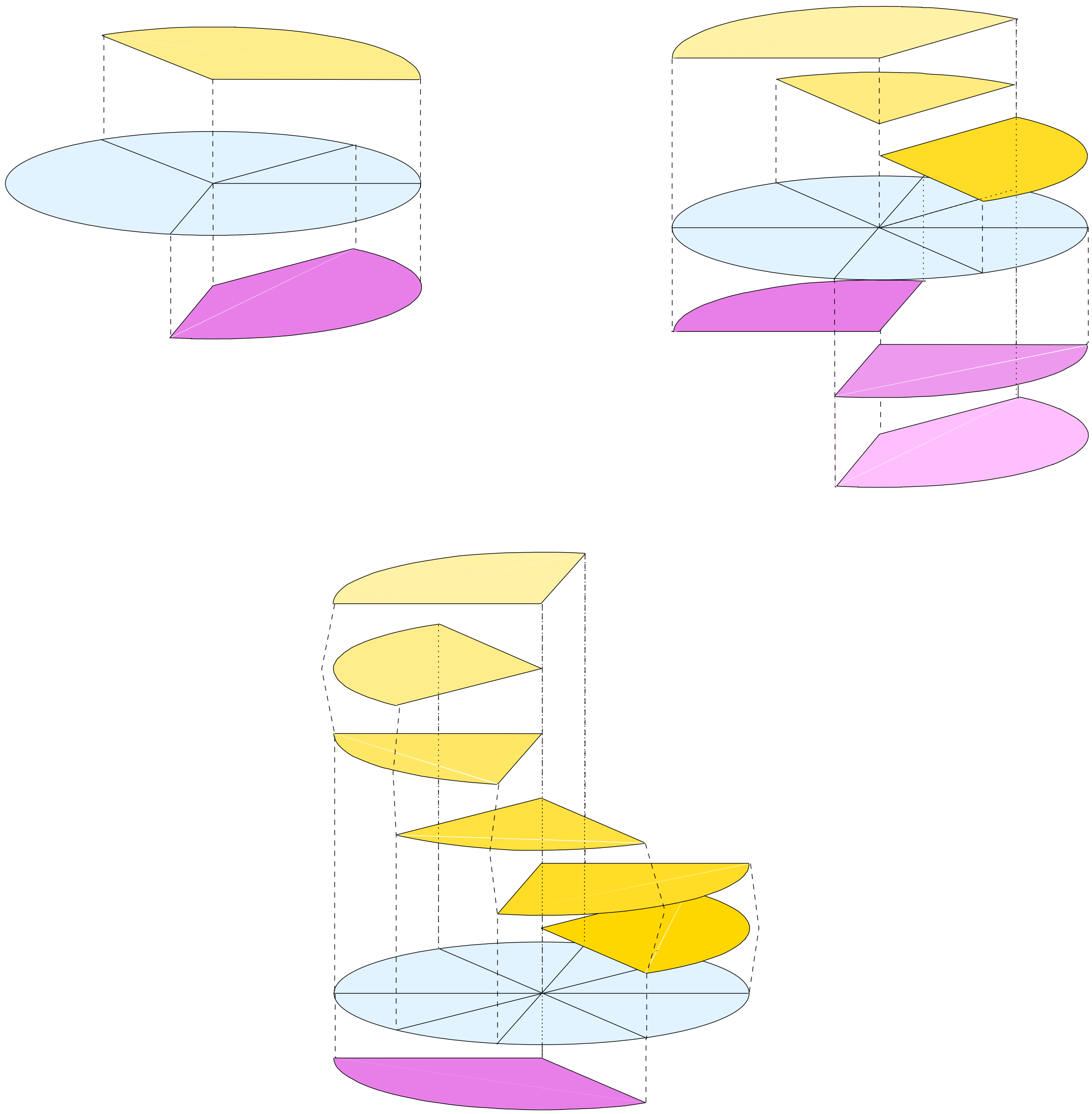
III.3)- Branched Oriented Flat Manifolds

Step 1:

1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.



