The GAP LABELLING THEOREM for TILINGS

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Main References

J. BELLISSARD, *The Gap Labelling Theorems for Schrödinger's Operators*, in *From Number Theory to Physics*, pp. 538-630, Les Houches March 89, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

J. BELLISSARD, D. HERRMANN, M. ZARROUATI, *Hull of Aperiodic Solids and Gap Labelling Theorems*, In *Directions in Mathematical Quasicrystals*, CRM Monograph Series, Volume **13**, (2000), 207-259, M.B. Baake & R.V. Moody Eds., AMS Providence.

J. BELLISSARD, J. KELLENDONK, A. LEGRAND, *Gap Labelling for three dimensional aperiodic solids* , C. R. Acad. Sci. (Paris), **t.332**, Série I, p. 521-525, (2001).

J. BELLISSARD, R. BENEDETTI, J. M. GAMBAUDO, *Spaces of Tilings, Finite Telescopic Approximations, and Gap-Labelling, preprint*, August 2000.

Content

I - The Hull as a Dynamical System

J. Bellissard, D. Hermmann, M. Zarrouati, *Hull of Aperiodic Solids and Gap Labelling Theorems*
To appear in *Directions in Mathematical Quasicrystals*, M.B. Baake & R.V. Moody Eds, AMS, (2000).

I.1)- Point Sets

Equilibrium positions of atomic nuclei make up a point set $\mathcal{L} \subset \mathbb{R}^d$ *the set of lattice sites*. $\mathcal{L}$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. A *Delone* set: $\mathcal{L}$ is uniformly discrete and *relatively dense* : $\exists R > 0$ s.t. each ball of radius R contains at least two points of $\mathcal{L}$.
4. A *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone sets.

Examples:

1. A random Poissonian set in $\mathbb{R}^d$ is almost surely discrete but not uniformly discrete nor relatively dense.
2. Due to Coulomb repulsion and Quantum Mechanics, **lattices of atoms are always uniformly discrete**.
3. Impurities in semiconductors are not relatively dense.
4. In amorphous media $\mathcal{L}$ is Delone.
5. In a quasicrystal $\mathcal{L}$ is Meyer.

I.2)- Point Measures

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $\mathcal{C}_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x-y) \quad \in \mathfrak{M}(\mathbb{R}^d) \; .$$

The *Hull* is the closure in $\mathfrak{M}(\mathbb{R}^d)$

$$\Omega = \overline{\left\{ \mathrm{T}^a \nu^{\mathcal{L}} ; a \in \mathbb{R}^d \right\}} \; ,$$

where $\mathrm{T}^a \nu$ is the translated of ν by a.

Facts:

1. Ω is compact and $\mathbb{R}^d$ acts by homeomorphisms.
2. If $\omega \in \Omega$, there is a uniformly discrete point set $\mathcal{L}_\omega$ in $\mathbb{R}^d$ such that ω coincides with $\nu_\omega = \nu^{\mathcal{L}_\omega}$.
3. If $\mathcal{L}$ is *Delone* (resp. *Meyer*) so are the $\mathcal{L}_\omega$'s.

I.3)- Properties

(a) Minimality

Proposition 1 *$\mathbb{R}^d$ acts minimaly on Ω if and only if, for any $\omega \in \Omega$ and $F \subset \Omega$ closed, the subset $\mathcal{L}_\omega^F = \{x \in \mathcal{L}_\omega ; \mathrm{T}^{-x}\omega \in F\}$ is a Delone set.*

(b) Transversal

The closed subset $X = \{\omega \in \Omega \; ; \; \nu_\omega(\{0\}) = 1\}$ is called the *canonical transversal*. Let G be the subgroupoid of $\Omega \rtimes \mathbb{R}^d$ induced by X.

A Delone set $\mathcal{L}$ has *finite type* if $\mathcal{L} - \mathcal{L}$ is closed and discrete.

(c) Cantorian Transversal

Proposition 2 *If $\mathcal{L}$ has finite type, then the transversal is completely discontinuous (Cantor).*

