Laplace Operator

on

Compact Metric Spaces

Jean BELLISSARD

Westfälische Wilhelms-Universität, Münster
Department of Mathematics

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics
e-mail: jeanbel@math.gatech.edu

Sponsoring

CRC 701, Bielefeld, Germany

SFB 878, Münster, Germany

Groups, Geometry & Actions

Contributors

J. Pearson, PhD Thesis, Georgia Institute of Technology, April 2008

I. Palmer, PhD Thesis, Georgia Institute of Technology, May 2010

J. Dever, PhD Thesis, Georgia Institute of Technology, May 2018

Main References

J. Pearson, J. Bellissard
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
J. Noncommut. Geo., **3**, (2009), 447-480.

I. C. Palmer
Noncommutative Geometry and Compact Metric Spaces,
PhD Thesis, Georgia Institute of Technology, May 10, 2010.

A. Julien, J. Kellendonk, J. Savinien
On the Noncommutative Geometry of Tilings, in *Mathematics of Aperiodic Order*,
J. Kellendonk, D. Lenz, J. Savinien Eds., Chap. 8, pp. 259-306, Birkhäuser, 2015
(in particular Theorem 2.6, p. 270).

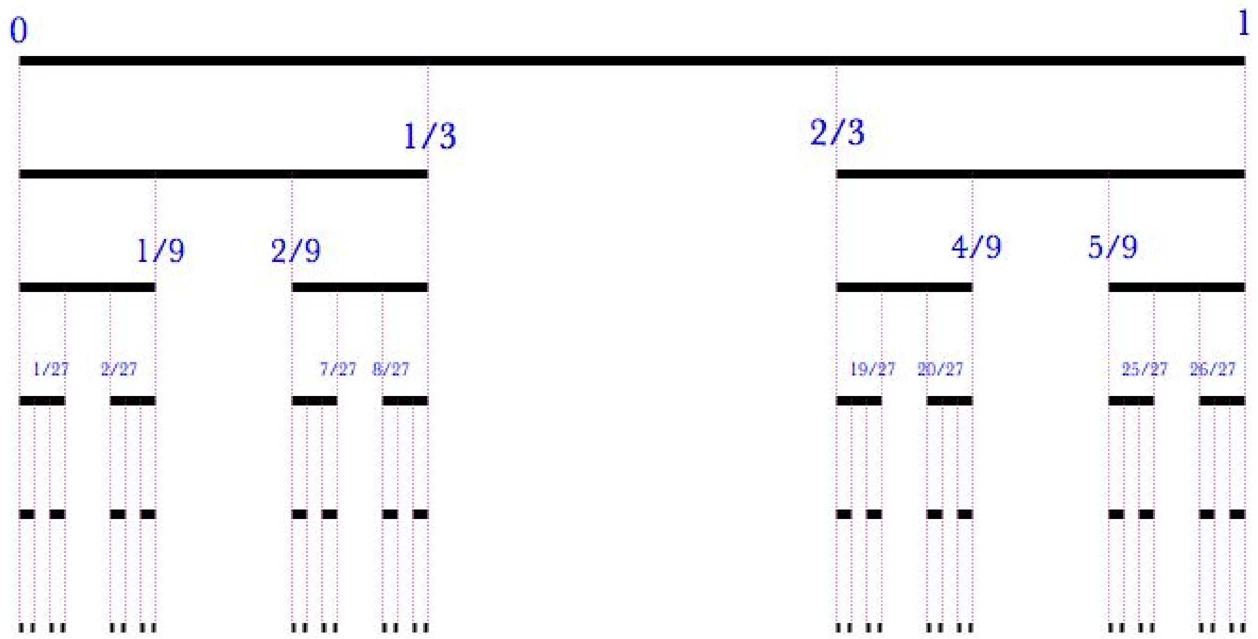
J. W. Dever
Local Space and Time Scaling Exponents for Diffusion on Compact Metric Spaces,
PhD Thesis, `arXiv:1806.04269`, June 11, 2018.

