TRANSVERSE GEOMETRY

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Main References

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Content

1. Spectral Triple on a Compact Metric Space
2. Spectral Triple for Crossed Products
3. Laplacians
4. Atomic Motion

Main Questions

- **Metrics Tiling Space:** for $\mathcal{L}$ a UD-set and $\xi, \xi' \in \mathrm{Trans}\mathcal{L}$ an example of a metric is the *combinatorial metric*

$$d(\xi, \xi') = \inf\{\epsilon > 0\,;\, d_H(\mathcal{L}_\xi \cap \overline{\mathrm{B}}(0;1/\epsilon), \mathcal{L}_{\xi'} \cap \overline{\mathrm{B}}(0;1/\epsilon)) < \epsilon\}$$

- Can one construct a *spectral triple* on the transversal then ?
  If yes, can one treat the transversal as a *noncommutative Riemannian manifold* ?
- Can one extend this spectral triple to the C*-algebra of the *groupoid of the transversal* ?
  It will be seen that there are obstructions
- Can one define a Laplacian on the transversal ?
  If yes can one use it to describe the dynamic of the solid ?

I - Palmer's Spectral Triples

I. Palmer, *Noncommutative Geometry of compact metric spaces*, PhD Thesis, May 3rd, 2010.

Open Covers

Let (X, d) be a *compact metric space* with an infinite number of points. Let $\mathcal{A} = C(X)$.

- An *open cover* $\mathcal{U}$ is a family of open sets of X with union equal to X. Then $\operatorname{diam}\mathcal{U} = sup\{\operatorname{diam}(U)\,;\, U \in \mathcal{U}\}$. All open covers used here will be at most *countable*
- A *resolving sequence* is a family $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \operatorname{diam}(\mathcal{U}_n) = 0$$

- A resolving sequence is *strict* if all $\mathcal{U}_n$'s are finite and if

$$\operatorname{diam}(\mathcal{U}_n) < \inf\{\operatorname{diam}(U)\,;\, U \in \mathcal{U}_{n-1}\} \quad \forall n$$

Choice Functions

Given a resolving sequence $\xi = (\mathcal{U}_n)_{n\in\mathbb{N}}$ a *choice function* is a map $\tau : \mathcal{U}(\xi) = \coprod_n \mathcal{U}_n \mapsto \mathrm{X} \times \mathrm{X}$ such that

- $\tau(U) = (x_U, y_U) \in U \times U$
- there is $C > 0$ such that

$$\mathrm{diam}(U) \geq d(x_U, y_U) \geq \frac{\mathrm{diam}(U)}{1 + C\,\mathrm{diam}(U)} \qquad \forall U \in \mathcal{U}(\xi)$$

The *set* of such choice functions is denoted by $\Upsilon(\xi)$.

A Family of Spectral Triples

- Given a *resolving sequence* ξ, let $\mathcal{H}_\xi = \ell^2(\mathcal{U}(\xi)) \otimes \mathbb{C}^2$
- For τ a *choice* let $D_{\xi,\tau}$ be the *Dirac operator* defined by

$$D_{\xi,\tau}\psi\,(U) = \frac{1}{d(x_U, y_U)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

- For $f \in C(X)$ let $\pi_{\xi,\tau}$ be the *representation* of $\mathcal{A} = C(X)$ given by

$$\pi_{\xi,\tau}(f)\psi\,(U) = \begin{bmatrix} f(x_U) & 0 \\ 0 & f(y_U) \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

Regularity

Theorem *Each* $\mathfrak{T}_{\xi,\tau} = (\mathcal{H}_\xi, \mathcal{A}, D_{\xi,\tau}, \pi_{\xi,\tau})$ *defines a spectral metric space such that* $\mathcal{A}_0 = C_{\text{Lip}}(X,d)$ *is the space of Lipshitz continuous functions on X. Such a triple is even when endowed with the grading operator*

$$G\psi(U) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

In addition, the family $\{\mathfrak{T}_{\xi,\tau}\,;\, \tau \in \Upsilon(\xi)\}$ *is regular in that*

$$d(x,y) = \sup\{|f(x - f(y)|\,;\; \sup_{\tau\in\Upsilon(\xi)} \|[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]\| \leq 1\}$$

Summability

Theorem *There is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral metric spaces if and only if the Hausdorff dimension of* X *is finite.*

If so, the spectral dimension s_D *satisfies* $s_D \geq \dim_H(X)$.

