Persistence of Topological Invariants under Disorder

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Main Works

J. Bellissard,
"Gap labeling theorems for Schrödinger operators" ,
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J. Bellissard, H. Schulz-Baldes, A. van Elst,
J. Math. Phys., **35**, (1994), 5373-5471.

H. Schulz-Baldes, J. Bellissard,
Rev. Math. Phys., **10**, (1998), 1-46.

E. Prodan, B. Leung, J. Bellissard,
J. Phys. A, **46**, (2013), 485202 (51 pages).

E. Prodan,
Appl. Math. Res. Express, AMRX, **2**, (2013), 176-255.

E. Prodan, J. Bellissard,
Ann. Phys., **368**, (2016), 1-15.

Content

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I - Motivations

The Integer Quantum Hall Effect

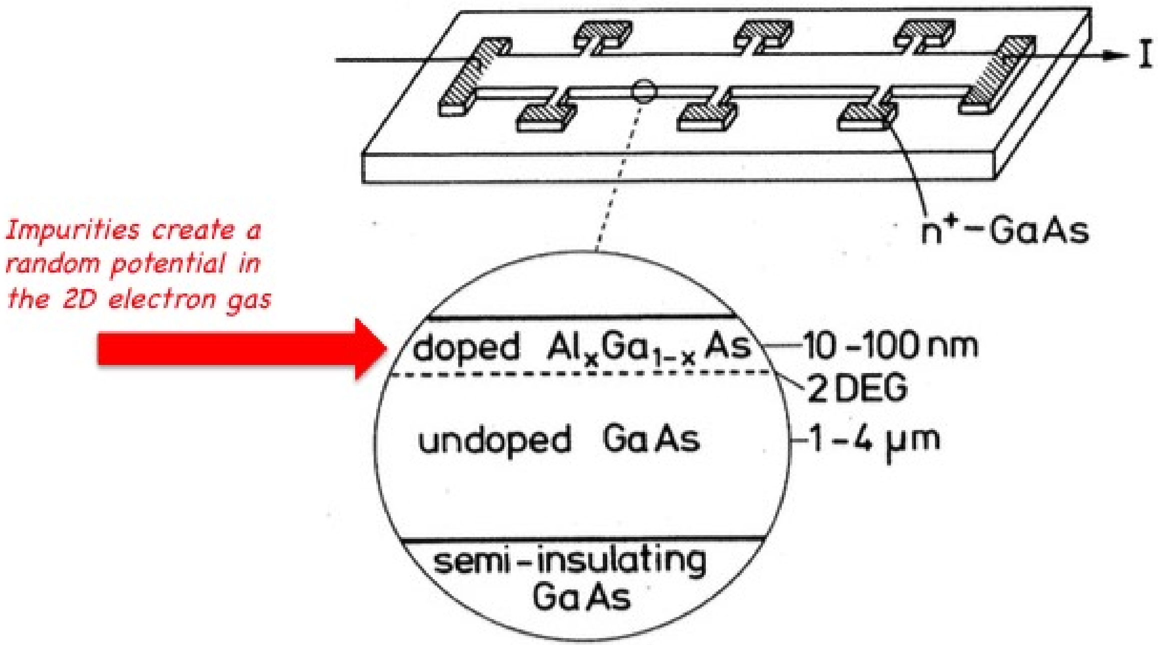


FIG. 3. Typical shape and cross section of a GaAs-$Al_xGa_{1-x}As$ heterostructure used for Hall-effect measurements.