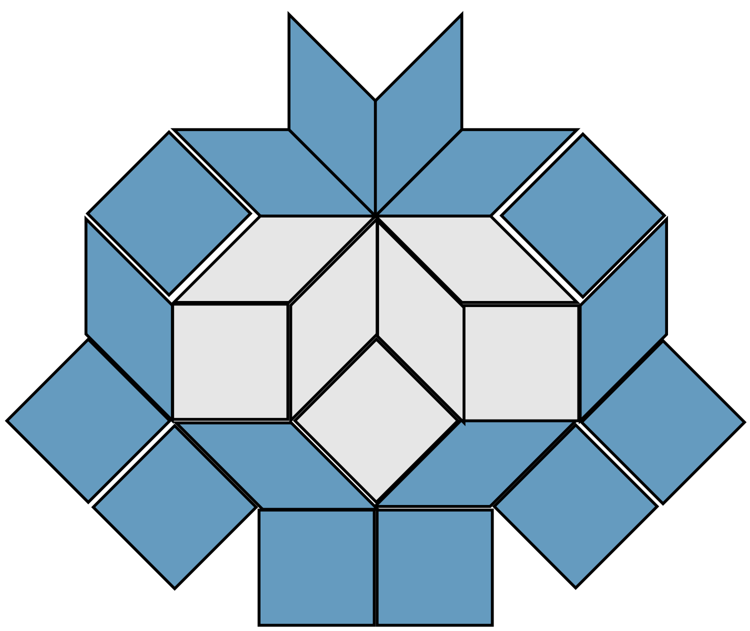
- The branching process -



- Vertex branching for the octagonal tiling -

Step 2:

1. Choose an increasing sequence $\{R_n\}_{n>0}$ of positive real numbers with $R_n \uparrow \infty$;
2. for each $n \geq 1$ consider all patches of diameter less than R_n;
3. add to each patch in $\mathcal{T}$, the tiles touching it from outside along its frontier. Call such a patch *modulo translation* a *a colored patch*;
4. proceed then as in Step 1 by replacing prototiles by colored patches to get the BOF-manifold B_n.



- A colored patch -

Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n;
2. call Ω the *projective limit* of the sequence
$$\cdots \xrightarrow{f_{n+1}} B_{n+1} \xrightarrow{f_n} B_n \xrightarrow{f_{n-1}} \cdots$$
3. there are commuting vector fields $X_1, \cdots X_d$ on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action T on Ω.

Theorem 2 *The dynamical system*
$$(\Omega, \mathbb{R}^d, \mathrm{T}) = \varprojlim (B_n, f_n)$$
obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

IV - NC Brillouin Zone

J. BELLISSARD, *The Gap Labelling Theorems for Schrödinger's Operators*, in *From Number Theory to Physics*, pp. 538-630, Les Houches March 89, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

IV.1)- Algebra

$(\Omega, \mathbb{R}^d, \tau)$ is a topological dynamical system. One orbit at least is dense. The crossed product

$$\mathcal{A} = \mathcal{C}(\Omega) \rtimes_\tau \mathbb{R}^d$$

is (almost) the smallest C^*-algebra containing both the space of continuous functions on Ω and the action of $\mathbb{R}^d$ submitted to the commutation rules (for $f \in \mathcal{C}(\Omega)$)

$$T(a) f T(a)^{-1} = f \circ \tau^{-a}$$

1. For a crystal $\Omega = \mathbb{V}$, $\mathbb{R}^d$ acts by quotient action.
2. $\mathcal{C}(\mathbb{V}) \rtimes_\tau \mathbb{R}^d \simeq \mathcal{C}(\mathbb{B}) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators and $\mathbb{B}$ is the dual of the period group of $\mathcal{L}$. $\mathbb{B}$ is called the *Brillouin zone*.