If $\dim_H(X) < \infty$ *there is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral triples with spectral dimension* $s_D = \dim_H(X)$.

Hausdorff Measure

Theorem *There exist a resolving sequence leading to a family $\mathfrak{T}_{\xi,\tau}$ of spectrally regular spectral metric spaces if and only if the Hausdorff measure of X is positive and finite.*

In such a case the Connes state coincides with the normalized Hausdorff measure on X.

Then the Connes state is given by the following limit *independently* of the choice τ

$$\frac{\int_X f(x)\mathcal{H}^{s_D}(dx)}{\mathcal{H}^{s_D}(X)} = \lim_{s\downarrow s_D} \frac{1}{\zeta_{\xi,\tau}(s)} \operatorname{Tr}\left(\frac{1}{|D_{\xi,\tau}|^s}\pi_{\xi,\tau}(f)\right) \qquad f \in C(X)$$

II - Crossed Product

J. Bellissard, M. Marcolli, K. Reihani, arXiv: 1008.4617, (2010).

Compact Spectral Metric Spaces

Let $X = (\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.
It will be called *compact* whenever $\mathcal{A}$ is unital.
It will be called a *spectral metric space* if

- The *D-commutant* $\mathcal{A}'_D = \{a \in \mathcal{A}\,;\, [D, a] = 0\}$ is reduced to $\mathbb{C}\mathbf{1}$
- The *Lipshitz ball* $B_{Lip} = \{a \in \mathcal{A}\,;\, \|[D, a]\| \leq 1\}$ has a precompact image in $\mathcal{A}/\mathcal{A}'_D$.

Theorem (Pavlovic, Rieffel) *A compact spectral triple is a spectral metric space if and only if the Connes distance on the state space*

$$d(\rho, \omega) = \sup\{|\rho(a) - \omega(a)|\,;\, a \in \mathcal{A},\ \|[D, a]\| \leq 1\}$$

is bounded and generates the weak-topology.*

Quasi-isometries

Let $\mathrm{Qiso}(X)$ be the set of quasi-isometries of the compact spectral metric space $X = (\mathcal{A}, \mathcal{H}, D)$. Then

Proposition *A $*$-automorphism of $\mathcal{A}$ is a quasi-isometry if and only if it generates a bi-Lipshitz transformation of the state space, namely there is $C > 0$ such that*

$$\frac{1}{C}\, d(\rho, \omega) \le d(\rho \circ \alpha, \omega \circ \alpha) \le C\, d(\rho, \omega)$$

for every pair of states (ρ, ω).

Equicontinuity

Let $X = (\mathcal{A}, \mathcal{H}, D)$ be a compact spectral metric space. A quasi-isometry $\alpha \in \mathrm{Qiso}(X)$ is called *equicontinuous* whenever

$$\sup_{n \in \mathbb{Z}} \| [D, \alpha^n(a)] \| < \infty \qquad \forall a \in C^1(X)$$

Theorem *A quasi-isometry is equicontinuous if and only if the group it generates in the set of $*$-automorphism of $\mathcal{A}$ has a compact closure*

$\alpha \in \mathrm{Qiso}(X)$ is called an *isometry* whenever

$$\| [D, a] \| = \| [D, \alpha(a)] \| \qquad \forall a \in C^1(X)$$

Proposition (Rieffel) $\alpha \in \mathrm{Qiso}(X)$ *is an isometry if and only if it defines an isometry in the state space for the Connes metric.*

Main Result

Let $\mathcal{A}$ be a unital separable C*-algebra.
Let α be a $*$-automorphism of $\mathcal{A}$.
Then, let u denotes the unitary implementing α in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$.