The Integer Quantum Hall Effect

Quantized Hall Plateaux

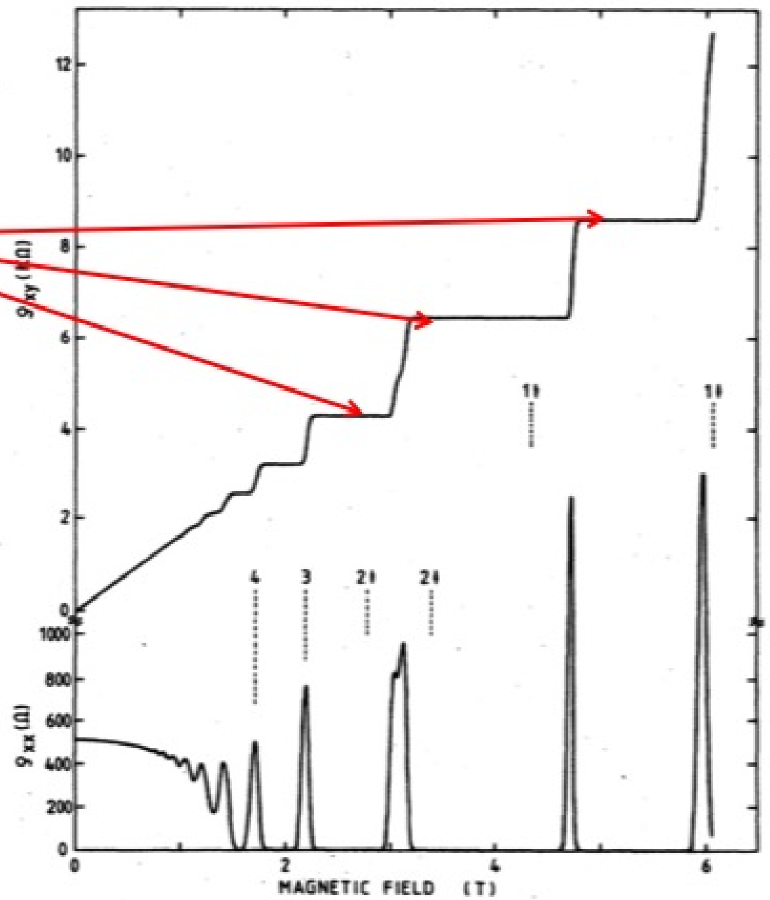


FIG. 14. Experimental curves for the Hall resistance $R_H = \rho_{xy}$ and the resistivity $\rho_{xx} \sim R_x$ of a heterostructure as a function of the magnetic field at a fixed carrier density corresponding to a gate voltage $V_g = 0$ V. The temperature is about 8 mK.

The quantized Hall effect*

Klaus von Klitzing

Rev. Mod. Phys., Vol. 58, No. 3, July 1986

The Integer Quantum Hall Effect

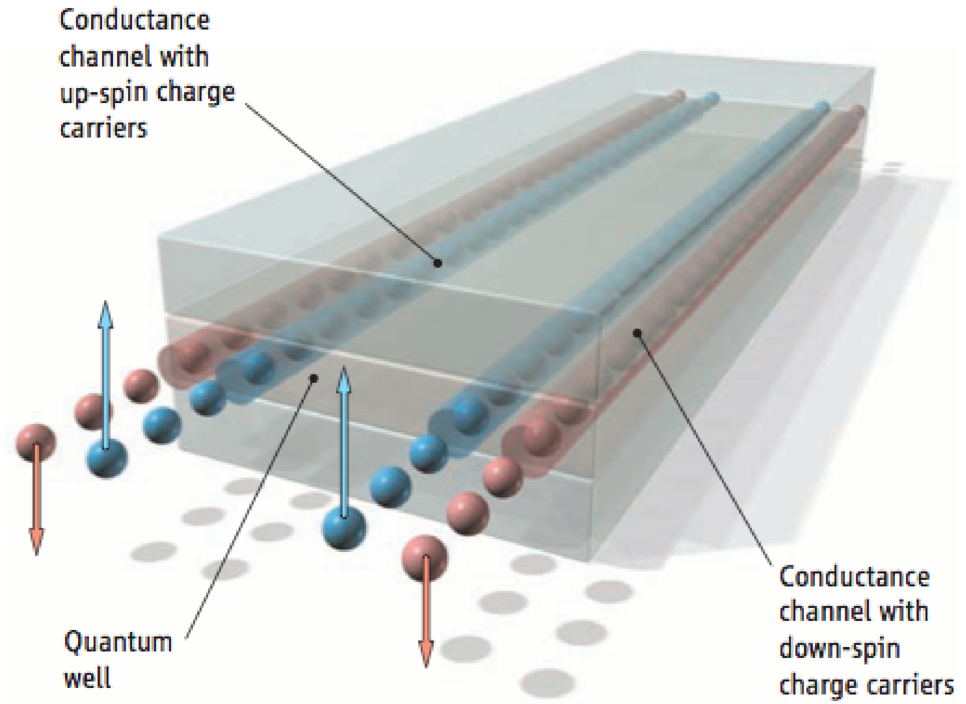
- A Coulomb potential is created by the impurities doping part of the Hall bar, in the 2D electron gas lying at the interface. Impurities are Poisson distributed (Thermal equilibrium). Hence the Coulomb *potential is actually spatially random* (but frozen in time).