Content

I - Michon's Trees

G. Michon, "Les Cantors réguliers", *C. R. Acad. Sci. Paris Sér. I Math.*, (19), **300**, (1985) 673-675.

Cantor Sets



The triadic Cantor set

Cantor Sets

Definition *A Cantor set is a compact, completely disconnected set without isolated points*

Theorem *Any Cantor set is homeomorphic to* $\{0, 1\}^{\mathbb{N}}$.

L. Brouwer, "On the structure of perfect sets of points", *Proc. Akad. Amsterdam*, **12**, (1910), 785-794.

Hence without extra structure there is only one Cantor set.

ϵ-connectivity

Given (C, d) a metric space, for $\epsilon > 0$ let $\overset{\epsilon}{\sim}$ be the equivalence relation defined by

$$x \overset{\epsilon}{\sim} y \quad \Leftrightarrow \quad \exists x_0 = x, x_1, \cdots, x_{n-1}, x_n = y \quad d(x_{k-1}, x_k) < \epsilon$$

Theorem *Let (C, d) be a metric Cantor set. Then there is a sequence $\epsilon_1 > \epsilon_2 > \cdots \epsilon_n > \cdots \geq 0$ converging to 0, such that $\overset{\epsilon}{\sim} = \overset{\epsilon_n}{\sim}$ whenever $\epsilon_n \geq \epsilon > \epsilon_{n+1}$.*

For each $\epsilon > 0$ there is a finite number of equivalence classes and each of them is close and open.

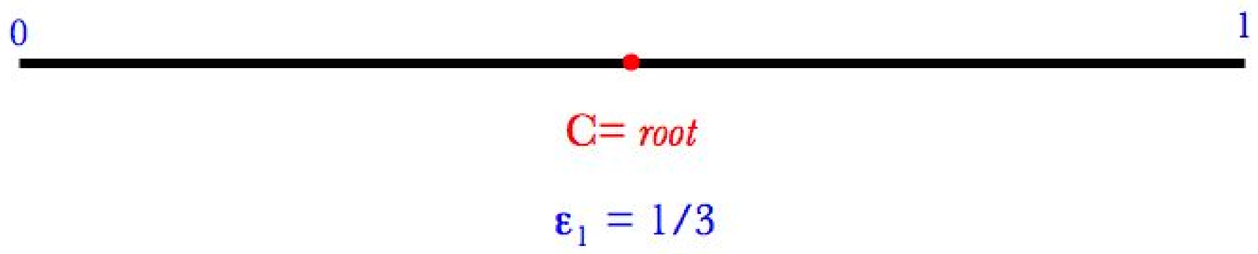
Moreover, the sequence $[x]_{\epsilon_n}$ of clopen sets converges to $\{x\}$ as $n \to \infty$.

Michon's Tree

- $\mathcal{V}_0 = \{C\}$ (called the *root*),
- for $n \geq 1$, $\mathcal{V}_n = \{[x]_{\epsilon_n}; x \in C\}$,
- $\mathcal{V}$ is the disjoint union of the $\mathcal{V}_n$'s,
- $\mathcal{E} = \{(v, v') \in \mathcal{V} \times \mathcal{V} \ ; \ \exists n \in \mathbb{N}\, , \, v \in \mathcal{V}_n, v' \in \mathcal{V}_{n+1}\, , \, v' \subset v\}$,
- $\delta(v) = \mathrm{diam}\{v\}$.

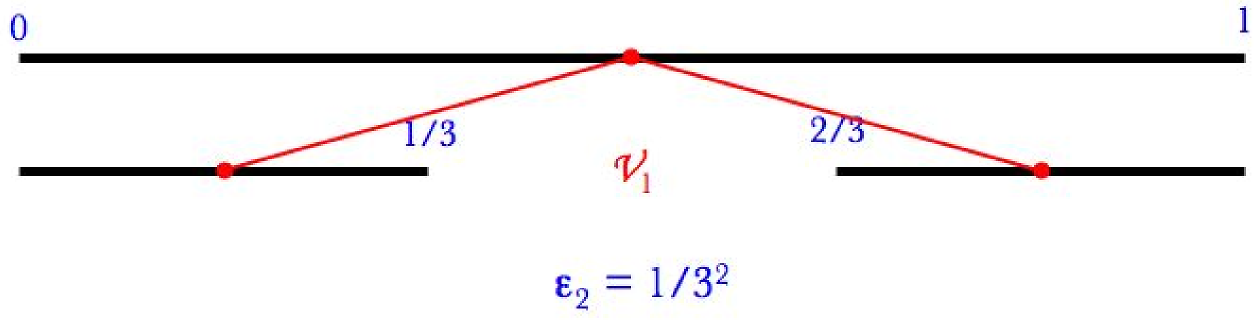
The family $\mathcal{T} = (C, \mathcal{V}, \mathcal{E}, \delta)$ defines a weighted rooted tree, with root C, set of vertices $\mathcal{V}$, set of edges $\mathcal{E}$ and weight δ