II - Tilings & Point Sets

C. Radin, *Miles of Tiles*, Student Mathematical Library, Vol 1, Amer. Math. Soc., Providence, (1999).

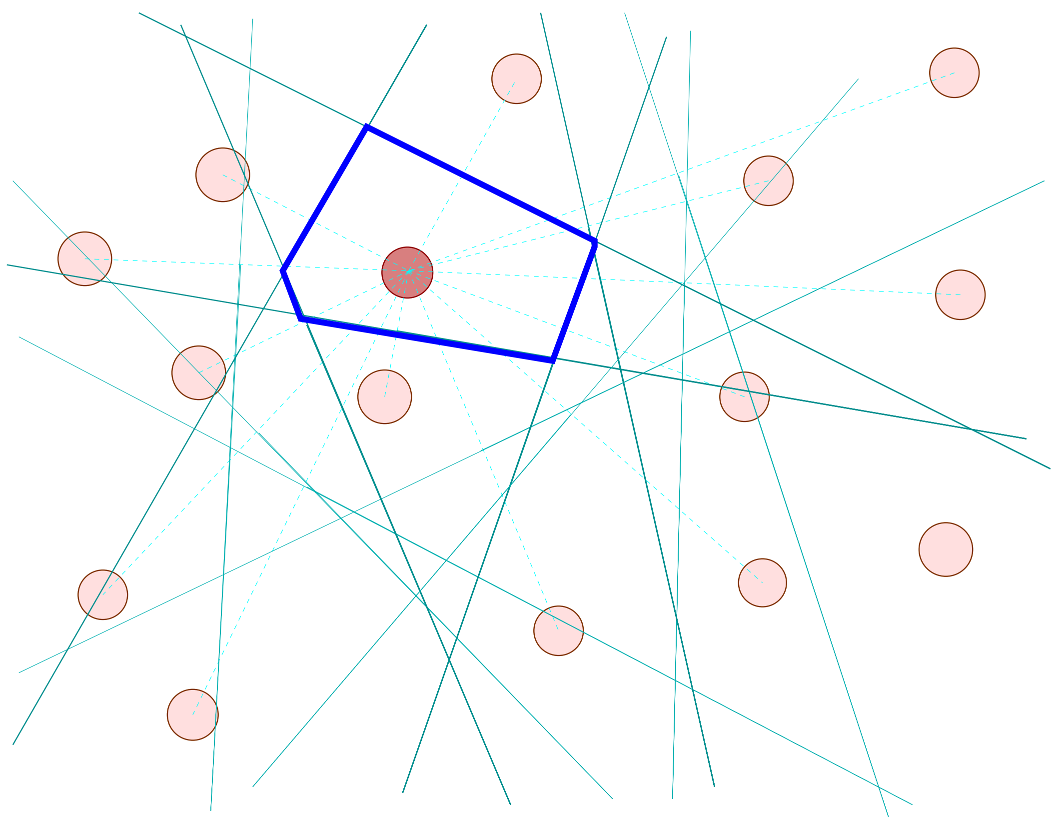
M. Senechal, *Crystalline Symmetries : An informal mathematical introduction*, Institute of Physics. Alan Hilger, Ltd., (1990).

J. Bellissard, R. Benedetti, J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling*, preprint August (2001).

II.1)- Voronoi Cells

For $\mathcal{L}$ *Delone* and $x \in \mathcal{L}$, the *Voronoi cell of x* is

$$V_x = \{y \in \mathbb{R}^d ; |y - x| < |y - x'|, \forall x' \in \mathcal{L} \setminus \{x\}\}$$