$\mathcal{A}$ is the *Noncommutative version of the space of $\mathcal{K}$-valued function over the Brillouin zone.*

IV.2)- Construction of $\mathcal{A}$

Endow $\mathcal{A}_0 = \mathcal{C}_c(\Omega \times \mathbb{R}^d)$ with (here $A, B \in \mathcal{A}_0$):

1. Product
$$A \cdot B(\omega, x) = \int_{y \in \mathbb{R}^d} d^d y \, A(\omega, y) \, B(\tau^{-y}\omega, x - y)$$

2. Involution
$$A^*(\omega, x) = \overline{A(\tau^{-x}\omega, -x)}$$

3. A faithfull family of representations in $\mathcal{H} = L^2(\mathbb{R}^d)$
$$\pi_\omega(A)\, \psi(x) = \int_{\mathbb{R}^d} d^d y \; A(\tau^{-x}\omega, y - x) \cdot \psi(y)$$
if $A \in \mathcal{A}_0$, $\psi \in \mathcal{H}$.

4. C^*-norm
$$\|A\| = \sup_{\omega \in \Omega} \|\pi_\omega(A)\| \; .$$

Definition 1 *The C^*-algebra $\mathcal{A}$ is the completion of $\mathcal{A}_0$ under this norm.*

IV.3)- Calculus

Integration: Let $\mathbb{P}$ be an $\mathbb{R}^d$-invariant ergodic probability measure on Ω. Then set (for $A \in \mathcal{A}_0$)):

$$\mathcal{T}_{\mathbb{P}}(A) = \int_\Omega d\mathbb{P}\ A(\omega, 0) = \overline{< 0|\pi_\omega(A)0 >}^{dis.}$$

Then $\mathcal{T}_{\mathbb{P}}$ extends as a *positive trace* on $\mathcal{A}$.

Trace per unit volume: thanks to Birkhoff's theorem:

$$\mathcal{T}_{\mathbb{P}}(A) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \mathrm{Tr}(\pi_\omega(A) \restriction_\Lambda) \quad \text{a.e. } \omega$$

Differential calculus: A commuting set of $*$-derivations is given by

$$\partial_i A(\omega, x) = \imath x_i A(\omega, x)$$

defined on $\mathcal{A}_0$. Then $\pi_\omega(\partial_i A) = -\imath[X_i, \pi_\omega(A)]$ where $X = (X_1, \cdots, X_d)$ are the coordinates of the position operator.

IV.4)- Electrons

Schrödinger's equation (ignoring interactions) on $\mathbb{R}^d$

$$H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y \in \mathcal{L}_\omega} v(.-y) \ ,$$

acting on $\mathcal{H} = L^2(\mathbf{R}^d)$. Here $v \in L^1(\mathbb{R}^d)$ is real valued, decays fast enough, is the *atomic potential*.

Lattice case (*tight binding representation*)

$$\tilde{H}_\omega \psi(x) = \sum_{y \in \mathcal{L}_\omega} h(\mathrm{T}^{-x}\omega, y-x)\, \psi(y) \ ,$$

Proposition 3 *1. There is $R(z) \in \mathcal{A}$, such that, for every $\omega \in \Omega$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$(z - H_\omega)^{-1} \ = \ \pi_\omega(R(z)) \ .$$

2. There is $\tilde{H} \in C^*(\Gamma_{tr})$ such that $\tilde{H}_\omega = \pi_\omega(\tilde{H})$.

3. If $\Sigma_{\mathrm{H}} = \bigcup_{\omega\in\Omega} \mathrm{Sp}(H_\omega)$, then $R(z)$ is holomorphic in $z \in \mathbb{C} \setminus \Sigma_H$. The bounded components of $\mathbb{R} \setminus \Sigma_{\mathrm{H}}$ are called spectral gaps (same with $\tilde{H}$).

IV.5)- Density of States

- Let $\mathbb{P}$ be an invariant ergodic probability on Ω. Let

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbf{R}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega|_\Lambda \leq E\}$$

It is called the *Integrated Density of states* or *IDS*.