Michon's Tree



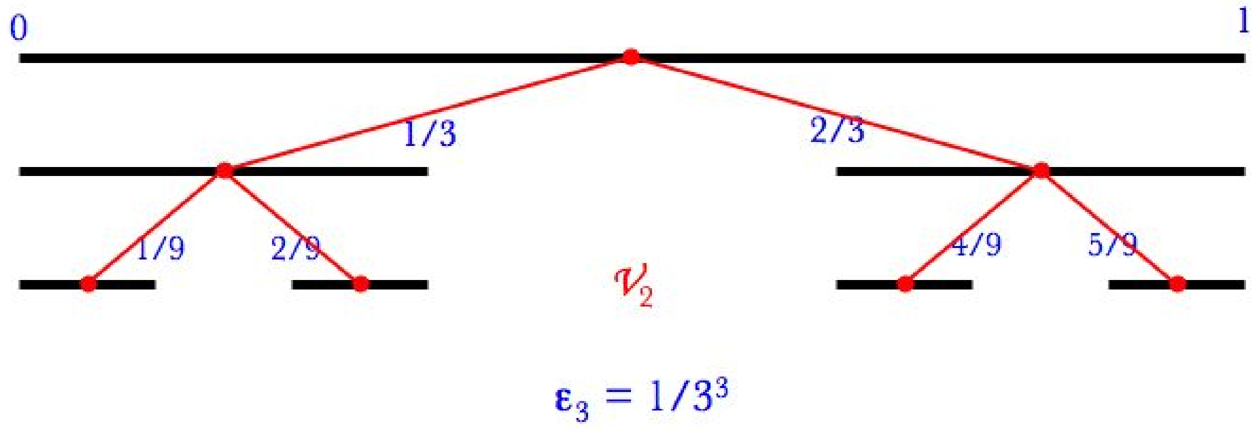
The Michon tree for the triadic Cantor set

Michon's Tree



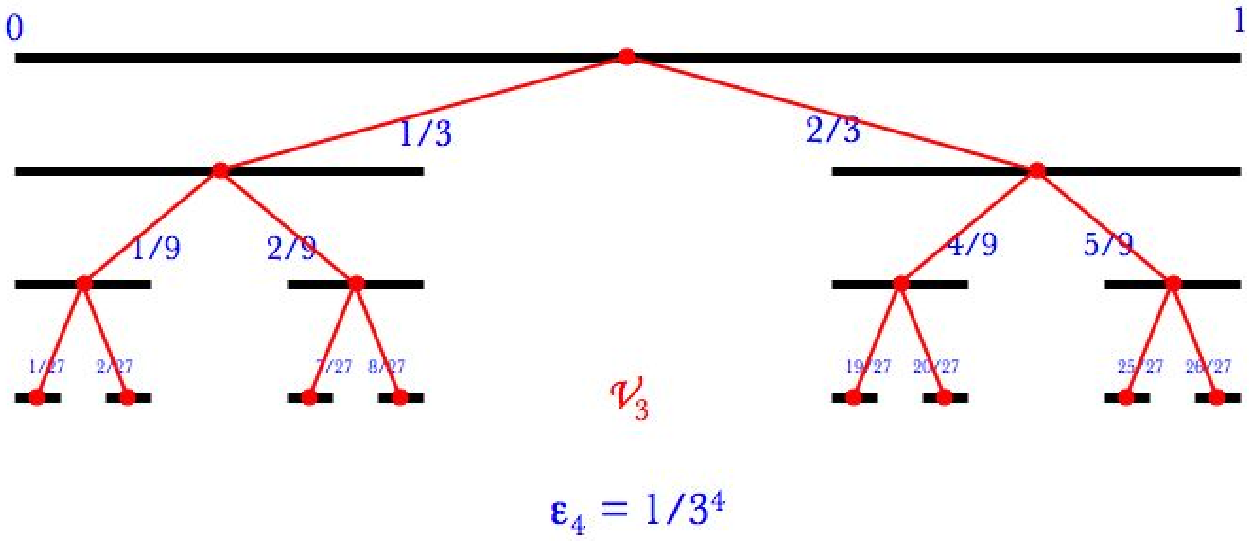
The Michon tree for the triadic Cantor set

