GAP LABELING THEOREMS

Jean BELLISSARD

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics
e-mail: jeanbel@math.gatech.edu

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Main References

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Content

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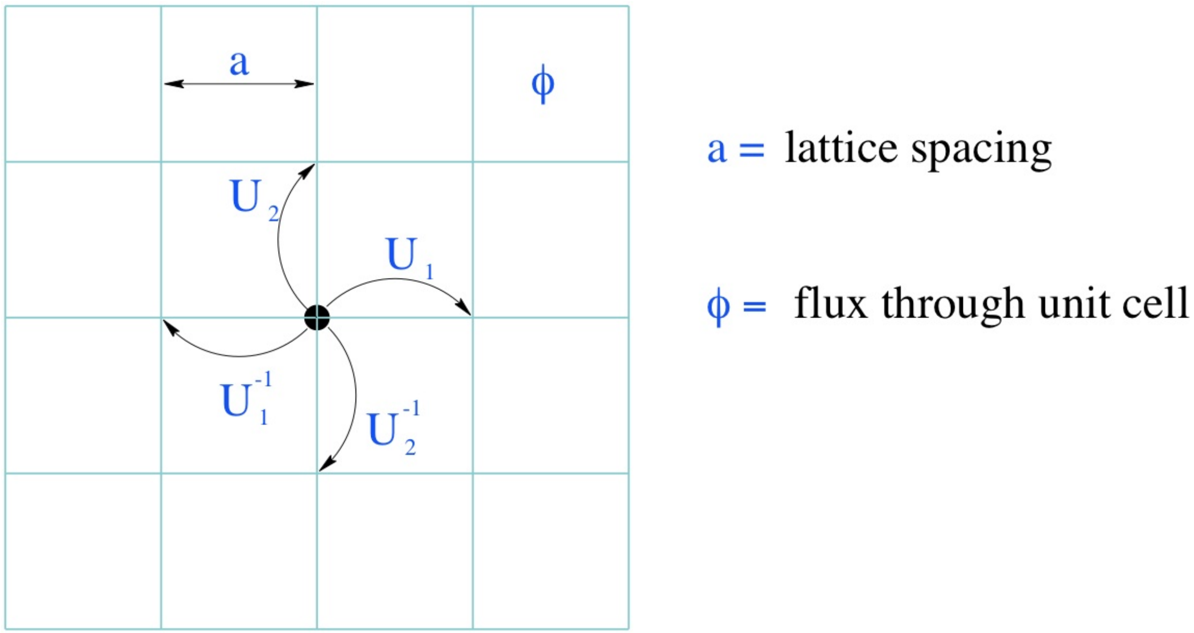
I - Harper's Model

P. G. Harper, Proc. Phys. Soc. A, **68**, 874-878, (1955)

D. R. Hofstadter, Phys. Rev. B, **14**, 2239-2249, (1976)

2*D*-Crystal Electrons in Magnetic Field

- Perfect *square lattice*, nearest neighbor hoping terms, *uniform magnetic field* B perpendicular to the plane of the lattice
- Translation operators U_1, U_2



2*D*-Crystal Electrons in Magnetic Field

- Commutation rules *(Rotation Algebra)*

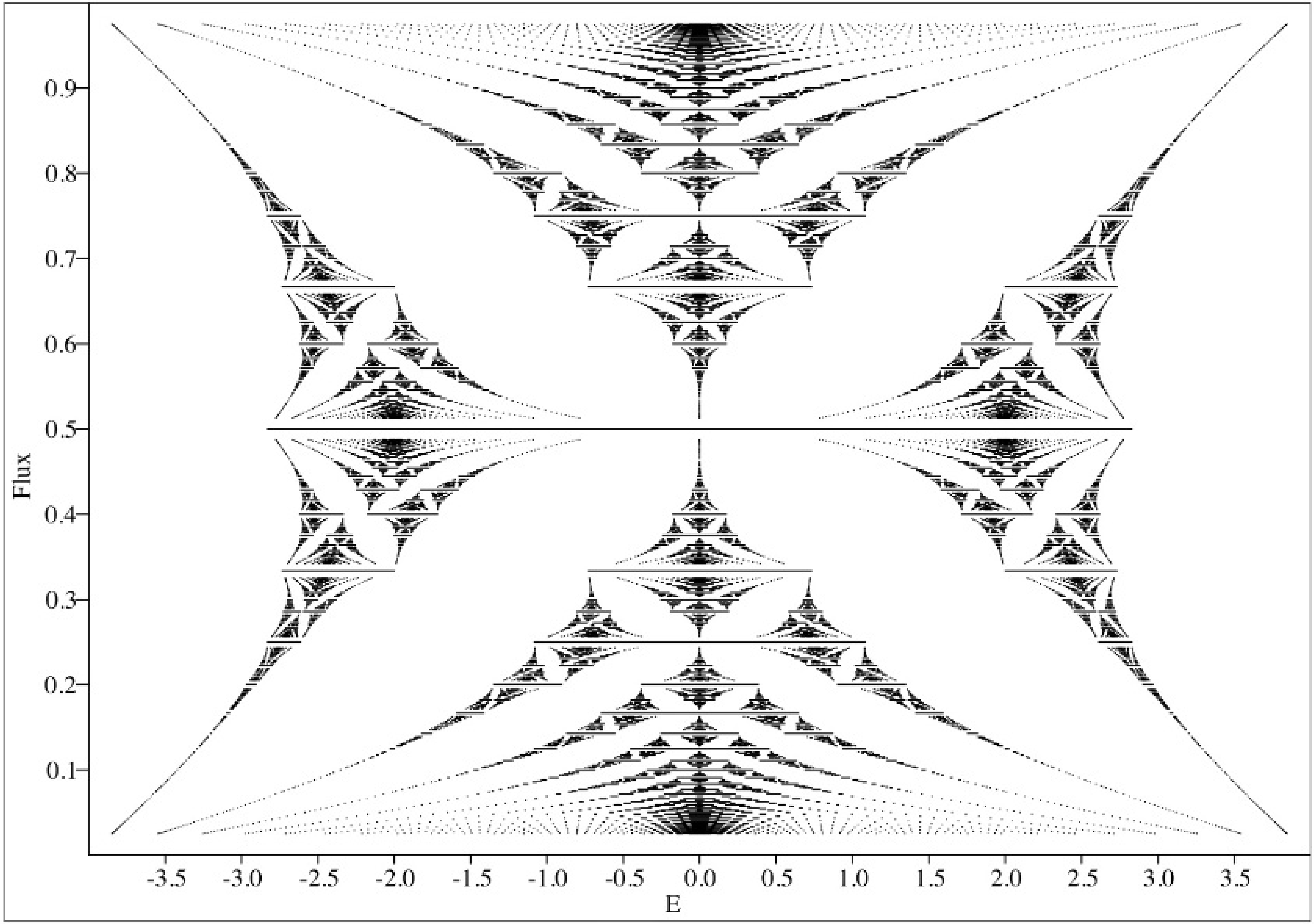
$$U_1\, U_2 = e^{2\pi\alpha}\, U_2\, U_1 \qquad \alpha = \frac{\phi}{\phi_0} \qquad \phi = Ba^2 \quad \phi_0 = \frac{e^2}{h}$$