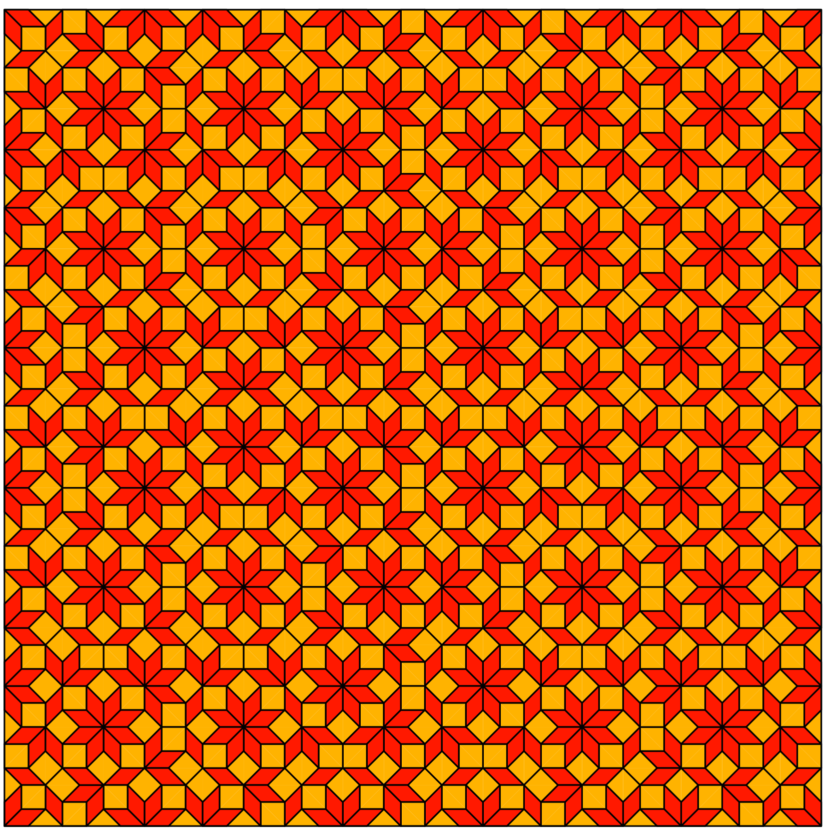


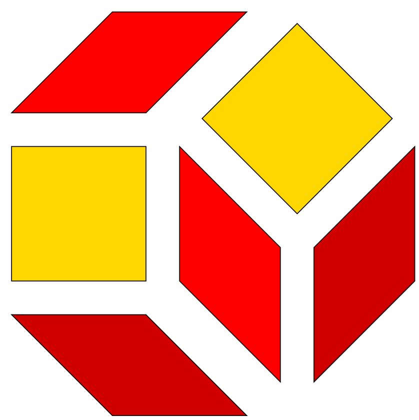
- Building a Voronoi Cell -

The V_x's are open polyhedrons with uniformly bounded diameter. They are mutually disjoint and their closure cover $\mathbb{R}^d$: it is a *tiling of $\mathbb{R}^d$*

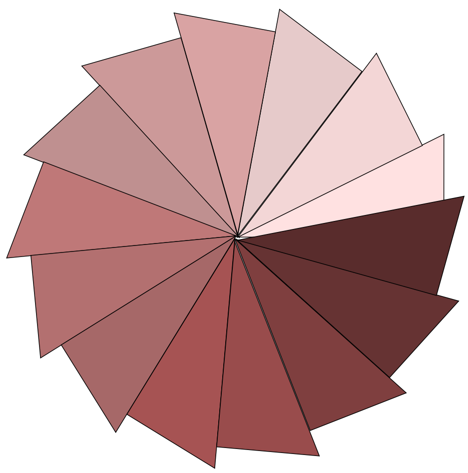
II.2)- The Finite Pattern Condition

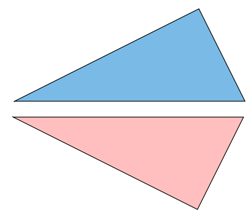
A tiling $\mathcal{T}$ fulfills the *finite pattern condition* (FPC) if the number of tiles *modulo translations* is finite. A *patch* is the set of tiles of $\mathcal{T}$ contained in some ball. The number of patches of given radius is finite iff $\mathcal{T}$ is FPC. Then the Hull is transversally Cantor.