- *Without the random potential, no Hall plateaux could be observed!*

- The main limitation to observing the QHE is provided the dissipation mechanisms, mainly the electron-phonon interaction. As a result, QHE can be observed at room temperature in graphene, since the electron-phonon coupling is almost vanishing.

Two Dimensional Topological Insulators

M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, S. C. Zhang
Science, **318**, (2007), 766-770



Schematic of the spin-polarized edge channels in a quantum spin Hall insulator.

2D-$HgTe$ semi-conductor with inverted band structure provides a way to create a spin polarized channel of electronic current, protected by topological invariant

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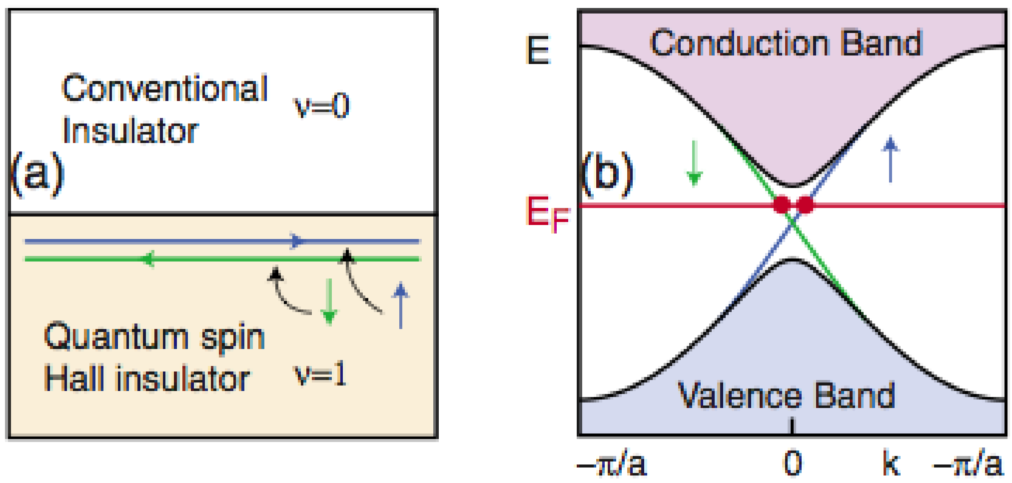
Science, **318**, (2007), 766-770



Edge states:
Right edge states interpolating between valence and conduction bands
Colors show the spin