- Kinetic Energy *(Hamiltonian)*

$$H = t\left(U_1 + U_2 + U_1^{-1} + U_2^{-1}\right)$$

- Landau gauge $\psi(m,n) = e^{2\imath\pi mk}\varphi(n)$.
  Hence $H\psi = E\psi$ means

$$\varphi(n+1) + \varphi(n-1) + 2\cos 2\pi(n\alpha - k)\varphi(n) = \frac{E}{t}\,\varphi(n)$$



Flux
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
-3.5
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
E

$2D$-Crystal Electrons in Magnetic Field

For $\alpha = p/q$, the following properties hold

- The spectrum has q nonoverlapping bands, *touching only at* $E = 0$

  (Bellissard-Simon '82,..., Avila-Jitomirskaya '09)

- The spectral gaps are bounded below by e^{-Cq} for some $C > 0$

  (Helffer-Sjöstrand '86-89, Choi-Elliot-Yui. 90)

$2D$-Crystal Electrons in Magnetic Field

For $\alpha \notin \mathbb{Q}$,

- The The spectrum is a Cantor set

  (Bellissard-Simon '82,..., Avila-Jitomirskaya '09)

- The spectrum has zero Lebesgue measure

  (Avron-van Mouche-Simon, '90, ..., Avila-Jitomirskaya '09)

- The gap edges are Lipshitz continuous as long as they do not close, otherwise they are Hölder with exponent $1/2$

  (Bellissard '94, Avron-van Mouche-Simon, '90, Haagerup et al.)

- The derivative of gap edges *w.r.t.* α is discontinuous at each rational

  (Wilkinson '84, Rammal '86, Bellissard-Rammal '90)

Rotation Algebra

- The C^*-algebra $\mathcal{A}_\alpha$ generated by two unitaries U_1, U_2 such that $U_1 U_2 = e^{2\imath\pi\alpha} U_2 U_1$ is called the *rotation algebra* *(Rieffel '81)*

- $\mathcal{A}_\alpha$ has a *trace* defined by

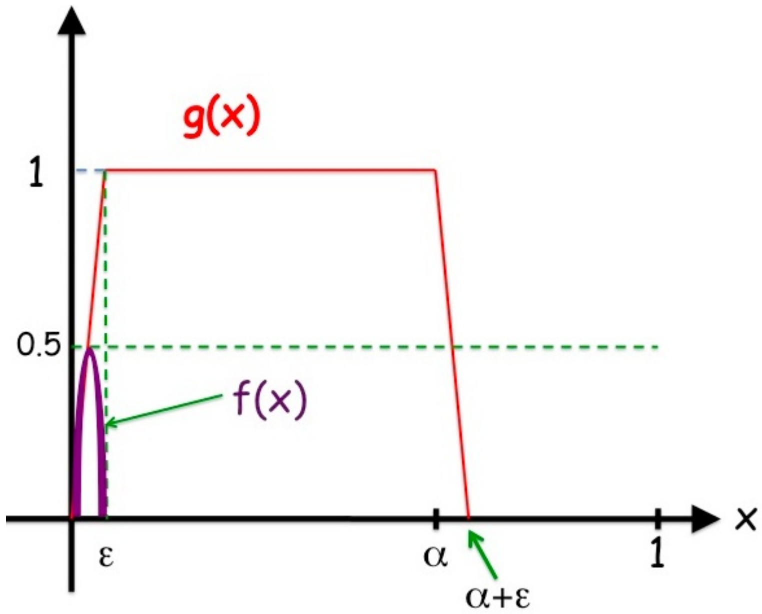
$$\mathcal{T}(U_1^m \, U_2^n) = \delta_{m,0} \, \delta_{n,0}$$

- $\mathcal{A}_\alpha$ admits two *$\ast$-derivations* ∂_1, ∂_2 defined by *(Connes '82)*

$$\partial_i U_j = 2\imath\pi \; \delta_{i,j} U_j$$

Rotation Algebra

- Rieffel's projection $P_{\mathrm{R}} = -f(U_2)U_1 + g(U_2) - U_1^{-1}f(U_2)$



$$\mathcal{T}(P_{\mathrm{R}}) = \alpha$$

$$\frac{1}{2\imath\pi}\mathcal{T}\left(P_{\mathrm{R}}\left[\partial_1 P_{\mathrm{R}}, \partial_2 P_{\mathrm{R}}\right]\right) = 1$$

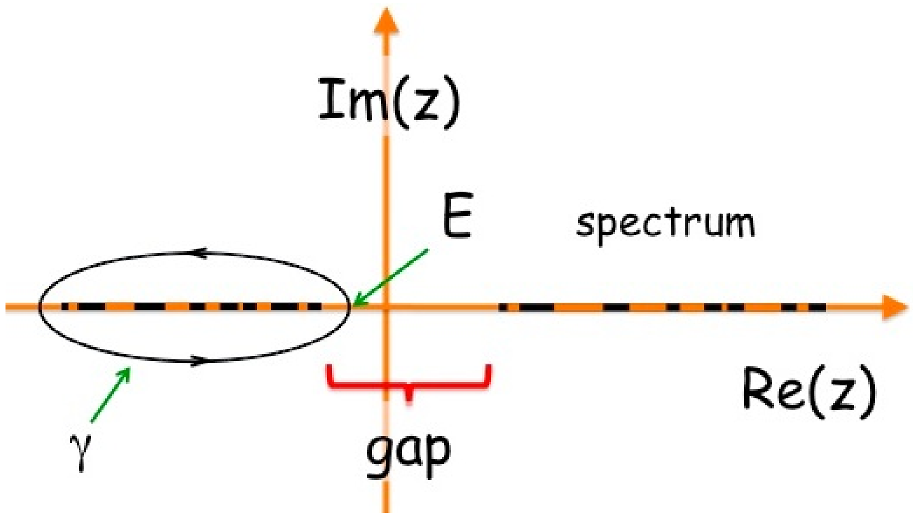
- If $P \in \mathcal{A}_\alpha$ is a *projection*, then