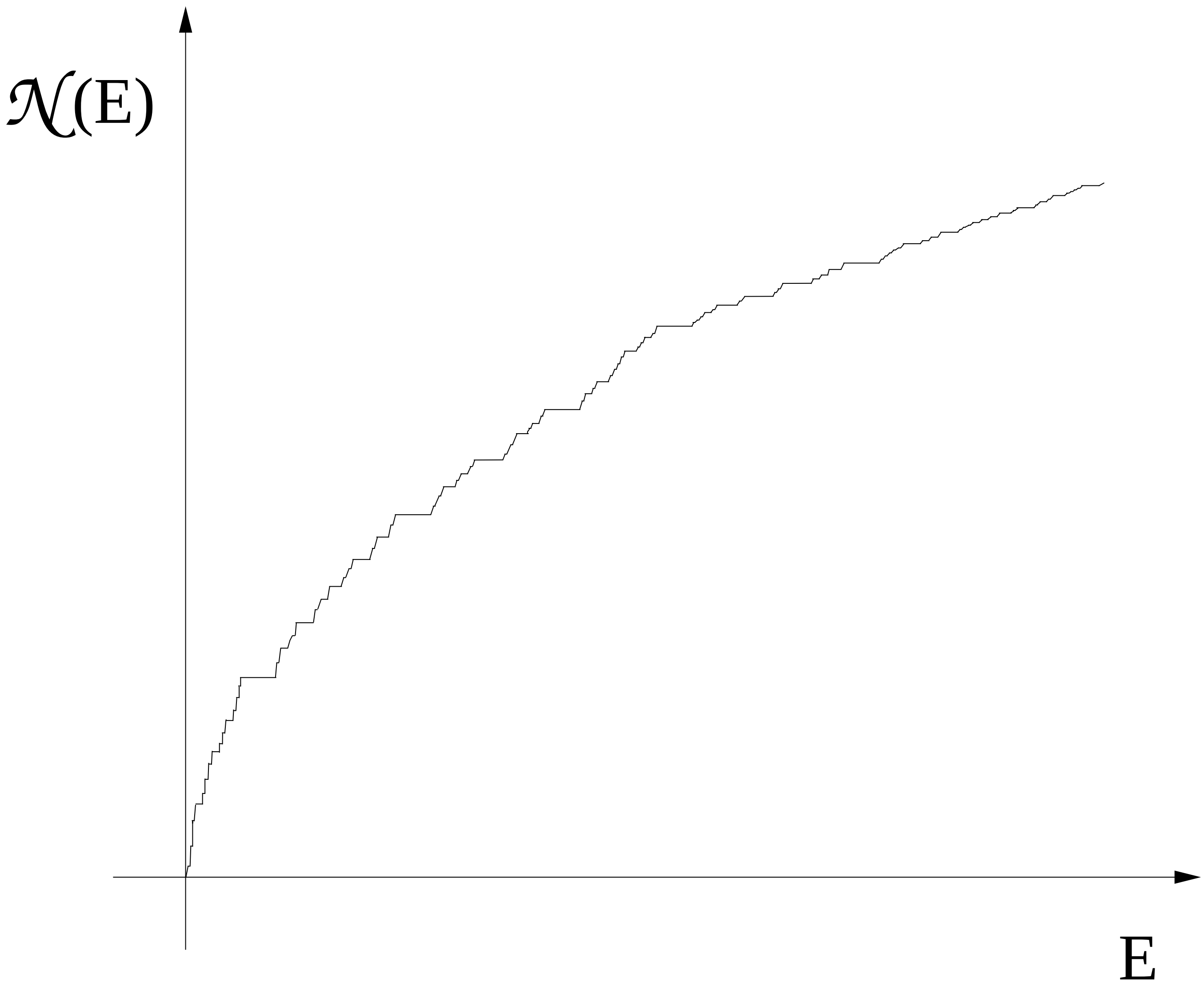
- The limit above exists $\mathbb{P}$-almost surely and

$$\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}(\chi(H \leq E)) \qquad \textit{(Shubin, '76)}$$

$\chi(H \leq E)$ is the eigenprojector of H in $\mathcal{L}^\infty(\mathcal{A})$.