Slope =velocity

Two Dimensional Topological Insulators

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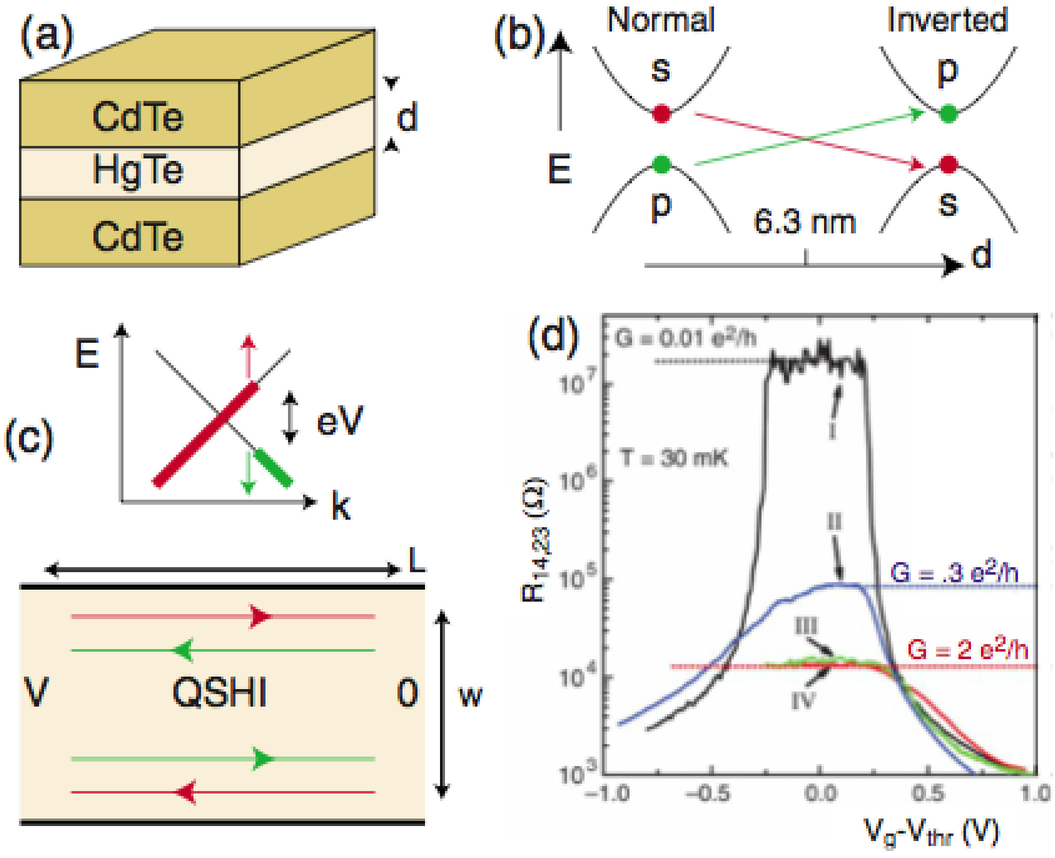
Science, **318**, (2007), 766-770



Edge states have quantized conductance

Three Dimensional Compounds

D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava & M. Z. Hasan,
Nature, **452**, (2008), 970-975

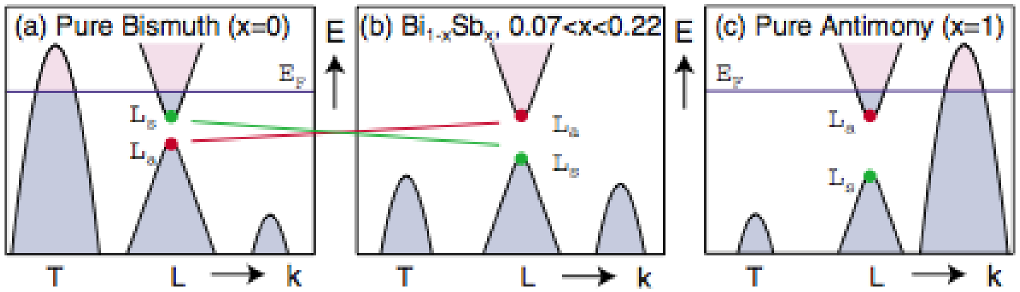


FIG. 8. (Color online) Schematic representation of the band structure of $Bi_{1-x}Sb_x$, which evolves from semimetallic behavior for $x<0.07$ to semiconducting behavior for $0.07<x<0.22$ and back to semimetallic behavior for $x>0.18$. The conduction and valence bands $L_{s,a}$ invert at $x\sim0.04$.

3D-$Bi_{0.9}Sb_{0.1}$
Inverted band structure

Topological Insulators

- Topological Insulators (TI) were predicted by Theoreticians for *perfectly periodic compounds*, using *Bloch Theory*.

- Most TI exhibit more *disorder* than anticipated. Still, the *topological effect survives*.

Main Question

If NO translation symmetry $\Rightarrow$ *NO Bloch Theory !*

What can be done then ?

II - Non-commutative Calculus

The Hull

- Spatial randomness and, more generally, *lack of periodicity*, is expressed through the concept of *Hull*. It represents the *set of all possible atomic distributions* for the material under study, in the *infinite volume limit*.
- The Hull can be shown to be a space Ω equipped with a topology, making it *compact*, namely that can be *approximated by finite sets*.
- The translation group $\mathbb{R}^d$ acting in the d-dimensional space in which the system is considered, acts on Ω, namely it moves the disorder around in space: $\omega \in \Omega \mapsto \mathrm{T}^x \omega$, under translating by x.
- The Hull is also equipped with a *probability distribution* $\mathbb{P}$, which is physically coming from *Equilibrium Thermodynamics* in the limit of zero temperature *(solid approximation)*. This probability is *ergodic-invariant* by the translation group *(homogeneity)*.