(Rieffel '81, Pimsner-Voiculescu '80, Connes '82)

$$\mathcal{T}(P) = n\alpha - [n\alpha] \qquad n = \mathrm{Ch}(P) = \frac{1}{2\imath\pi}\mathcal{T}\left(P\left[\partial_1 P, \partial_2 P\right]\right) \in \mathbb{Z}$$

Gap Labels

- If $H = U_1 + U_1^{-1} + U_2 + U_2^{-1}$, and if E belongs to a gap of the spectrum of H, set



$$P_E = \frac{1}{2\imath\pi} \oint_\gamma \frac{dz}{zI - H}$$

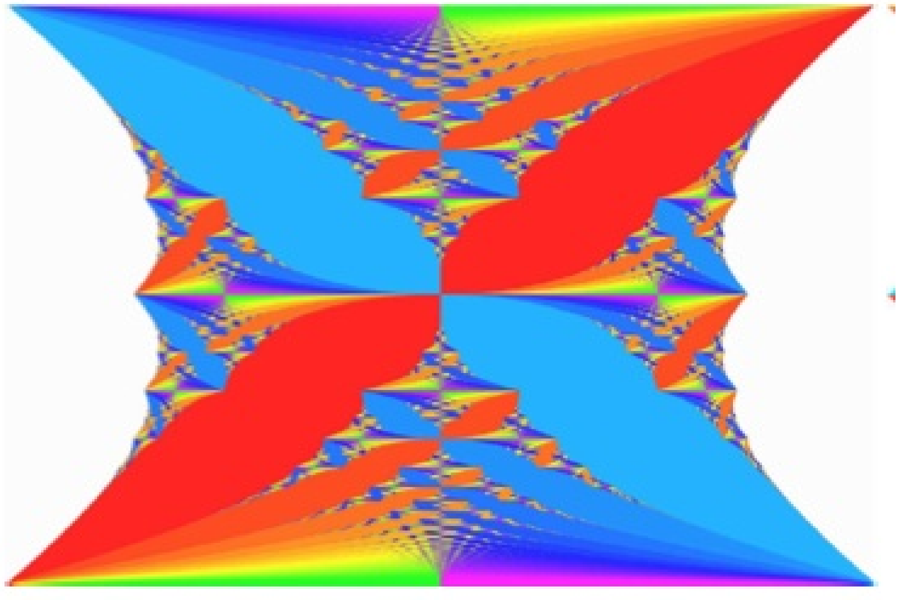
- Then $P_E \in \mathcal{A}_\alpha$!! Hence $\mathcal{T}(P_E) = n\alpha - [n\alpha]$ for some $n \in \mathbb{Z}$!!

Gap Labels

- The spectral projection of the Harper model between any two gaps can be labelled by an integer, using the previous results,

  (Claro-Wannier '78)

- This integer corresponds to the quantization of the Hall conductivity in such systems *(Thouless-Kohmoto-den Nijs-Nightingale '82)*



Each color corresponds to the integer gap label, for the eigenprojection between the l.h.s *and the gap.*

(Avron-Osadchy-Seiler '03)

II - Almost Periodic Potentials

J. Moser, Comment. Math. Helv., **56**, 198-224, (1981).

R. Johnson, J. Moser, Comm. Math. Phys., **84**, 403-438, (1982)

Schrödinger's Operator with Nowhere Dense Spectrum

- 1D-Schrödinger equation with a periodic potential

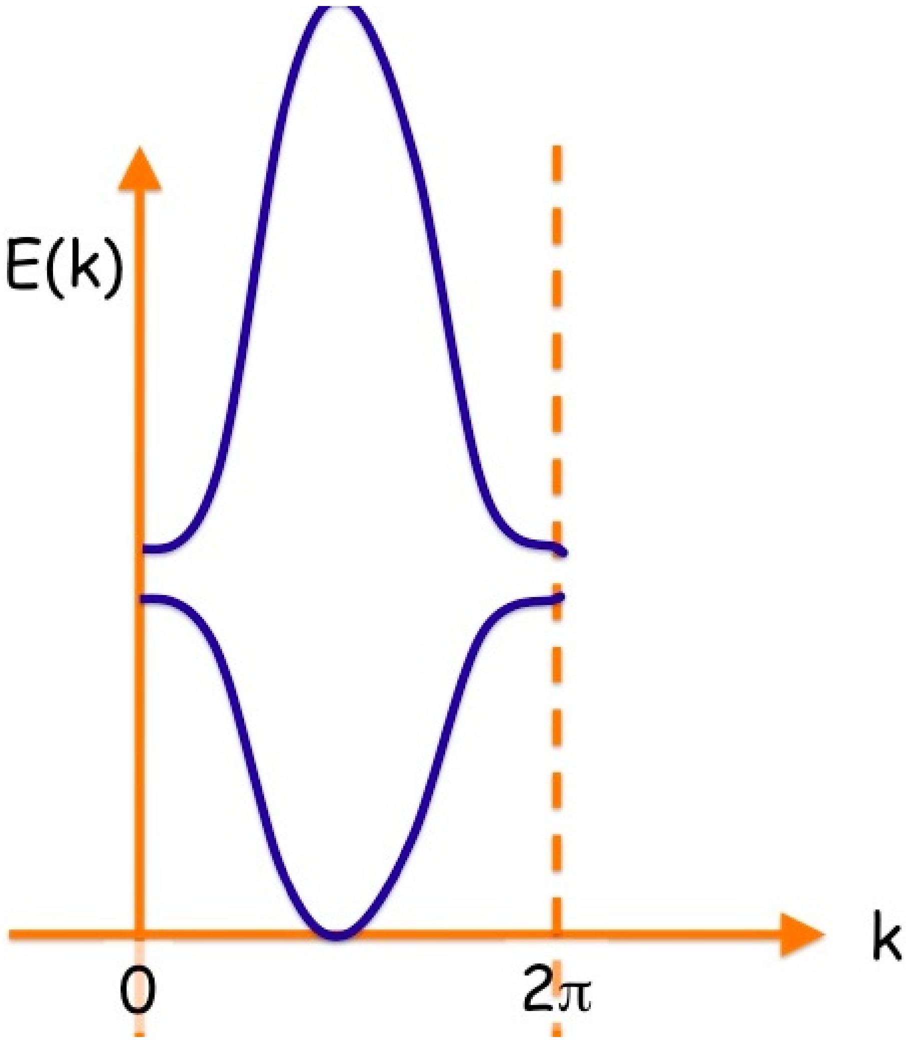
$$-\frac{d^2\varphi}{dx^2} + V(x)\varphi(x) = E\varphi(x) \qquad V(x+1) = V(x) \qquad V \;\text{ smooth}$$