Michon's Tree



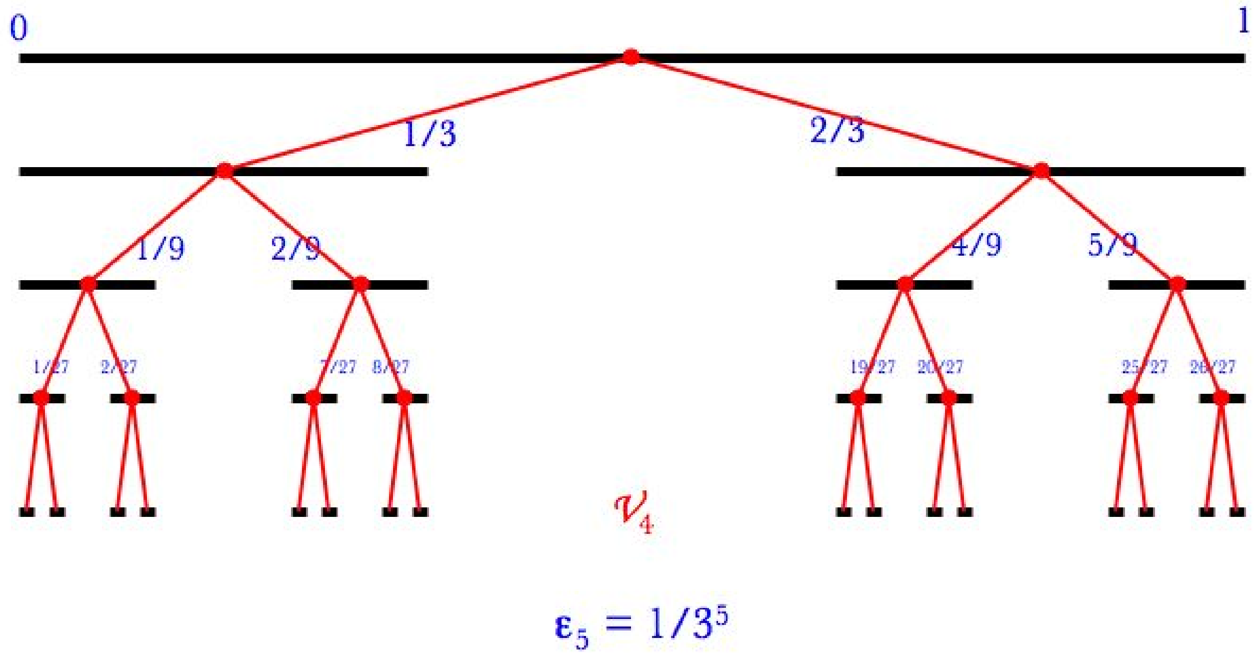
The Michon tree for the triadic Cantor set