Random Operators

- The Hilbert space of quantum states describing the degrees of freedom under study, such as *electrons, spin, phonons*, will be denoted by $\mathcal{H}$.

- $\mathcal{H}$ is equipped with a *unitary representation* of the translation group. $U(x)$ will denote the translation by x acting on this Hilbert space.

- Quantum observables in the solid, like the *energy, charge, spin, currents, strain*, are described by families $A = (A_\omega)_{\omega\in\Omega}$ of operators indexed by the Hull.

Random Operators

- *Random Operators* are submitted to two conditions
  - **Homogeneity:**

$$U(x)A_\omega U(x)^{-1} = A_{T^x\omega}$$

  - **Strong Continuity:** for all state $|\phi\rangle \in \mathcal{H}$, the map

$$\omega \in \Omega \mapsto A_\omega|\phi\rangle \in \mathcal{H}$$

is *norm continuous*.

Random Operators

- Random operators can be *added, multiplied by scalars*, their *product* and their *adjoint* are well defined, hence they form a $*$-algebra.

- There is a *norm* permitting to define *limits, convergence of series*, so that they form what is called a C^*-algebra.

- Such series permit to define the *inverse*, the *Green's functions* and other physically relevant quantities.

- If the material is *perfectly periodic* in space, the Hull corresponds to the *Bravais zone*. By using *Fourier* transform *(Bloch Theory)*, the observable can be seen as *matrix-valued functions of the quasi-momentum*.

Random Operators

- **Trace per Unit Volume:** ergodicity means that space average is ensemble average *(Boltzmann, Gibbs, Birkhoff '33)*

$$\mathcal{T}_{\mathbb{P}}(A) = \lim_{\mathrm{vol}(\Lambda)\uparrow\infty} \frac{1}{\mathrm{vol}(\Lambda)} \mathrm{Tr}\,(A_\omega \restriction_\Lambda) = \int_\Omega \langle x|A_\omega|x\rangle \, d\mathbb{P}(\omega)\,.$$

- **Derivation:** if $Q = (Q_1, \cdots, Q_d)$ denotes the *position operators*

$$\partial_j(A)_\omega = \imath \left[Q_j, A_\omega\right] \qquad\qquad j = 1, \cdots, d\,.$$

Analogy with the Periodic Case

- **Trace per Unit Volume:**

$$\mathcal{T}_{\mathbb{P}}(A) \qquad \longleftrightarrow \qquad \int_{\mathbb{B}} \widehat{A}(k)dk$$

$\mathbb{B}$ denotes the *quasi-momentum space*, namely the *Brillouin zone*.

- **Derivation:**

$$\partial_j A \qquad \longleftrightarrow \qquad \frac{\partial \widehat{A}(k)}{\partial k_j}$$

III - Transport

H. Schulz-Baldes, J. Bellissard, *A kinetic theory for quantum transport in aperiodic media*, J. Stat. Phys., **91**, (1998), no. 5-6, 991-1026.

J. Bellissard, A. van Elst, H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, J. Math. Phys., **35**, (1994), no. 10, 5373-5451.

The Green-Kubo Formula

Using the dictionary between the periodic and
the non periodic case,
it becomes possible to write a formula
for the electric conductivity tensor.

The Green-Kubo Formula

- The *current operator* for electrons, for instance, is given by the product of the charge by the velocity. The *velocity* is the derivative of the position *w.r.t.* time namely

$$(J_j)_\omega = -e\,\frac{dQ_j}{dt} = \frac{(-e)}{\hbar}\;\imath\left[H_\omega, Q_j\right] = \frac{e}{\hbar}\,(\partial_j H)_\omega$$

  it is straightforward to check that, at least if H is bounded, the *current is a random operator*.