- *Bloch-Floquet*: find solutions with $\varphi(x+1) = e^{\imath k}\varphi(x)$, giving

$$E = E_n(k) \qquad E_n(k+2\pi) = E_n(k) \quad n \in \mathbb{N}$$

Band spectrum with gaps at $k = m\pi \; m \in \mathbb{Z}$

Schrödinger's Operator with Nowhere Dense Spectrum



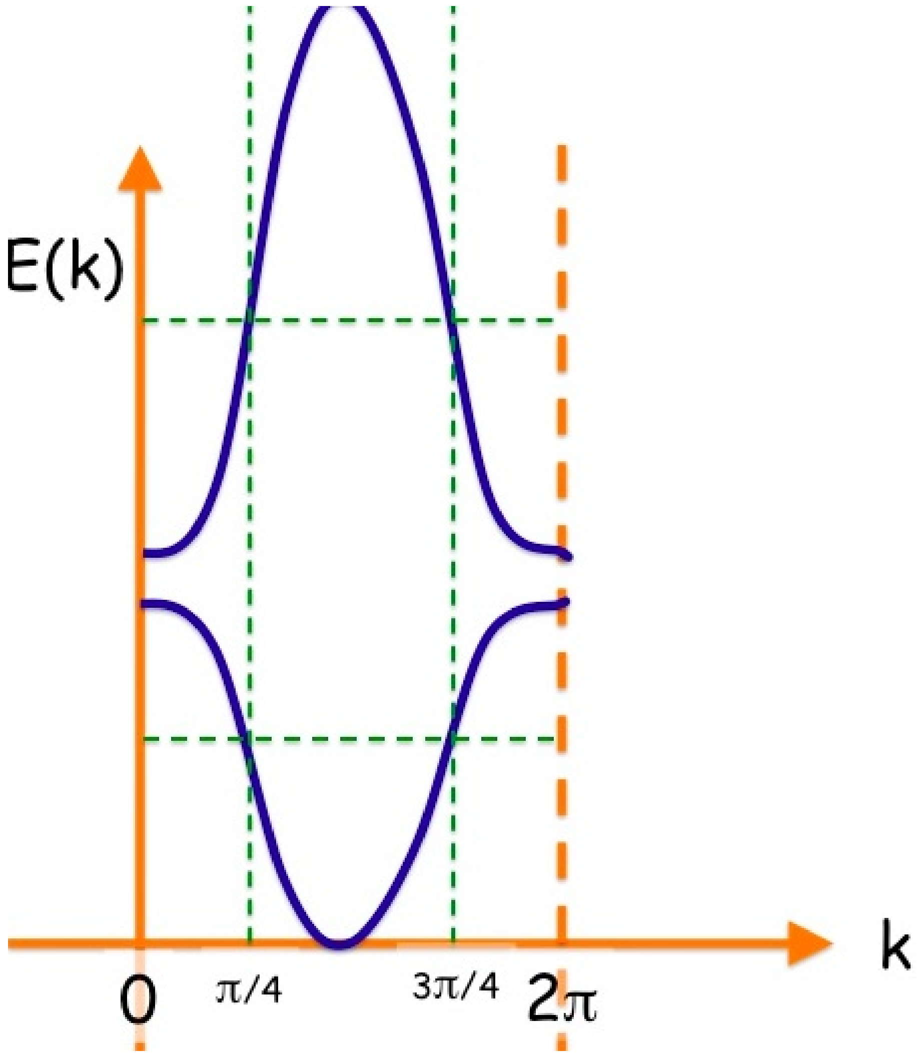
Schrödinger's Operator with Nowhere Dense Spectrum

- Add to $V(x)$ a contribution $V_1(x/2)$ with $V_1(x+1) = V_1(x)$ and $\|V_1\|_\infty < \|V\|_\infty$.

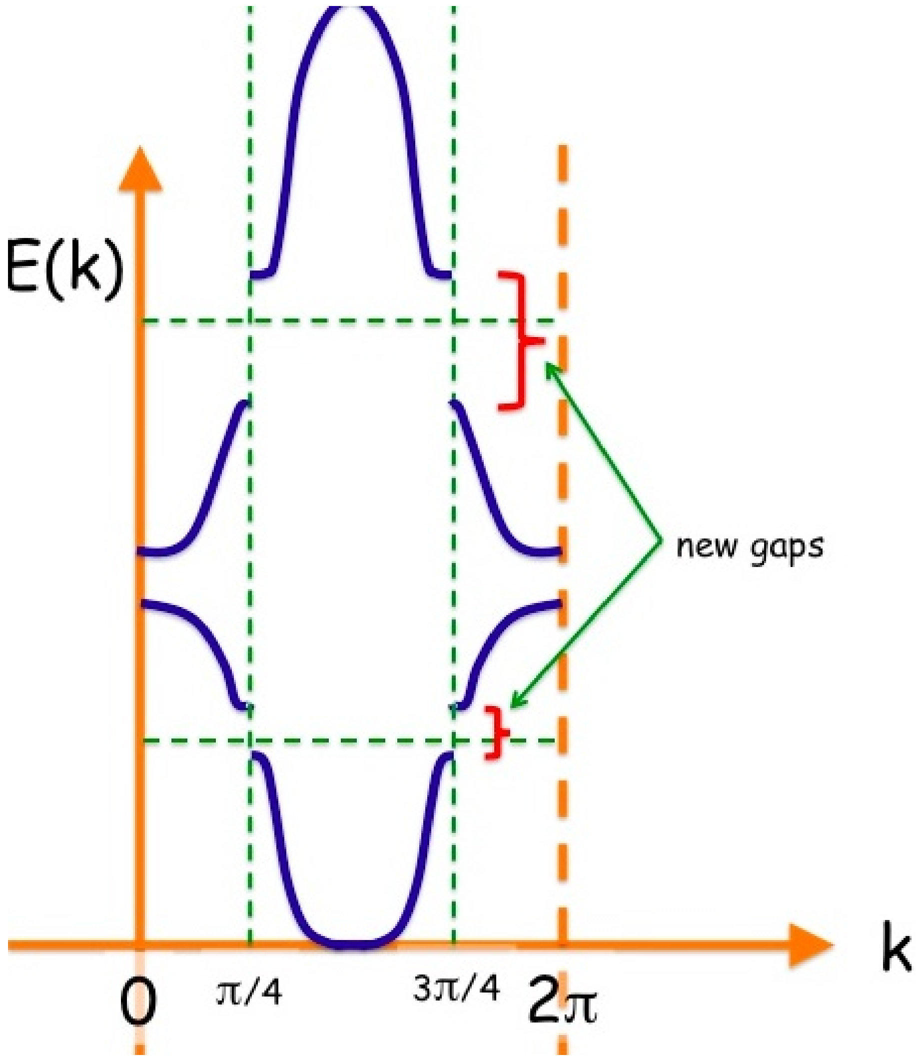
- It leads to the opening of *new gaps* at $k = m\pi/2$ instead