Theorem *There is a spectral metric space* $X = (\mathcal{A}, \mathcal{H}, D)$ *based on* $\mathcal{A}$ *for which* α *is equicontinuous if and only if there is a spectral metric space* $Y = (\mathcal{A} \rtimes_\alpha \mathbb{Z}, \mathcal{K}, \hat{D})$*, based on the crossed product, such that*

- *The dual action on* $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ *is equicontinuous*
- $u^{-1}[\hat{D}, u]$ *is bounded and commutes to the elements of* $\mathcal{A}$
- *The Connes metrics induced by* X *and by* Y *on the state space of* $\mathcal{A}$ *are equivalent*

Tiling Spaces

- For any reasonable metric on the tiling space, the action of the groupoid of the transversal of a tiling is *never isometric* !! This action makes the transversal a *Smale space* *(Putnam)*

- The previous result shows that it is not possible to construct a spectral triple in a canonical way on the C^*-algebra of the transversal.

Examples

Arnold's cat map: $\mathcal{A} = C(\mathbb{T}^2)$, $\mathcal{H} = L^2(\mathbb{T}^2) \otimes \mathbb{C}^2$, and

$$D = \begin{bmatrix} 0 & -\imath\partial_1 - \partial_2 \\ -\imath\partial_1 + \partial_2 & 0 \end{bmatrix}, \qquad \phi(x) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x,$$

with $\alpha(f) = f \circ \phi^{-1}$. Then α is a *quasi-isometry* that is *not equicontinuous*

$$\| [D, \alpha^n(f)] \| \overset{|n|\uparrow\infty}{\sim} \left(\frac{\sqrt{5}+1}{2} \right)^{|n|}$$

More generally any *strictly hyperbolic map* on a compact metric space (*Smale spaces*) will give rise to a similar situation.

The Metric Bundle

If M is a smooth manifold, the *metric bundle* is a principle bundle over M such that the fiber over each point is the cone of possible positive definite metrics on the tangent space.

Connes and Moscovici have shown that this bundle admits a tautological *Riemannian structure* that is *invariant by the diffeomorphisms* of M. In particular each diffeomorphim becomes an *isometry* for this structure.

The Metric Bundle

If ϕ is a diffeomorphism of M, it is sufficient to restrict this bundle to the *orbits* of ϕ with its Riemannian structure.

The *C^*-algebra of this orbit* is the tensor product $C(M) \otimes c_0(\mathbb{Z})$. The action of ϕ on the $\mathbb{Z}$-part is reduced to the *shift*.

Metric on $\mathbb{Z}$

Let $d_{\mathbb{Z}}$ be a *bounded translation invariant metric* on $\mathbb{Z}$. Then a spectral triple, based on $c_0(\mathbb{Z})$, can be defined as follows

- **Clifford matrices:** $\gamma_1, \cdots, \gamma_4$ acting on the Hilbert space $\mathcal{E}$
- **Hilbert Space:** $\ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathcal{E}$
- **Operators:**,

$$(\nabla f)_{n,r} = \frac{f_{n,r} - f_{n-r,r}}{d_{\mathbb{Z}}(n, n-r)}, \qquad (Xf)_{n,r} = \left(n\gamma_3 + \frac{1}{d_{\mathbb{Z}}(0,r)^2}\gamma_4 \right) f_{n,r}$$

- **Dirac operator:**

$$D_{\mathbb{Z}} = \frac{\gamma_1 + \imath\gamma_2}{2} \nabla + \frac{\gamma_1 - \imath\gamma_2}{2} \nabla^* + X.$$

Metric on $\mathbb{Z}$

Ref.: F. Latrémolière, *Taiwanese J. of Math.*, **11**, (2007), 447-469.

Proposition: $(c_0(\mathbb{Z}), \ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathcal{E}, D_{\mathbb{Z}})$ *is a spectral triple.*

Its Lipshitz Ball B_{Lip} *is bounded and, for any strictly positive sequence* $h \in c_0(\mathbb{Z})$, $hB_{Lip}h$ *is precompact.*

In particular, while the state space of $c_0(\mathbb{Z})$ *is not weak*-compact, the Connes distance is bounded and generates the weak*-topology.*

The Spectral Metric Bundle

Theorem: *Let* $X = (\mathcal{A}, \mathcal{H}, D)$ *be a compact spectral metric space. Let* $\alpha \in \mathrm{Qiso}(X)$ *be non-equicontinuous.*