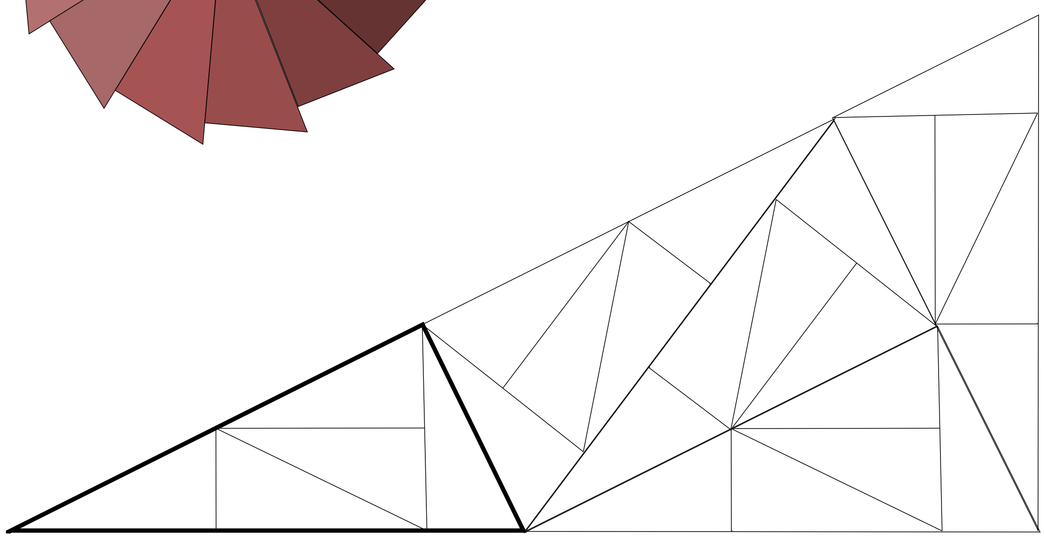




- The octagonal tiling is FPC -





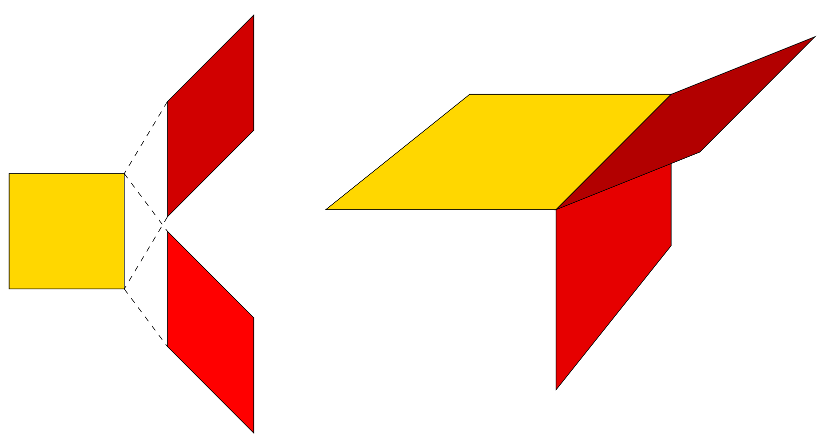


- The pinwheel tiling is NOT FPC ! -

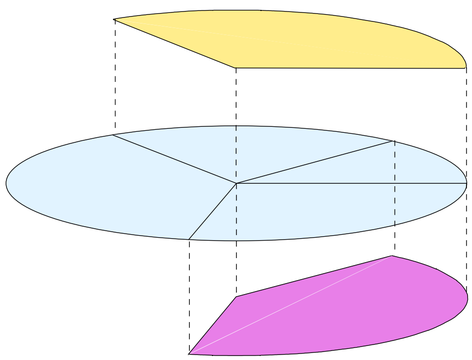
II.3)- Branched Oriented Flat Manifolds

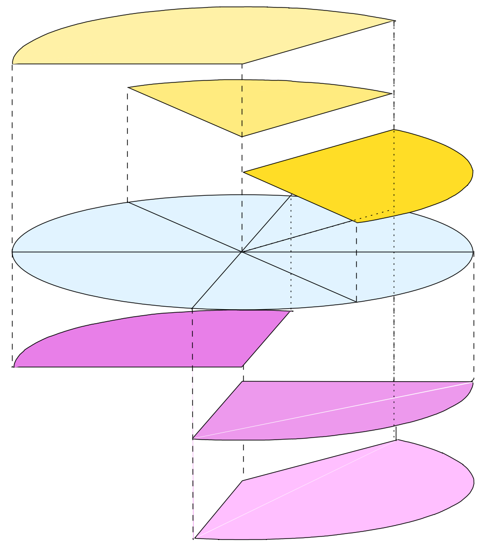
Step 1:

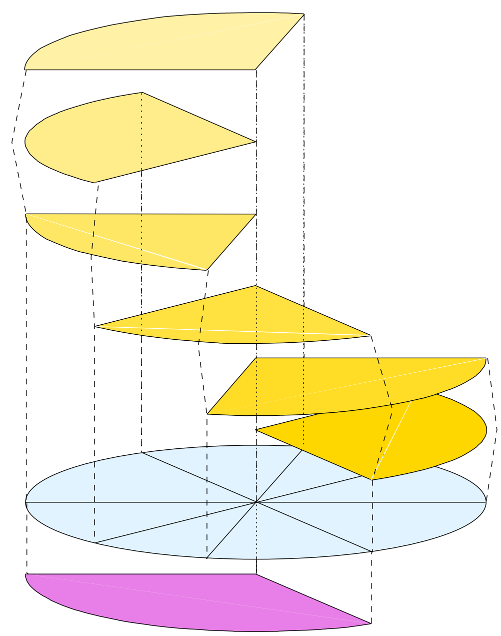
1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.



- The branching process -



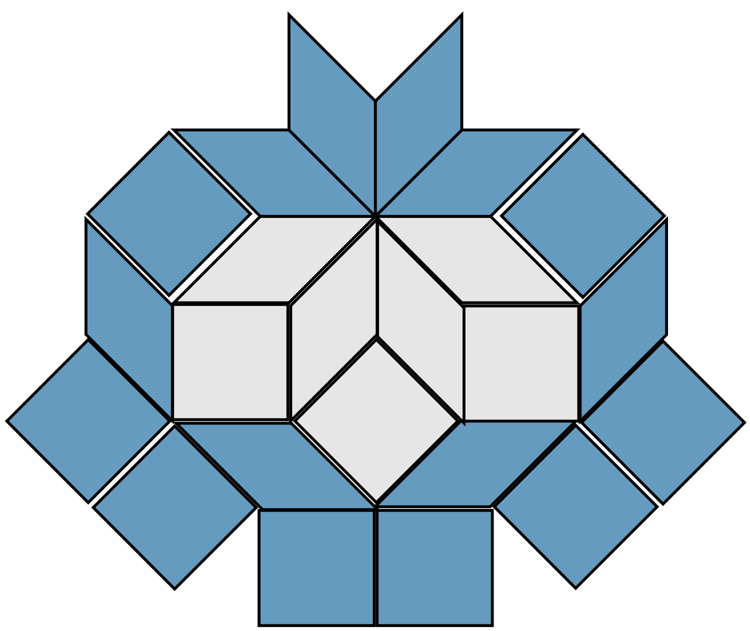




- *Vertex branching for the octagonal tiling* -

Step 2:

1. Choose an increasing sequence $\{R_n\}_{n>0}$ of positive real numbers with $R_n \uparrow \infty$;
2. for each $n \geq 1$ consider all patches of diameter less than R_n;
3. add to each patch in $\mathcal{T}$, the tiles touching it from outside along its frontier. Call such a patch *modulo translation* a *a colored patch*;
4. proceed then as in Step 1 by replacing prototiles by colored patches to get the BOF-manifold B_n.



- A colored patch -

Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n;
2. call Ω the *projective limit* of the sequence
$$\cdots \stackrel{f_{n+1}}{\rightarrow} B_{n+1} \stackrel{f_n}{\rightarrow} B_n \stackrel{f_{n-1}}{\rightarrow} \cdots$$
3. there are commuting vector fields $X_1, \cdots X_d$ on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action T on Ω.

Theorem 1 *The dynamical system*
$$(\Omega, \mathbb{R}^d, \mathrm{T}) = \lim_{\leftarrow}(B_n, f_n)$$
obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

III - Gap Labelling and K-theory

J. BELLISSARD, *The Gap Labelling Theorems for Schrödinger's Operators*, in *From Number Theory to Physics*, pp. 538-630, Les Houches March 89, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

III.1)- Algebra

Let $\mathcal{A}_0 = \mathcal{C}_c(\Omega \times \mathbb{R}^d)$ be seen as a dense subalgebra of $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. For any $\omega \in \Omega$, let π_ω be the left regular representation on $\mathcal{H} = L^2(\mathbf{R}^d)$ defined by:

$$\pi_\omega(A)\psi(x) = \int_{\mathbb{R}^d} d^d y \; A(\mathrm{T}^{-x}\omega, y - x) \; \psi(y) \; ,$$

and $\psi \in \mathcal{H}$.

If $\mathbb{P}$ is an $\mathbb{R}^d$-invariant ergodic probability measure on Ω, let $\mathcal{T}_{\mathbb{P}}$ be the trace on $\mathcal{A}$ defined by

$$\mathcal{T}_{\mathbb{P}}(A) \;=\; \int_{\Omega} \mathbb{P}(d\omega) \; A(\omega, 0)) \; ,$$

for $A \in \mathcal{A}_0$

III.2)- Hamiltonian

Schrödinger's equation (ignoring interactions)

$$H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y\in\mathcal{L}_\omega} v(. - y) \ ,$$

acting on $\mathcal{H} = L^2(\mathbf{R}^d)$. Here $v \in L^1(\mathbb{R}^d)$ is real valued, decays fast enough, is the *atomic potential.*

Proposition 3 *There is $R(z) \in \mathcal{A}$, such that, for every $\omega \in \Omega$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$(z - H_\omega)^{-1} \ = \ \pi_\omega(R(z)) \ .$$

If $\Sigma_{\mathrm{H}} = \bigcup_{\omega\in\Omega} \mathrm{Sp}(H_\omega)$, then $R(z)$ is holomorphic in $z \in \mathbb{C} \setminus \Sigma_H$.

The bounded connected components of $\mathbb{R}\backslash\Sigma_{\mathrm{H}}$ are called *spectral gaps*.

III.3)- Density of States

- Let $\mathbb{P}$ be an invariant ergodic probability on Ω. Let

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbf{R}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega|_\Lambda \leq E\}$$

It is called the *Integrated Density of states* or *IDS*.