- The size of the new gaps can be controlled

Schrödinger's Operator with Nowhere Dense Spectrum



Schrödinger's Operator with Nowhere Dense Spectrum



Schrödinger's Operator with Nowhere Dense Spectrum

- Add to $V(x)$ a sequence $V_j(x/2^j)$ with $V_j(x+1) = V_j(x)$ and

$$\sum_{j=0}^{\infty} e^{rj} \|V_j\|_\infty < \infty$$

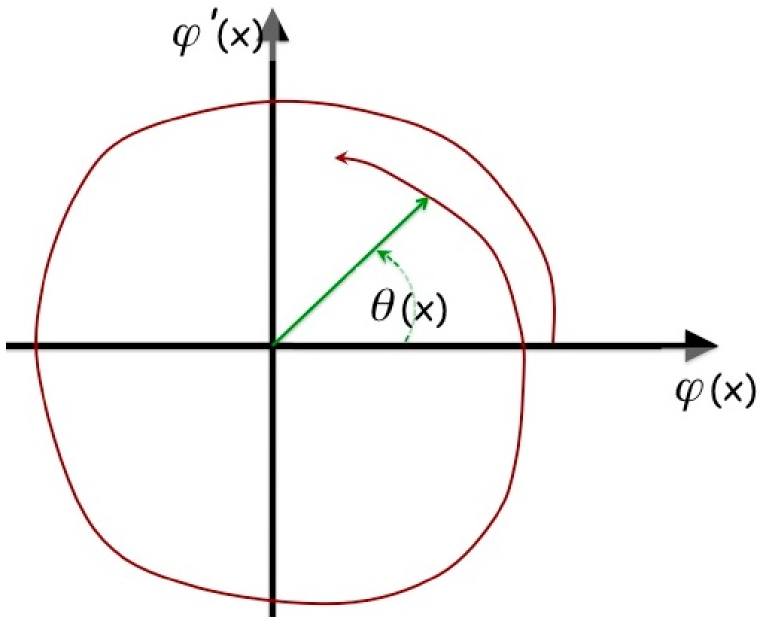
  for $r > 0$ large enough.

- It leads to the opening of an *infinite number* of very small gaps at $k = m\pi/2^j$. This leads to a *Cantor spectrum*

Rotation Number

- If $\varphi(x)$ is a solution of the Schrödinger equation at energy E, the rotation angle is defined by

$$e^{\imath\theta(x)} = \frac{\varphi(x) + \imath\varphi'(x)}{|\varphi(x) + \imath\varphi'(x)|}$$



The rotation number $\rho(E)$ is defined by

$$\rho(E) = \lim_{L\to\infty} \frac{1}{2\pi L} \int_{-L}^{+L} d\theta(x)$$

Rotation Number

Theorem *(Johnson-Moser '82)*

If V is a quasiperiodic potential and E belongs to a spectral gap, then $\rho(E)$ belongs to the $\mathbb{Z}$-module generated by the frequencies of V

III - The Hull of a Hamiltonian

J. BELLISSARD, Lecture Notes in Physics, **257**, 99-156, (1986).

J. BELLISSARD, in *From Number Theory to Physics*, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

Homogeneity

- Let $H = -\Delta + V$, be a Schrödinger operator on $\mathbb{R}^d$. Let $U(a)$ be the unitary operator on $L^2(\mathbb{R}^d)$ representing the translation by $a \in \mathbb{R}^d$.
- H is called *homogeneous* if the family $\Omega_0(z) = \{U(a)\,(H - zI)^{-1}\,U(a)^{-1}\,;\, a \in \mathbb{R}^d\}$ is *strongly precompact* for at least one $z \in \mathbb{C}$ such that $\Im m(z) \neq 0$.
- **Example:** *if $V \in L^\infty_{\mathbb{R}}(\mathbb{R}^d)$ then H is homogeneous*
- The strong closure of $\Omega_0(z)$ is denoted by Ω: it is a compact metrizable set, independent of z modulo homeomorphisms called the *Hull* of H.

Homogeneity

- The translation group $\mathbb{R}^d$ acts on Ω by homeomorphisms T and $(\Omega, \mathbb{R}^d)$ is a *topological dynamical system*.

- Let H be homogeneous. Then, each $\omega \in \Omega$ defines a selfadjoint operator H_ω on $L^2(\mathbb{R}^d)$ through taking the strong resolvent limit. Then

  – $\omega \in \Omega \to (H_\omega - zI)^{-1}$ is *strongly continuous* (for $z \notin \mathbb{R}$)

  – $U(a)H_\omega U(a)^{-1} = H_{\mathrm{T}^a\omega}$ *(covariance)*

- $H_\omega = -\Delta + V_\omega$ where, *if V is continuous*, $V_\omega(x) = v(\mathrm{T}^{-x}\omega)$ with $v \in C(\Omega)$.

C^*-algebra

The crossed product algebra $\mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d$ is constructed as follows. Let $\mathcal{A}_0 = C_c(\Omega \times \mathbb{R}^d)$

- **Product:** if $A, B \in \mathcal{A}_0$ then

$$AB(\omega, x) = \int_{\mathbb{R}^d} A(\omega, y)\, B(\mathrm{T}^{-y}\omega, x - y)\, d^d y$$

- **Adjoint:** if $A \in \mathcal{A}_0$ then

$$A^*(\omega, x) = \overline{A(\mathrm{T}^{-x}\omega, -x)}$$

- **Left Regular Representation:** if $A \in \mathcal{A}_0$ and if $\psi \in L^2(\mathbb{R}^d)$ then

$$\pi_\omega(A)\, \psi(x) = \int_{\mathbb{R}^d} A(\mathrm{T}^{-x}\omega, y - x)\, \psi(y)\, d^d y$$

C^*-algebra

- **C^*-norm:** if $A \in \mathcal{A}_0$ then

$$\|A\| = \sup_{\omega \in \Omega} \|\pi_\omega(A)\|$$

- **C^*-algebra:** $\mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d$ is the completion of $\mathcal{A}_0$ *w.r.t.* the norm $\|\cdot\|$

- **Theorem:** *(Bellissard '86, using Woronowicz, Baaj, Doplicher et al, Georgescu '02)*
  If H is homogeneous, it is affiliated to $\mathcal{A}$:
  namely, there is a $*$-homomorphism

$$f \in C_0(\mathbb{R}) \mapsto f(H) \in \mathcal{A}$$

such that $\pi_\omega(f(H)) = f(H_\omega)$ *for all* $\omega \in \Omega$.