Then there is a spectral triple $Y = (\mathcal{A} \otimes c_0(\mathbb{Z}), \mathcal{K}, D_{\mathcal{K}})$ *which is a non-compact spectral metric space for which the Connes metric is bounded on which* α *can be extended as an isometry.*

Moreover, $\mathcal{K}$ *support a representation of* $C = \mathcal{A} \otimes c_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ *which makes* $Z = (C, \mathcal{K}, D_{\mathcal{K}})$ *a spectral metric space on which the dual action is equicontinuous with respect to the weak-uniform topology.*

III - Laplacians

A. Beurling & J. Deny, *Dirichlet Spaces*, Proc. Nat. Acad. Sci., **45**, (1959), 208-215.

M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland (1980).

J. Pearson, J. Bellissard, J. Noncommut. Geom., **3**, (2009), 447-480.

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Choices and Tangent Space

The main remark is that, if $\tau(U) = (x, y)$ then

$$[D, \pi(f)]_\tau \; \psi \; (U) = \frac{f(x) - f(y)}{d(x, y)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(U)$$

The commutator with the Dirac operator is a coarse graining version of a *directional derivative*. In particular

- $\tau(U)$ can be interpreted as a coarse grained version of a *normalized tangent vector* at U.
- the set $\Upsilon(\xi)$ can be seen as the set of *sections of the tangent sphere bundle*.
- $[D, \pi(f)]_\tau$ could be written as $\nabla_\tau f$

Choice Averaging

To mimic the previous formula, a *probability* over the set $\Upsilon(\xi)$ is required.

For each open set $U \in \mathcal{U}(\xi)$, the set of choices is given by the set of pairs $(x, y) \in U \times U$ such that $d(x, y) > \mathrm{diam}(U)\,(1 + C\,\mathrm{diam}(U))^{-1}$. This is an *open set*.

Thus the probability measure ν_U defined as the *normalized measure* obtained from *restricting* $\mathcal{H}^{s_D} \otimes \mathcal{H}^{s_D}$ to this set is the right one.

This leads to the probability

$$\nu = \bigotimes_{U \in \mathcal{U}(\xi)} \nu_U$$

The Quadratic Form

This leads to the quadratic form (omitting the indices ξ, τ)

$$Q_\alpha(f,g) = \lim_{s \to s_D} \int_{\Upsilon(\xi)} d\nu(\tau) \, \frac{1}{\zeta(s)} \, \mathrm{Tr} \left(\frac{1}{|D|^s} [D^\alpha, \pi(f)]^* \, [D^\alpha, \pi(g)] \right)$$

Theorem *If X is a Riemannian manifold equipped with the geodesic distance the quadratic form $Q_{\alpha=1}$ coincides with the Laplace-Beltrami one.*

Open Problem *Under some regularity condition on (X, d), like the doubling ball property, this quadratic form is closable and Markovian.*

Open Problems

The quadratic form Q_α is *presumably* closable and Markovian in the following cases

- when (X, d) is the *Sierpinsky gasket* with $\alpha = 5/3$ or the *Sierpinsky carpet* with some α embedded in $\mathbb{R}^D$ (in particular $D = 2$) endowed with the Euclidean metric *(Barlow-Brass, Kusuoka, Sabot)*
- when (X, d) admits a path metric equivalent to d *(Cheeger)*
- when (X, d) is a *Koch curve* embedded in $\mathbb{R}^D$
- when (X, d) is a Brownian path or a Brownian surface embedded in $\mathbb{R}^D$ *(Fukushima)*

Cantor sets

If (X, d) is an ultrametric Cantor set, the characteristic functions of clopen sets are continuous. For such a function $[D, \pi(f)]$ is a finite rank operator. This gives

Theorem *If (X, d) is an ultrametric Cantor set, the previous quadratic form vanishes identically.*

To replace the previous form, define, *for any real $s \in \mathbb{R}$*, the form