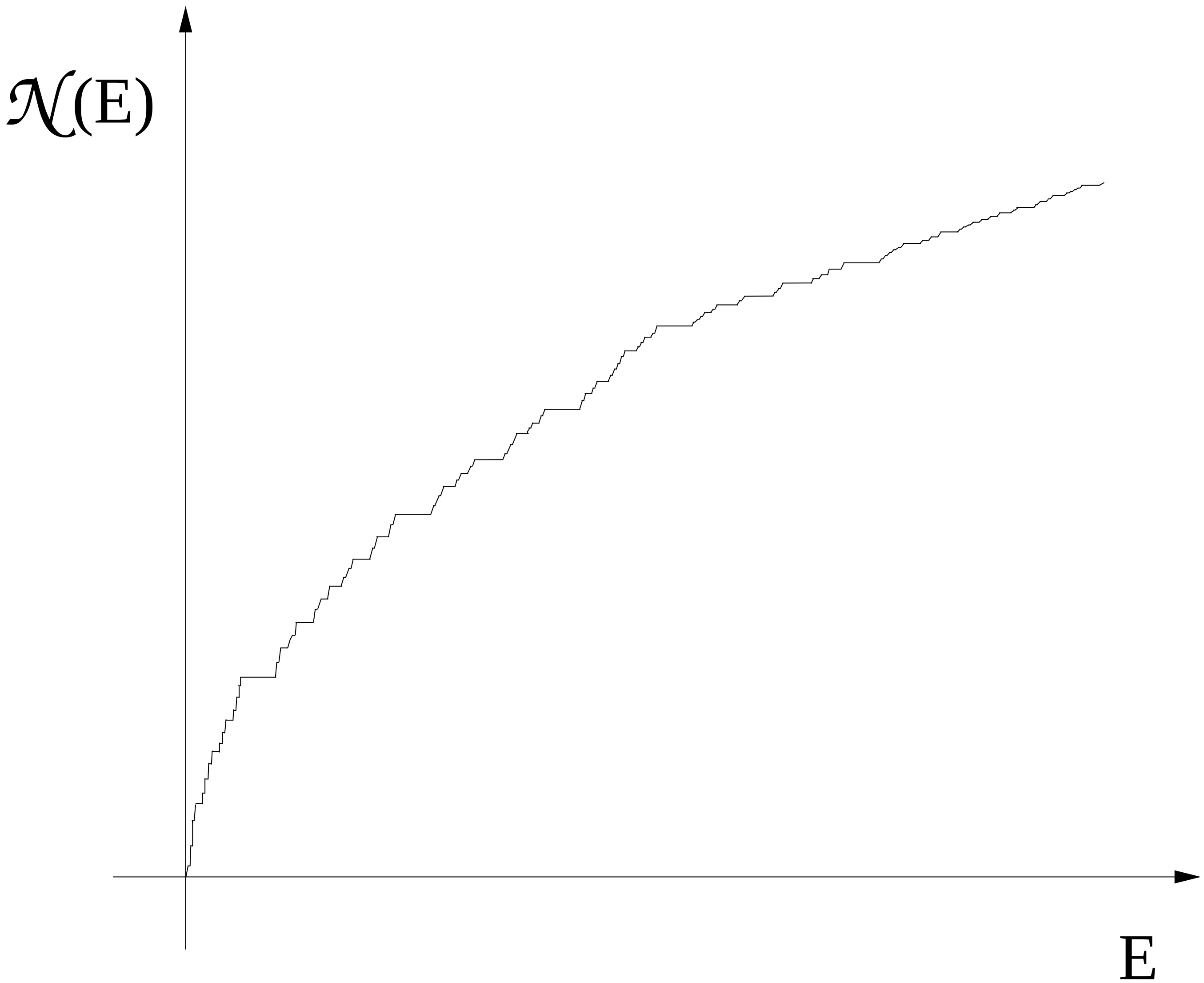
- The limit above exists $\mathbb{P}$-almost surely and

$$\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}(\chi(H \leq E)) \qquad \textit{(Shubin, '76)}$$