Energy Spectrum

- **$\mathcal{A}$-spectrum:** $\mathrm{Sp}_{\mathcal{A}}(H)$ is the complement of the domain of holomorphy of $R_H(z) = (H - zI)^{-1} \in \mathcal{A}$.
- **Gaps:** Since H is selfadjoint, $\mathrm{Sp}_{\mathcal{A}}(H) \subset \mathbb{R}$ and is closed. A *gap* is a connected component of its complement $\mathbb{R} \setminus \mathrm{Sp}_{\mathcal{A}}(H)$
- **Proposition:** *(Bellissard '86)*
  - $\mathrm{Sp}_{\mathcal{A}}(H)$ *is the union over* ω *of* $\mathrm{Sp}(H_\omega)$
  - *If the orbit of* $\omega \in \Omega$ *is dense then* $\mathrm{Sp}_{\mathcal{A}}(H) = \mathrm{Sp}(H_\omega)$
  - *If there is a periodic orbit in* Ω *then* $\mathrm{Sp}_{\mathcal{A}}(H)$ *cannot be nowhere dense*

Calculus

- **Trace:** let $\mathbb{P}$ be an ergodic, $\mathbb{R}^d$-invariant probability measure on Ω. A *trace* on $\mathcal{A}$ is defined by

$$\mathcal{T}_{\mathbb{P}}(A) = \int_{\Omega} A(\omega, 0)\, d\mathbb{P}(\omega) \qquad\qquad A \in \mathcal{A}_0$$

- **Trace per Unit Volume:** if Λ are cubes centered at the origin and if χ_Λ is the characteristic function of Λ, then, Birkhoff's ergodic theorem leads to

$$\mathcal{T}_{\mathbb{P}}(A) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \mathrm{Tr}\left(\pi_\omega(A)\chi_\Lambda\right) \qquad \mathbb{P}\text{-}\textit{almost all } \omega$$

Calculus

- **Dual Action***(Connes, Takai-Takesaki)* $\eta_k(A)(\omega, x) = e^{\imath k \cdot x}\, A(\omega, x)$, with $k \in \mathbb{R}^d$, defines a norm pointwise continuous *d-parameter group* of $*$-automorphisms.

- **Differential Structure:** The dual action is generated by the following $*$-derivations

$$\partial_j A(\omega, x) = \imath x_j\, A(\omega, x) \qquad\qquad x = (x_1, \cdots, x_d) \in \mathbb{R}^d$$

- **Position Operators:** let $X = (X_1, \cdots, X_d)$ be the operators on $L^2(\mathbb{R}^d)$ defined by $X_j\,\psi(x) = x_j \psi(x)$. Then

$$\pi_\omega\left(\partial_j A\right) = \imath\,\left[X_j, \pi_\omega(A)\right]$$

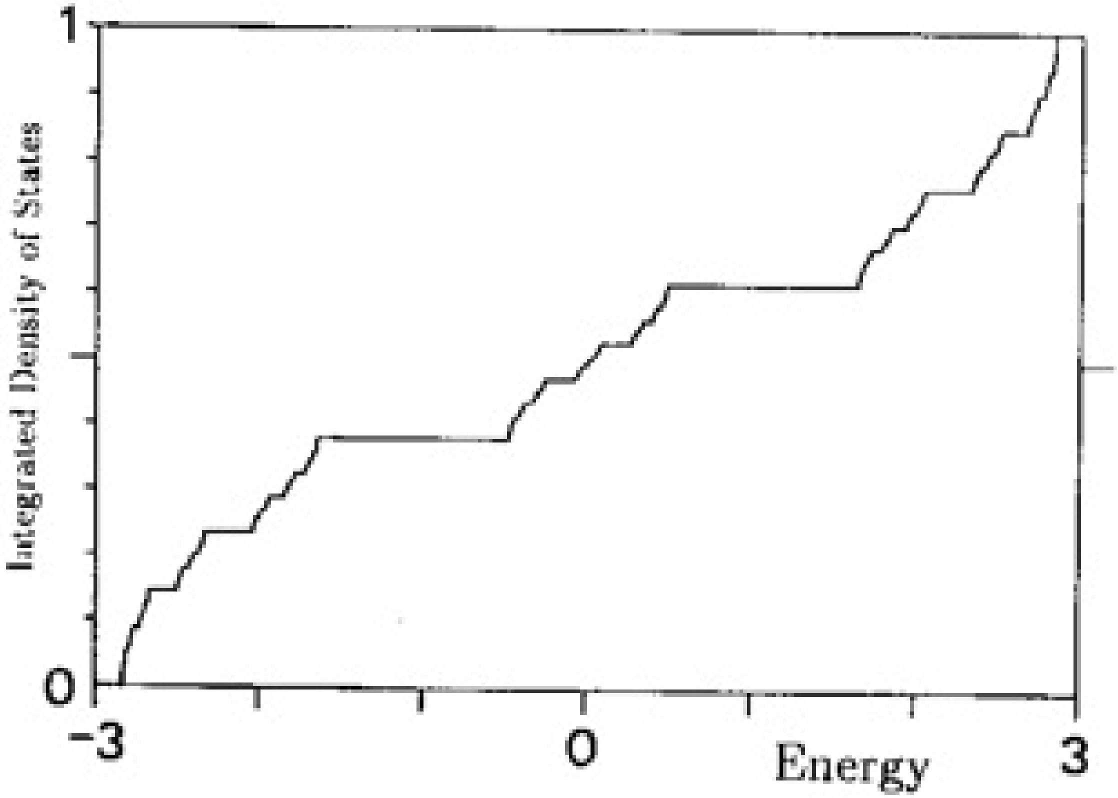
The Integrated Density of States

- **Integrated Density of States:** The restriction $H_{\omega,\Lambda}$ of H_ω to a bounded domain Λ is elliptic. Hence its spectrum is discrete. The *IDS* is defined by

$$\mathcal{N}_{\mathbb{P}}(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \# \left\{ \textit{eigenvalue of } H_{\omega,\Lambda} \leq E \right\} \qquad \mathbb{P}\textit{-almost all } \omega$$