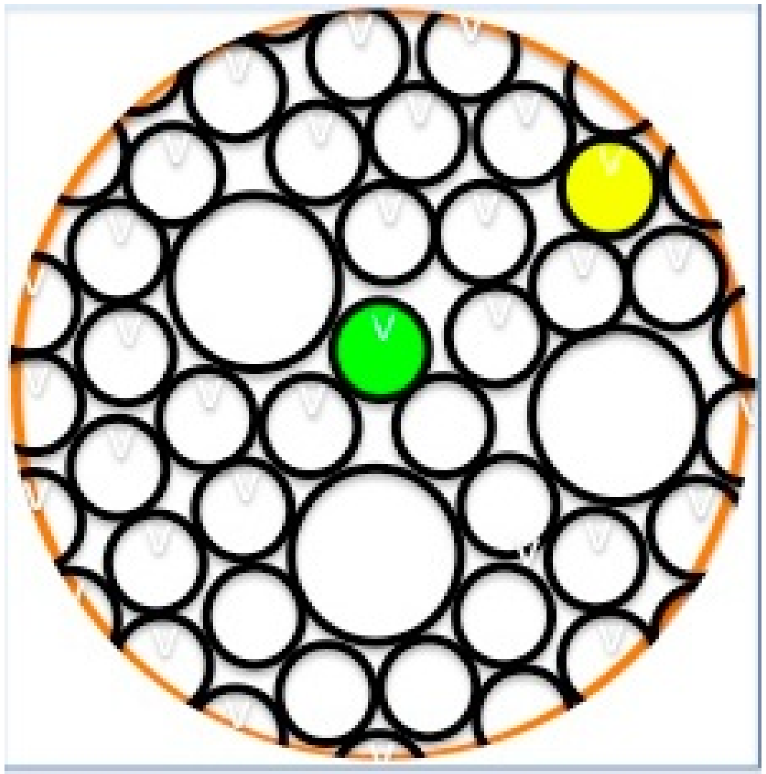
$$Q_s(f, g) = \int_{\Upsilon(\xi)} d\nu(\tau) \operatorname{Tr}\left(\frac{1}{|D|^s}[D, \pi(f)]^* \, [D, \pi(g)]\right)$$

Theorem *If (X, d) is an ultrametric Cantor set, the quadratic forms Q_s are closable in $L^2(X, \mathcal{H}^{s_D})$ and Markovian. The corresponding Laplacians have pure point spectrum. They are bounded if and only if $s > s_D + 2$ and have compact resolvent otherwise. The eigenspaces are common to all s's and can be explicitly computed.*