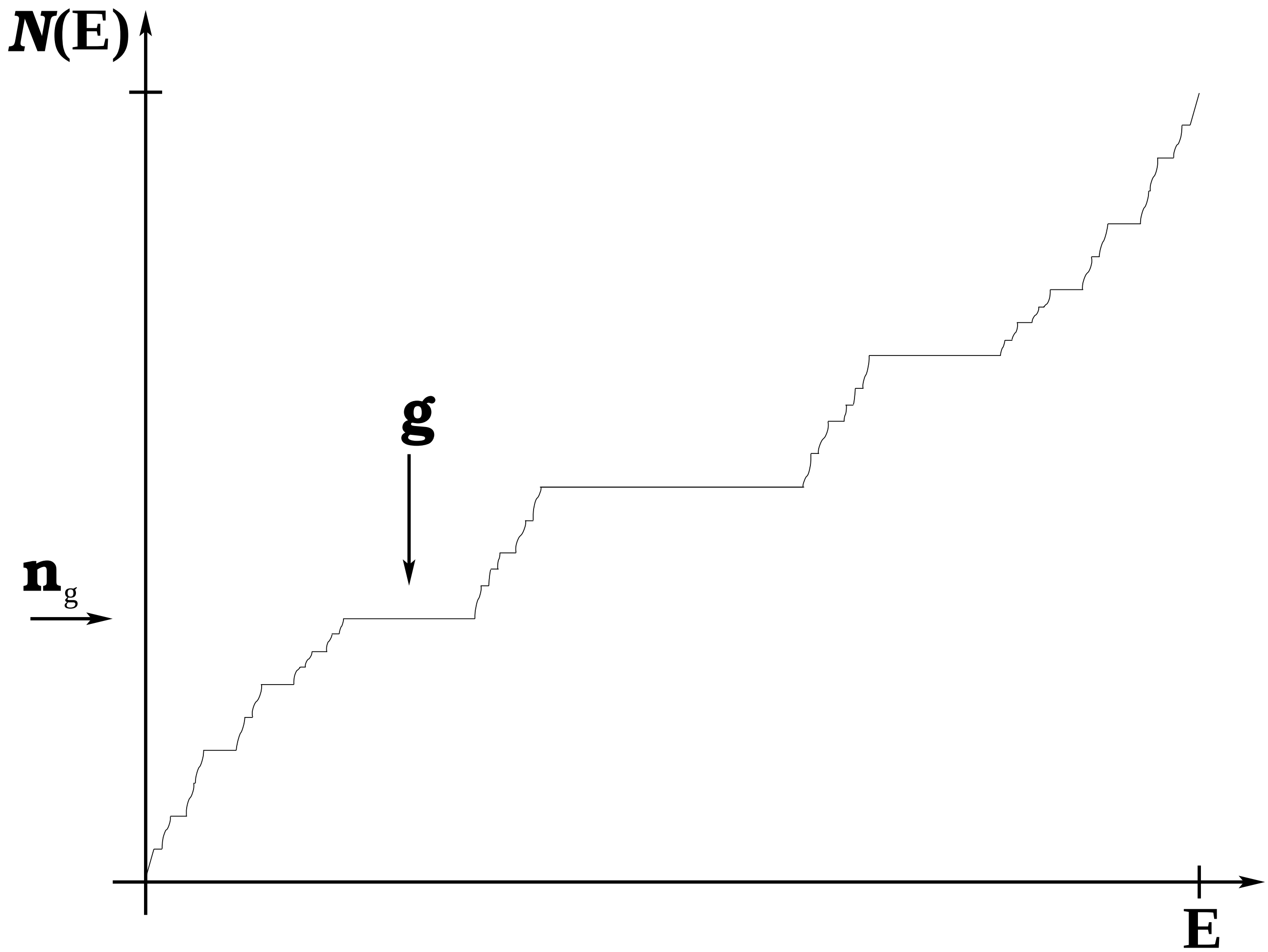
$\chi(H \leq E)$ is the eigenprojector of H in $\mathcal{L}^\infty(\mathcal{A})$.

- $\mathcal{N}$ is non decreasing, non negative and constant on gaps. $\mathcal{N}(E) = 0$ for $E < \inf \Sigma_{\mathrm{H}}$. For $E \to \infty$, $\mathcal{N}(E) \sim \mathcal{N}_0(E)$ where $\mathcal{N}_0$ is the IDS of the free case (namely $v = 0$).

- *Gaps can be labelled by the value the IDS takes on them*



- An example of IDS -



- An example of IDS -

III.4)- K-group labels

- If E belongs to a gap $\mathfrak{g}$, the characteristic function $E' \in \mathbf{R} \mapsto \chi(E' \leq E)$ is continuous on the spectrum of H. Thus:

  $P_{\mathfrak{g}} = \chi(H \leq E)$ *is a projection in* $\mathcal{A}$ *!*

- $\mathcal{N}(E) = \mathcal{T}_{\mathbb{P}}\left(P_{\mathfrak{g}}\right) \in \mathcal{T}_{\mathbb{P}}^{*}\left(K_0(\mathcal{A})\right)$ *!*

Theorem 2 (**Abstract gap labelling theorem**)

- *$S \subset \Sigma_{\mathrm{H}}$ clopen, $n_S = [\chi_S(H)] \in K_0(\mathcal{A})$. If $S_1 \cap S_2 = \emptyset$ then $n_{S_1 \cup S_2} = n_{S_1} + n_{S_2}$ (***additivity***).*

- *Gap labels are invariant under norm continuous variation of H (***homotopy invariance***).*

- *For $\lambda \in [0,1] \mapsto H(\lambda) \in \mathcal{A}$ continuous, if $S(\lambda) \subset \Sigma_{\mathrm{H}}$ clopen, continuous in λ with $S(0) = S_1 \cup S_2$, $S(1) = S_1' \cup S_2'$ and $S_1 \cap S_2 = \emptyset = S_1' \cap S_2'$ then $n_{S_1} + n_{S_2} = n_{S_1'} + n_{S_2'}$ (***conservation of gap labels under band crossings***).*

IV - Computing Gap Labels

J. Bellissard, *The Gap Labelling Theorems for Schrödinger's Operators*, in *From Number Theory to Physics*, pp. 538-630, Les Houches March 89, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

A. van Elst, *Rev. Math. Phys.*, **6**, (1994), 319-342.

J. Bellissard, J. Kellendonk, A. Legrand, *Gap Labelling for three dimensional aperiodic solids* *C. R. Acad. Sci.* (Paris), **t.332**, Série I, p. 521-525, (2001).