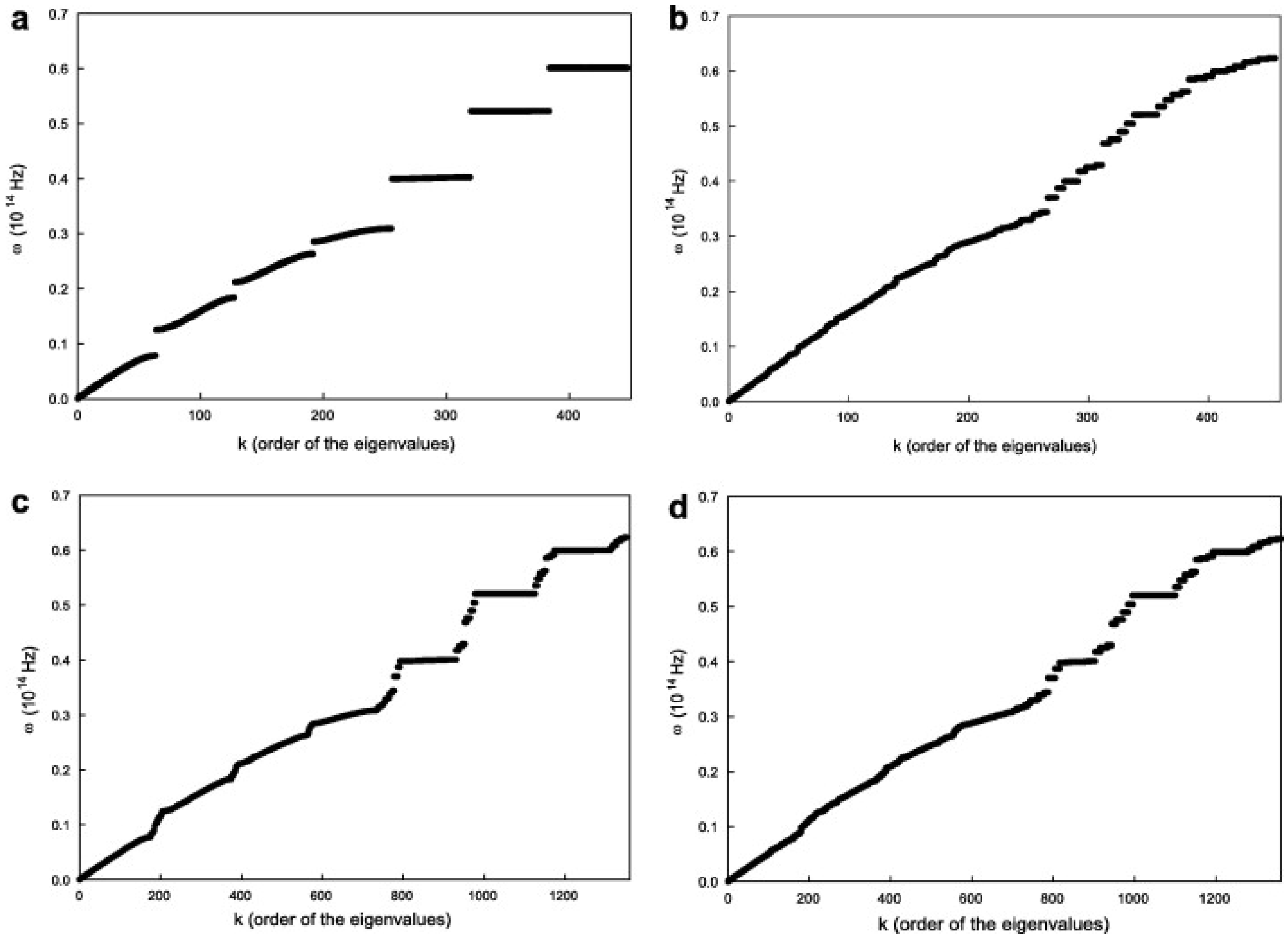
- **Properties:**
  - $\mathcal{N}_{\mathbb{P}}$ is nondecreasing *w.r.t.* E
  - $\mathcal{N}_{\mathbb{P}}(E) = 0$ for $E < \inf \mathrm{Sp} H$
  - $\mathcal{N}_{\mathbb{P}}(E) \sim E^{d/2}$ as $E \to +\infty$
  - $\mathcal{N}_{\mathbb{P}}$ is *constant on spectral gaps*.

The Integrated Density of States



An example of IDS

The Integrated Density of States



IDS for a

Rudin-Shapiro potential

(Montalbana et al. '07)

Gap Labels

- **Shubin's Formula:** *(Shubin '76, Bellissard '86)*

$$\mathcal{N}_{\mathbb{P}}(E) = \mathcal{T}_{\mathbb{P}}(P_E) \qquad\qquad P_E = \chi_{(-\infty,E]}(H)$$

- **Spectral Projections:**
  - If $E \in \mathrm{Sp}_{\mathcal{A}}(H)$ then P_E is a projector in $L^{\infty}(\mathcal{A}, \mathcal{T}_{\mathbb{P}})$
  - *If E is in a gap*, then $P_E \in \mathcal{A}$!
  - P_E does not change as E moves in the *same gap*

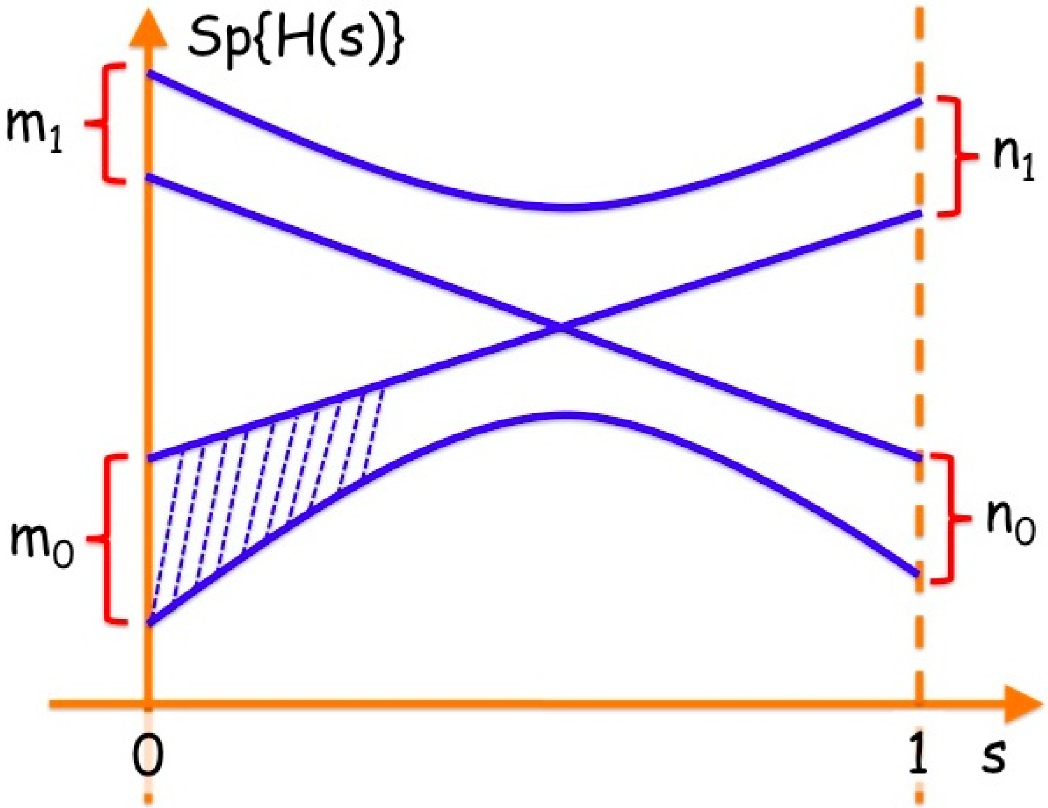
- If E is in a gap $\mathcal{T}_{\mathbb{P}}(P_E)$ depends only upon the equivalent class of P_E in $K_0(\mathcal{A})$