Michon's Tree



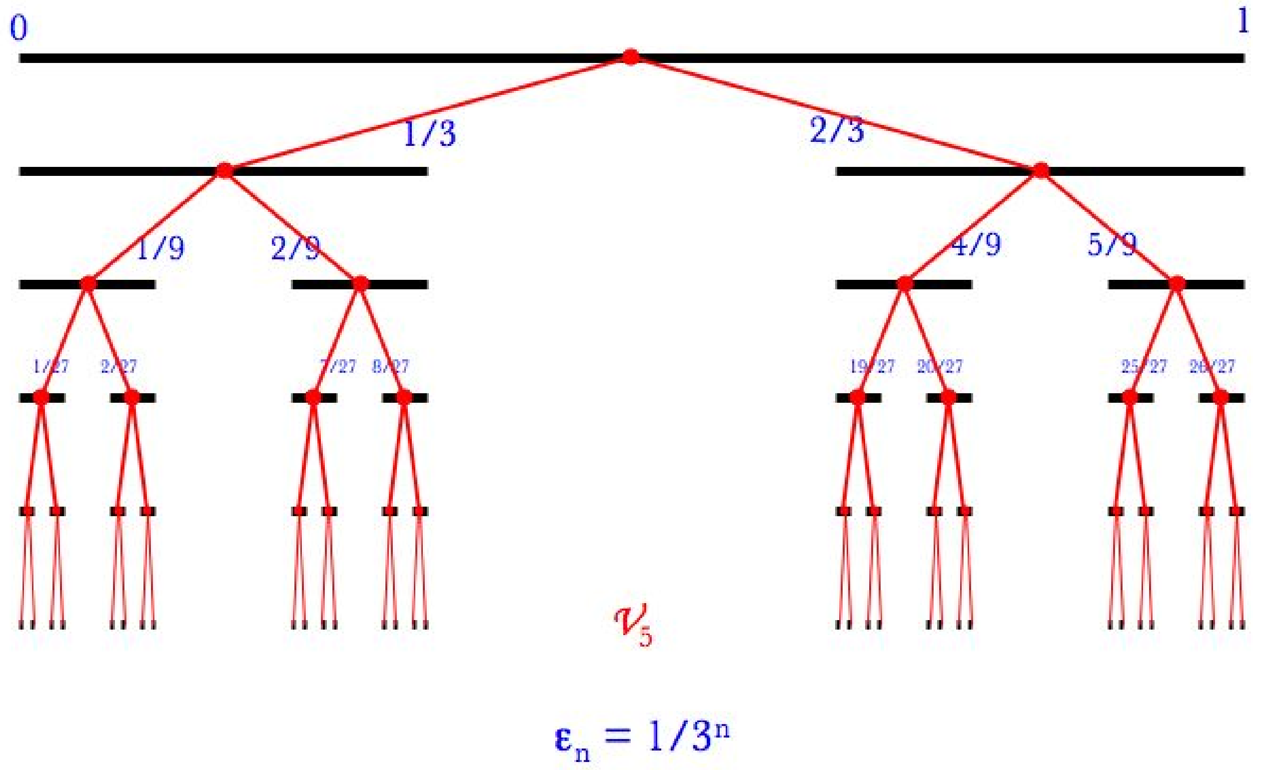
The Michon tree for the triadic Cantor set

Michon's Tree



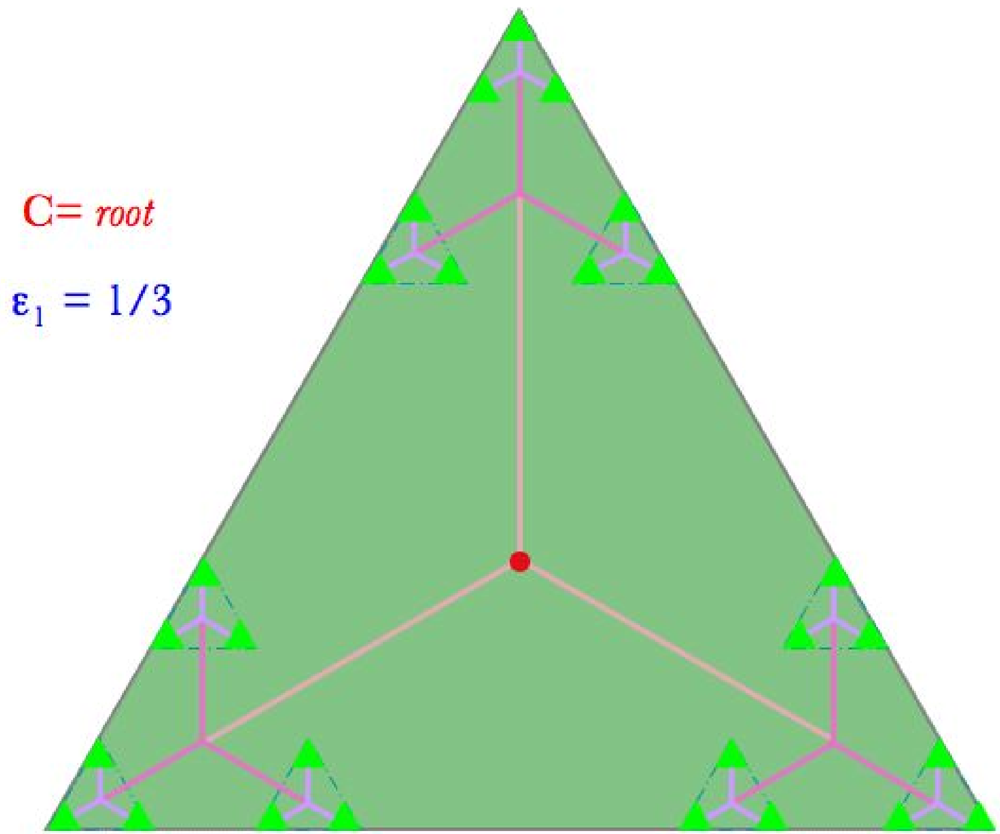
The Michon tree for the triadic Cantor set