J. Bellissard, R. Benedetti, J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling*, preprint August (2001).

M. Benameur, H. Oyono, *private communication*, August (2001).

J. Kaminker, I. Putnam, *private communication*, August (2001).

IV.1)- The Main Result

Theorem 3 *If $\mathcal{T}$ is an FPC-tiling in $\mathbb{R}^d$ with Hull $(\Omega, \mathbb{R}^d, \mathrm{T})$, then, for any $\mathbb{R}^d$-invariant probability measure $\mathbb{P}$ on Ω*

$$\mathcal{T}^*_{\mathbb{P}}(K_0(\mathcal{A})) = \int_X d\mathbb{P}_{tr}\ \mathcal{C}(X, \mathbb{Z})\ .$$

if $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$, X is the canonical transversal and $\mathbb{P}_{tr}$ the transverse measure induced by $\mathbb{P}$.

For $d = 1$ this result follows from the Pimsner & Voiculescu exact sequence *(Bellissard, '92)*.

For $d = 2$, a double use of the Pimsner & Voiculescu exact sequence provides the result *(van Elst, '95)*.

For $d \geq 3$ whenever $(\Omega, \mathbb{R}^d, \mathrm{T})$ is Morita equivalent to a $\mathbb{Z}^d$-action, a strategy using spectral sequences led to this theorem for $d = 3$ *(Bellissard, Kellendonk, Legrand, '00)*.
The theorem has also been proved for all d's recently an independently by *(Benameur, Oyono, 2001)* and *(Kaminker, Putnam, 2001)*.

IV.2)- Cycles on BOF-Manifolds

A BOF-Manifold B admits a unique finite decomposition into tiles. Each tile inherits the orientation of B, thus each face of a tile is also oriented. A *positive weight* on B is a map w affecting to each positively oriented tile T a positive number $w(T)$ such that

$$w(\overline{T}) = -w(T)$$

and for each oriented face F

$$\partial w(F) = \sum_{F\subset\partial T} w(T) = 0 \qquad (Kirchhoff's\ law)$$

Thus the set $W(B)$ of positive weights satisfies

$$W(B) \ = \ H_d(B,\mathbb{R})^+ \qquad\qquad d = dim(B)$$

if $H_*(B,\mathbb{R})$ is the homology of the CW-complex defined by the tiles and their faces of lower dimensions. The positive cone is defined by the positively oriented tiles.

IV.3)- Invariant Measures

D. SULLIVAN, *Invent. Math.* , **89**, (1976), 225-255.

Using

$$\Omega = \lim_{\leftarrow} B_n \quad \Rightarrow \quad H_*(\Omega, \mathbb{R}) = \lim_{\leftarrow} H_*(B_n, \mathbb{R})$$

Theorem 4 *1)- The set of $\mathbb{R}^d$-invariant positive bounded measures on Ω is affinely isomorphic to the positive part of $H_d(\Omega, \mathbb{R})$.*
2)- The direct limit $H^*(\Omega, \mathbb{R})$ of the cohomology groups $H^*(B_n, \mathbb{R})$ coincides with the de Rham cohomology of Ω along the orbits of the $\mathbb{R}^d$-action.
3)- The pairing between the invariant measures and $H^d(\Omega, \mathbb{R})$ is given by the Ruelle-Sullivan current

$$\langle \mathbb{P} | [\eta] \rangle = \int_\Omega d\mathbb{P} \, \langle \eta | X_1 \wedge \cdots \wedge X_d \rangle$$

where $X_1, \cdots, X_d$ generate the $\mathbb{R}^d$-action.

IV.4)- K-theory

A. CONNES, in *Noncommutative Geometry*, Academic Press, San Diego, (1994).

A. CONNES, in *Pitman Res. Notes in Math*, **123**, Longman Harlow (1986), pp. 52-144.

M.V. PIMSNER, in *Lecture Notes in Math* **1132**, (1983), 374-408.

By the Connes-Thom theorem,

$$K_i(\mathcal{C}(\Omega) \rtimes \mathbb{R}^d) \simeq K_{i+d}(\mathcal{C}(\Omega)) \, .$$

For a projector $P \in \mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$

$$\mathcal{T}_{\mathbb{P}}(P) \;=\; \langle \mathbb{P} | [\eta_P] \rangle$$

where $[\eta] \in H^d(\Omega, \mathbb{Z})$. In particular η can be chosen so that

$$\langle \eta | X_1 \wedge \cdots \wedge X_d \rangle \;\in \mathcal{C}(\Omega, \mathbb{Z})$$

Using the theorem 4, the Main Theorem is proved.

* * *