Gap Labels

- $\mathcal{T}_{\mathbb{P}}$ induces a group homomorphism $\mathcal{T}_* : K_0(\mathcal{A}) \to \mathbb{R}$. Since $K_0(\mathcal{A})$ is countable its image, the group of *gap labels*, is a countable subgroup of $\mathbb{R}$.
- **Gap Labeling Theorem:**
  - *the values of the IDS on gaps belongs to the image of* $K_0(\mathcal{A})$ *under the trace*
  - *if* $\mathfrak{g}_0 < \mathfrak{g}_1$ *are two gaps the difference* $\mathcal{N}_{\mathbb{P}}(\mathfrak{g}_1) - \mathcal{N}_{\mathbb{P}}(\mathfrak{g}_0)$ *is also a gap labels.*
  - *If* $H = H(s)$ *depends continuously (norm resolvent) on a parameter* s *then the gap label of a gap does not change as long as the gap does not close*

Gap Labels

- **Sum Rule:** let $H = H(s)$ depends continuously (norm resolvent) on a parameter $s \in [0, 1]$



$$m_0 + m_1 = n_0 + n_1$$

IV - Computing the Gap Labels

J. Bellissard, in *From Number Theory to Physics*, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

Quasiperiodic Potentials

- For $n > d$ integers, let κ be an $n \times d$ matrix of rank d with real coefficients.
  κ is *irrational* whenever $\mathrm{Im}(\kappa) \cap \mathbb{Z}^n = \{0\}$.
- For $\mathcal{V} \in C(\mathbb{T}^n)$ let $V(x) = \mathcal{V}(-\kappa x)$ for $x \in \mathbb{R}^d$.
  Then V is *quasiperiodic*. Conversely every quasiperiodic function on $\mathbb{R}^d$ can be written in this way.
- **Theorem:** *(Johnson-Moser '82, Bellissard '93 using Connes Index Theorem for foliations)*
  - *If $H = -\Delta + V$ then $\Omega = \mathbb{T}^n$*
  - $\mathbb{R}^d$-action: $\tau^{-a}\omega = \omega - \kappa a$ *is uniquely ergodic*
  - *Then $V_\omega(x) = \mathcal{V}(\omega - \kappa x)$ if $\omega \in \mathbb{T}^n$*
  - *The group of gap labels is the $\mathbb{Z}$-module spanned by* $\det(\beta)$*'s where β runs through the set of submatrices of maximal rank of κ*

Quasiperiodic Potentials

In $1D$, more can be said

- Let $\psi(\omega, x)$ be the *unique solution* of the Schrödinger equation

$$-\psi'' + V_\omega \psi = E\psi \qquad \lim_{x\to+\infty} \psi(x) = 0 \qquad \psi(0) = 1$$

- It defines a curve in the complex plane through

$$\Phi(\omega, x) = \psi(\omega, x) + \frac{\imath}{\sqrt{E}} \frac{d\psi(\omega, x)}{dx} \neq 0$$

- The covariance property shows that

$$F(\omega, x) = \Phi(\omega, x)^{-1} \frac{d\Phi(\omega, x)}{dx} \quad \Rightarrow \quad F(\omega, x) = F(\omega - \kappa x, 0)$$

Quasiperiodic Potentials

- **Sturm-Liouville:** *The number of eigenvalues smaller than or equal to E of the restriction to $[-L, L]$ of H_ω is given by*

$$\frac{1}{2\imath\pi} \int_{-L}^{+L} \Phi(\omega, x)^{-1} \frac{d\Phi(\omega, x)}{dx}\, dx \qquad \textbf{(rotation number)}$$

- Hence, using the *covariance* and *Birkhoff's ergodic theorem*, the IDS becomes

$$\mathcal{T}(P_E) = \mathcal{N}(E) = \frac{1}{2\imath\pi} \int_{\mathbb{T}^n} \Phi(\omega, 0)^{-1} \frac{d\Phi(\omega, 0)}{dx}\, d\omega$$

- **Comments:** (i) this gives a proof of the *Johnson-Moser* result, (ii) it is a special case of the *Connes index formula* for foliations.

Atomic Potentials with Finite Local Complexity

- Let $\mathcal{L} \subset \mathbb{R}^d$ be *uniformly discrete*:

$$\inf\{|x - y| \,;\, x \neq y\,,\, x, y \in \mathcal{L}\} > 0$$

- V is an *atomic potential* on $\mathcal{L}$ if

$$V(x) = \sum_{y \in \mathcal{L}} v(x - y) \qquad\qquad v \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$$

- A *patch* is a finite subset of $\mathcal{L}$, *modulo translations*. $\mathcal{L}$ has *finite local complexity* if, for any $R > 0$ the set of patches of diameter $R > 0$ is finite.

Atomic Potentials with Finite Local Complexity

- **Theorem:** *Let V be an atomic potential with finite local complexity and let $H = -\Delta + V$*
  - *The Hull of H is a compact space, foliated by the action of $\mathbb{R}^d$, with a completely discontinuous transversal* *(Kellendonk '97)*
  - *The set of gap labels is the $\mathbb{Z}$-module generated by the occurrence probabilities of patches* w.r.t. *the probability $\mathbb{P}$ on the Hull*

    (Bellissard '92, Kaminker-Putnam '01, Bellissard-Benedetti-Gambaudo '01-'06, Oyono-Oyono & Benameur '01-'07)

- **Examples:**
  - If $d = 1$ and $\mathcal{L}$ is the *Fibonacci chain*, the set of gap labels is $\mathbb{Z} + \sigma\mathbb{Z}$ where $\sigma = (\sqrt{5} - 1)/2$
  - Same results if $\mathcal{L}$ is the *Penrose* lattice or an *icosahedral quasicrystal* in $3D$.

It is time for coffee !