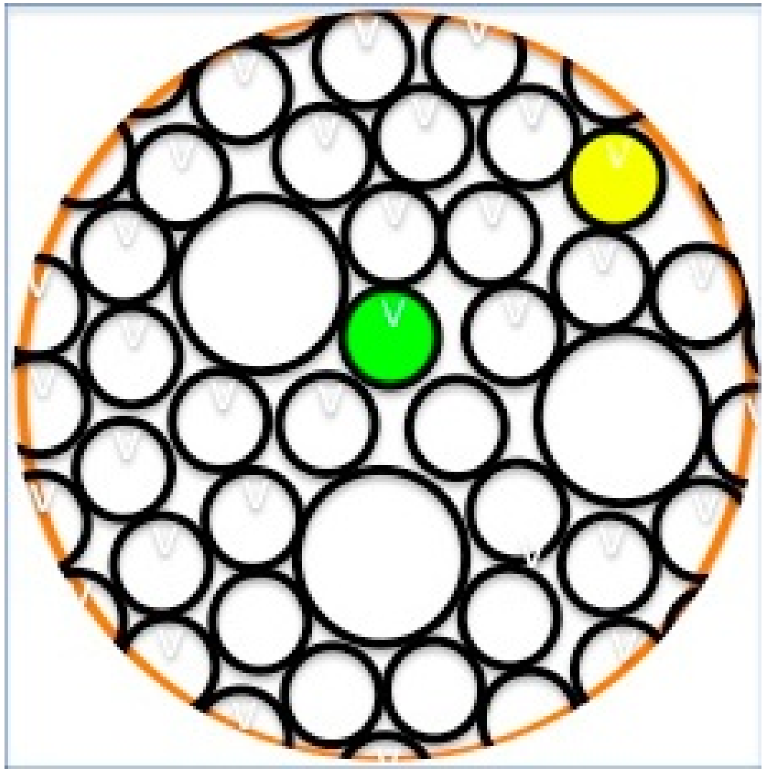
IV - Atomic Motion

A Model and some Speculations

Atoms Motion in Patches



Atoms Motion in Patches



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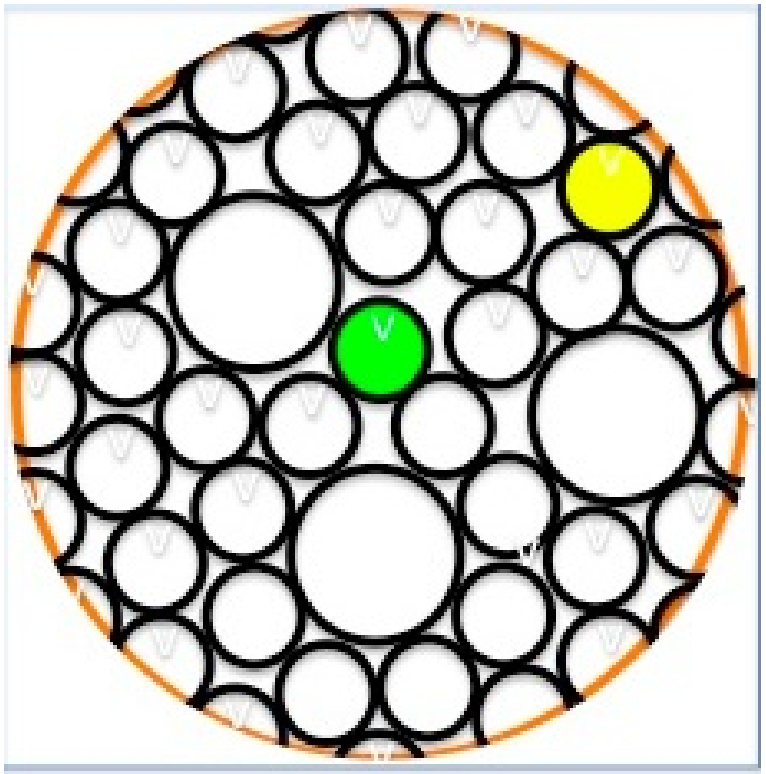

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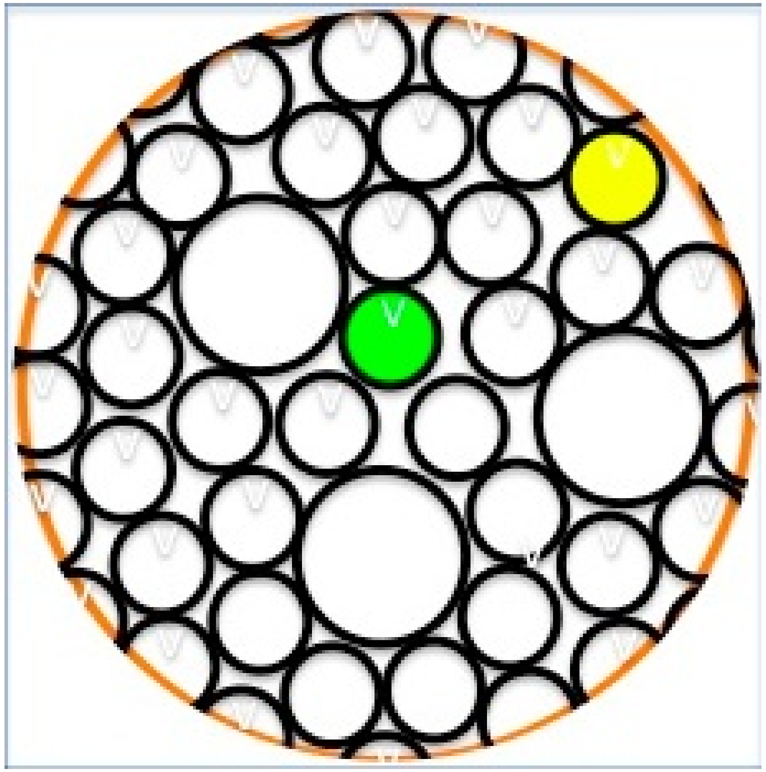