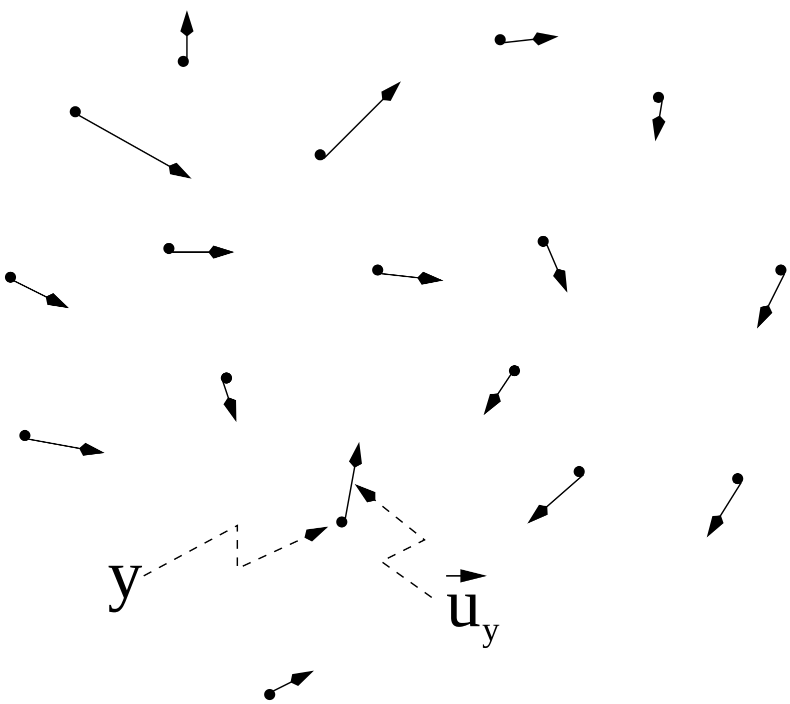
- $\mathcal{N}$ is non decreasing, non negative and constant on gaps. $\mathcal{N}(E) = 0$ for $E < \inf \Sigma_{\mathrm{H}}$. For $E \to \infty$, $\mathcal{N}(E) \sim \mathcal{N}_0(E)$ where $\mathcal{N}_0$ is the IDS of the free case (namely $v = 0$).

- *Gaps can be labelled by the value the IDS takes on them*



- An example of IDS -

IV.6)- Phonons



1. Phonons are *acoustic waves* produced by small displacements of the atomic nuclei.
2. These waves are polarized with d-directions of polarization: $d-1$ are *transverse*, one is *longitudinal*.
3. The nuclei motion is approximatively harmonic and quantized according to the *Bose-Einstein* statistics.
4. The charged nuclei interact with electrons, leading to an *electron-phonon interaction*.

1. For identical atoms with *harmonic motion*, the classical equations of motion are:

$$M \frac{d^2 \vec{u}_{(\omega, x)}}{dt^2} = \sum_{x \neq y \in \mathcal{L}_\omega} K_\omega(x, y) \left(\vec{u}_{(\omega, y)} - \vec{u}_{(\omega, x)} \right)$$

   where M is the atomic mass, $\vec{u}_{(\omega, x)}$ is its classical displacement vector and $K_\omega(x, y)$ is the matrix of *spring constants*.

2. $K_\omega(x, y)$ decays fast in $x - y$, uniformly in ω..

3. Covariance gives

$$K_\omega(x, y) = k(\tau^{-x}\omega, y - x)$$

   thus

$$k \in C^*(\Gamma_{tr}) \otimes M_d(\mathbb{C})$$

4. Then the spectrum of k/M gives the *eigenmodes* propagating in the solid. Its density (DPM) is given by Shubin's formula again.

V - Gap labels and K-theory

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V.1)- K-group

Let $\mathcal{A}$ be a separable C^*-algebra .