- The *electron density* is provided by the *Fermi-Dirac* distribution

$$n_{el} = \mathcal{T}_{\mathbb{P}}\left\{\frac{1}{1+e^{\beta(H-\mu)}}\right\} \qquad \beta = \frac{1}{k_B T}, \quad \mu = \text{chemical potential}$$

The Green-Kubo Formula

- Using the *relaxation time approximation* (RTA), for $j, m = 1, \cdots, d$, the conductivity tensor is given by

$$\sigma_{jm} = \frac{e^2}{h^2} \mathcal{T}_{\mathbb{P}} \left\{ \partial_j H \, \frac{1}{1/\tau_{rel} + \mathcal{L}_H - \imath \hat{\omega}} \left(\partial_m f_{\beta,\mu}(H) \right) \right\}$$

- Here τ_{rel} is the *average electron-phonon collision time*, $\hat{\omega} = 2\pi\nu$ where ν is the *frequency* of the external electric field and

$$\mathcal{L}_H(A) = \frac{dA}{dt} = \frac{\imath}{\hbar} [H, A] \,, \qquad\qquad f_{\beta,\mu}(H) = \frac{1}{1 + e^{\beta(H-\mu)}}$$

The Chern-Kubo Formula

- In the limit $T \downarrow 0$, at constant electron density n_{el}, the chemical potential becomes the Fermi level E_F.
- In the limit $T \downarrow 0$, this gives the *Fermi projection* defined as the projection onto the energies lower than the *Fermi energy* E_F

$$\lim_{T \downarrow 0} f_{\beta,\mu}(H) = P_F$$

- In the limit $T \downarrow 0$, for $d = 2$ this gives

$$\lim_{T \downarrow 0} \sigma_{11} = \lim_{T \downarrow 0} \sigma_{22} = 0\,, \qquad \lim_{T \downarrow 0} \sigma_{12} = \frac{e^2}{h} \mathcal{T}_{\mathbb{P}} \{P_F\, [\partial_1 P_F, \partial_2 P_F]\}\,.$$

The Chern-Kubo Formula

Theorem

(i) If $\mathcal{T}_{\mathbb{P}}\left(|\partial_j P_F|^2\right) < \infty$ *then the Chern number of the Fermi projection is an integer*

$$\mathrm{Chern}(P_F) = \mathcal{T}_{\mathbb{P}}\{P_F\,[\partial_1 P_F, \partial_2 P_F]\} \in \mathbb{Z}$$

(ii) The condition $\mathcal{T}_{\mathbb{P}}\left(|\partial_j P_F|^2\right) < \infty$ *is equivalent to having a finite localization length at the Fermi level*

(iii) If the dynamics is perturbed, for instance by a new addition to the potential energy such as disorder, as long as, either the Fermi level is in a gap or the localization length stays finite at E_F*, the Chern number does not change.*

IV - Further Applications

Numerical Computations

- Emil Prodan *(Prodan, '11)* has developed a numerical code, based on *periodic approximation* of the disordered Hamiltonian applied to the Green-Kubo formula, respecting the algebraic properties. It gives a *very stable* numerical method under perturbation by a random potential. The *error decays exponentially fast with the size of the period*.

- It gave the first numerical calculation of the *current-current correlation* function at equal energy. A well-known *scaling law*, observed at midpoint between Hall plateaux, was explained in terms of a singularity of this correlation function at some critical energy *(E. Prodan, J. Bellissard '16)*.

Numerical Computations

- The set of all possible topological invariants of a system characterized by its Hull, can be expressed through a countable, discrete Abelian group called the *K-group* of the corresponding C^*-algebra *(J.B. '81, '82)*. The Chern number occurring in the QHE belongs to this group.
- Alexei Kitaev *(Kitaev '09)* predicted the classification of topological insulators as a function of the space dimension and the symmetry type, in terms of the *real K-group* of the Fermi projection.
- A similar study for the topological invariants for *topological insulators*, using the previous noncommutative approach, was developed by several authors *(Prodan, Leung, J.B. '13, Shulz-Baldes, Kellendonk, Varghese-Thiang, Carey, Rennie, Bourne,...* since 2012. It shows that these invariants are *stable under perturbation by disorder*.

Thanks for listening !

Georgia Institute of Technology