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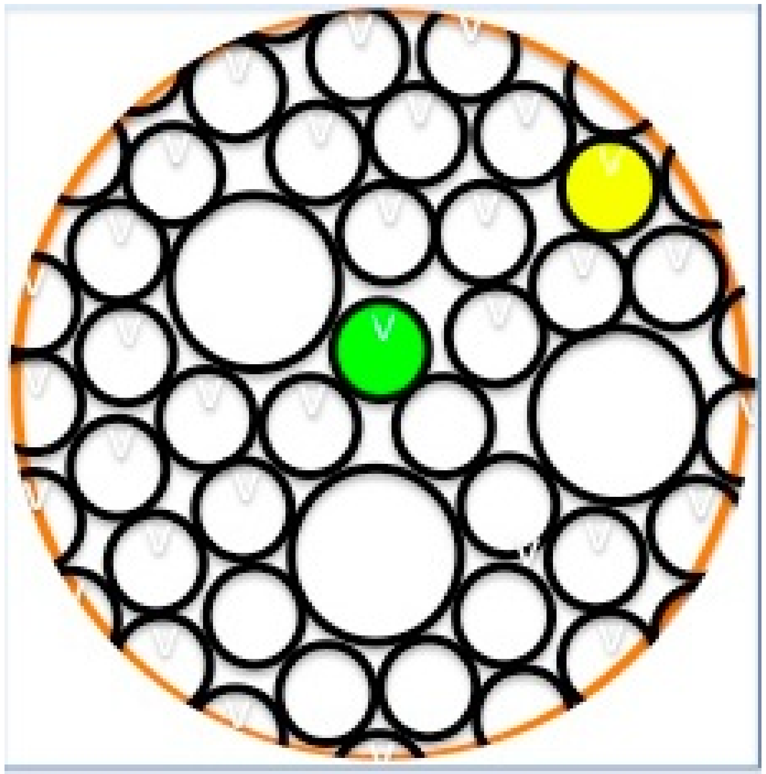

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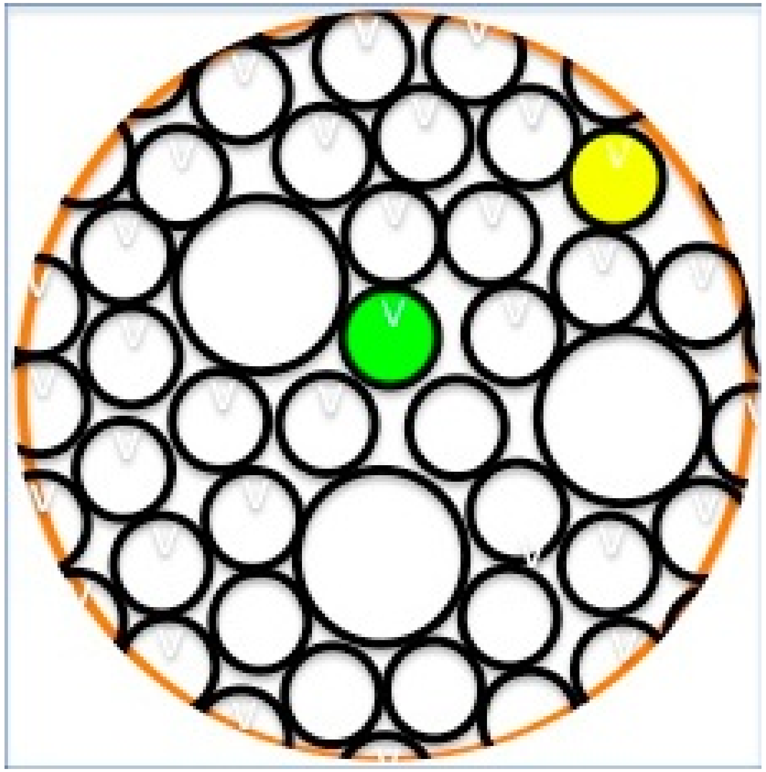