1. A *projection* is $P \in \mathcal{A}$ with $P = P^* = P^2$.
2. Two projections P and Q are*equivalent* if there is $U \in \mathcal{A}$ such that $P = UU^*, \ \ Q = U^*U$.
   $[P]$ denotes the equivalent classe of P.
3. Two projections P and Q are *orthogonal* if $PQ = QP = 0$. Then $P + Q$ is a projection called the *direct sum* $P \oplus Q$.
4. If $\mathcal{K} = \overline{\bigcup_n M_n(\mathbb{C})}^{\|\cdot\|}$, then $\mathcal{A}$ is *stable* if $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{K}$. In a stable C^*-algebra for any pair P, Q of projections, there are $P' \sim P$, $Q' \sim Q$ with $P'Q' = Q'P' = 0$.
   Thus $[P] + [Q] = [P' \oplus Q']$ is well defined.
5. $K_0(\mathcal{A})$ is the group generated by the $[P]$'s with the previous addition.
6. $[P]$ is invariant by homotopy.

V.2)- K-group labels

- If E belongs to a gap $\mathfrak{g}$, the characteristic function $E' \in \mathbf{R} \mapsto \chi(E' \leq E)$ is continuous on the spectrum of H. Thus:

  $P_{\mathfrak{g}} = \chi(H \leq E)$ *is a projection in* $\mathcal{A}$ *!*

- $\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}\left(P_{\mathfrak{g}}\right) \in \mathcal{T}^*_{\mathbb{P}}\left(K_0(\mathcal{A})\right)$ *!*

Theorem 3 (Abstract gap labeling theorem)

- $S \subset \Sigma_{\mathrm{H}}$ *clopen,* $n_S = [\chi_S(H)] \in K_0(\mathcal{A})$*. If* $S_1 \cap S_2 = \emptyset$ *then* $n_{S_1 \cup S_2} = n_{S_1} + n_{S_2}$ *(***additivity***).*

- *Gap labels are invariant under norm continuous variation of* H *(***homotopy invariance***).*

- *For* $\lambda \in [0,1] \mapsto H(\lambda) \in \mathcal{A}$ *continuous, if* $S(\lambda) \subset \Sigma_{\mathrm{H}}$ *clopen, continuous in* λ *with* $S(0) = S_1 \cup S_2$*,* $S(1) = S'_1 \cup S'_2$ *and* $S_1 \cap S_2 = \emptyset = S'_1 \cap S'_2$ *then* $n_{S_1} + n_{S_2} = n_{S'_1} + n_{S'_2}$ *(***conservation of gap labels under band crossings***).*

Theorem 4 *If $\mathcal{L}$ is an finite type Delone set in $\mathbb{R}^d$ with Hull $(\Omega, \mathbb{R}^d, \mathrm{T})$, then, for any $\mathbb{R}^d$-invariant probability measure $\mathbb{P}$ on Ω*

$$\mathcal{T}^*_{\mathbb{P}}\left(K_0(\mathcal{A})\right) \;=\; \int_X d\mathbb{P}_{tr}\ \mathcal{C}(X, \mathbb{Z})\ .$$

if $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$, X is the canonical transversal and $\mathbb{P}_{tr}$ the transverse measure induced by $\mathbb{P}$.

Main ingredient for the proof

the Connes measured index theorem
for foliated spaces

A. Connes, *Sur la théorie non commutative de l'intégration*, Lecture Notes in Math **725**, 19-143, Springer, Berlin (1979).

V.3)- History

For $d = 1$ this result follows from the Pimsner & Voiculescu exact sequence *(Bellissard, '92)*.

For $d = 2$, a double use of the Pimsner & Voiculescu exact sequence provides the result *(van Elst, '95)*.

For $d \geq 3$ whenever $(\Omega, \mathbb{R}^d, \mathrm{T})$ is Morita equivalent to a $\mathbb{Z}^d$-action, using spectral sequences *(Hunton, Forrest)* this theorem was proved for $d = 3$
(Bellissard, Kellendonk, Legrand, '00).

The theorem has also been proved for all d's recently and independently by
-*(Bellissard, Benedetti, Gambaudo, 2001)*,
-*(Benameur, Oyono, 2001)*,
-*(Kaminker, Putnam, 2002)*.