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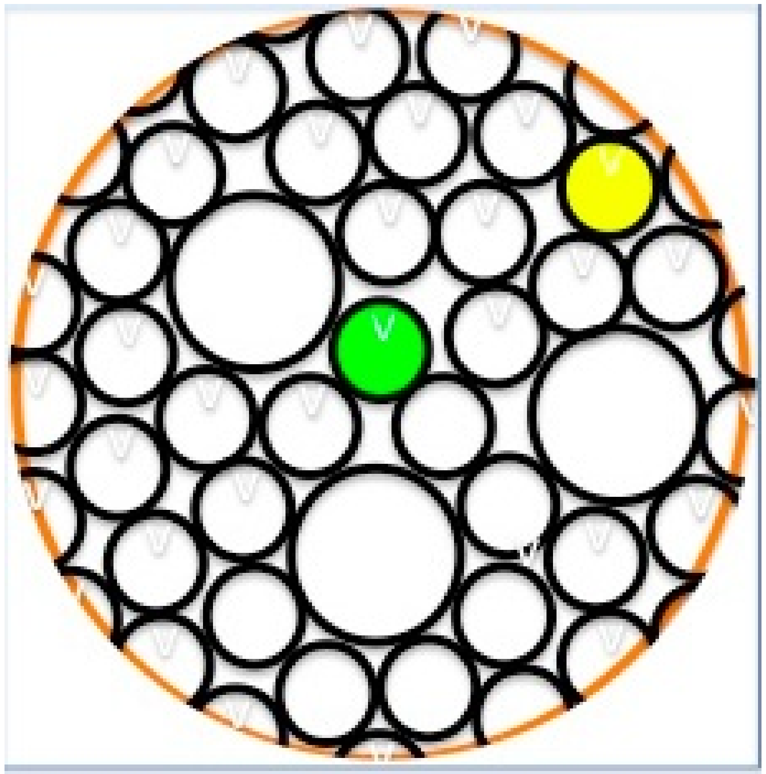

Random Jumps

- Atomic motion is due to *thermal noise* or *tunneling*. Both mechanisms are *random*.
- Local motion can be represented by a *jumps* from local patches to local patches.
- The *jump probability rate* $P(p \to q)$ between two patches p and q is proportional to
  - The *Gibbs* factor $\exp\{-\beta(F(q) - F(p))\}$, where $\beta = 1/k_{\mathrm{B}}T$ and F is the *free energy* of the patch.
  - The inverse of the *jump time* $\tau_{p \to q}$ depending upon the height of the *energy barrier* between p and q

Random Jumps

The *free energy* of a patch p is given by

$$F(p) = U(p) - \sum_{a \in \mathcal{A}} \mu_a N_a + \mathrm{tr}\{\Pi\, \Sigma(p)\}$$

where

- U is the *mechanical* energy,
- N_a is the *number* of atoms of species a,
- μ_a is the corresponding *chemical potential*,
- Π is the *stress* tensor
- $\Sigma(p)$ is the *deformation* of p

Random Jumps

- The *tunneling time* is proportional to

$$\tau_{\text{tunn}} \sim \exp\left\{\frac{S_{p\to q}}{\hbar}\right\}$$

where $S_{p\to q}$ is the tunneling action between the two patches

- The *noise transition time* is provided by Kramers law given, in the limit of high viscosity, by

$$\tau_{\text{noise}} \sim \exp\left\{\frac{W_{p\to q}}{\epsilon^2}\right\}$$

where $W_{p\to q}$ is the height of the energy barrier measured from p and ϵ is the noise intensity. For thermal noise $\epsilon^2 = k_{\text{B}}T$.

A Markov Process

- The probability $\mathbb{P}_t(q)$ at time t of the patch $q \in \Omega_r$ is given by

$$\frac{d\mathbb{P}_t(q)}{dt} = \sum_{p \in \Omega_r} P(p \to q)\ \mathbb{P}_t(p)$$

- This equation is *not rigorous*, unless Ω_r is finite (namely for tilings with *finite local complexity*, such as quasicrystals)

- Solving this equation gives a calculation of *transport coefficients*, such as the *elasticity tensor* or the response of the solid to stress.

Questions and Problems

- *Coarse graining* of each Ω_r allows to replace it by a *finite* number of local patches.

  How is the Markov process behaving in the continuum limit ?

- Letting the radius $r \to \infty$ is equivalent to look at *small scale* in the tiling space Ξ.

  How can one control this infinite volume limit ?

It is time for coffee !