Michon's Tree



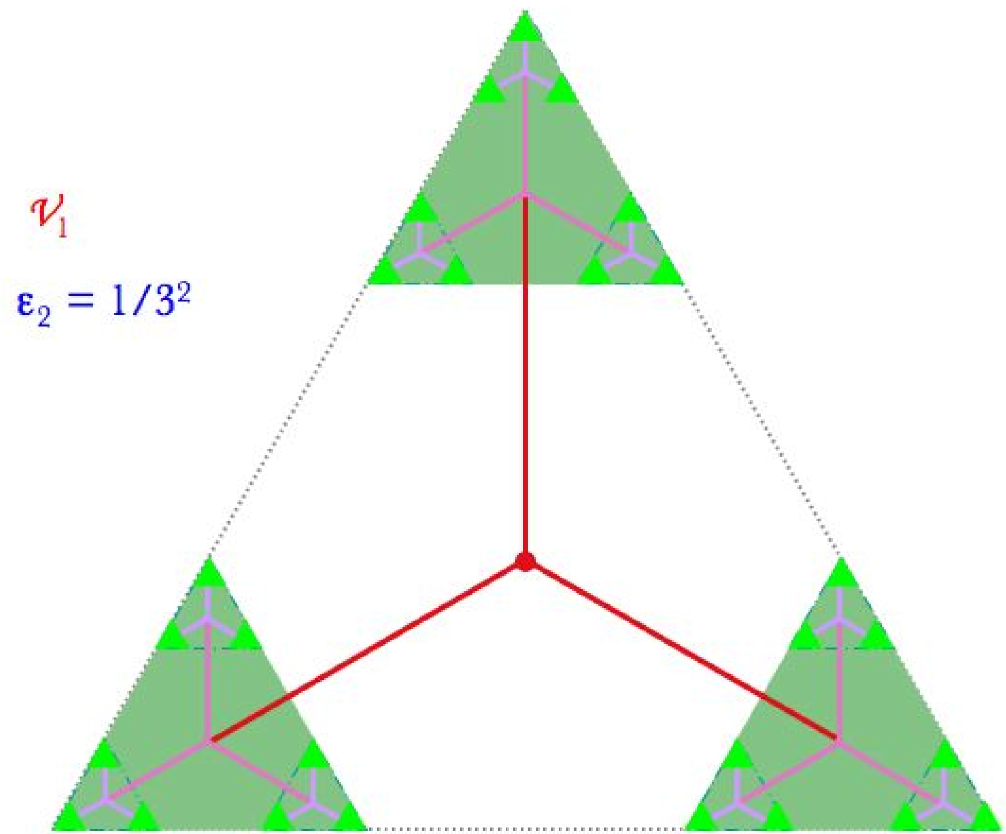
The Michon tree for the triadic Cantor set

Michon's Tree



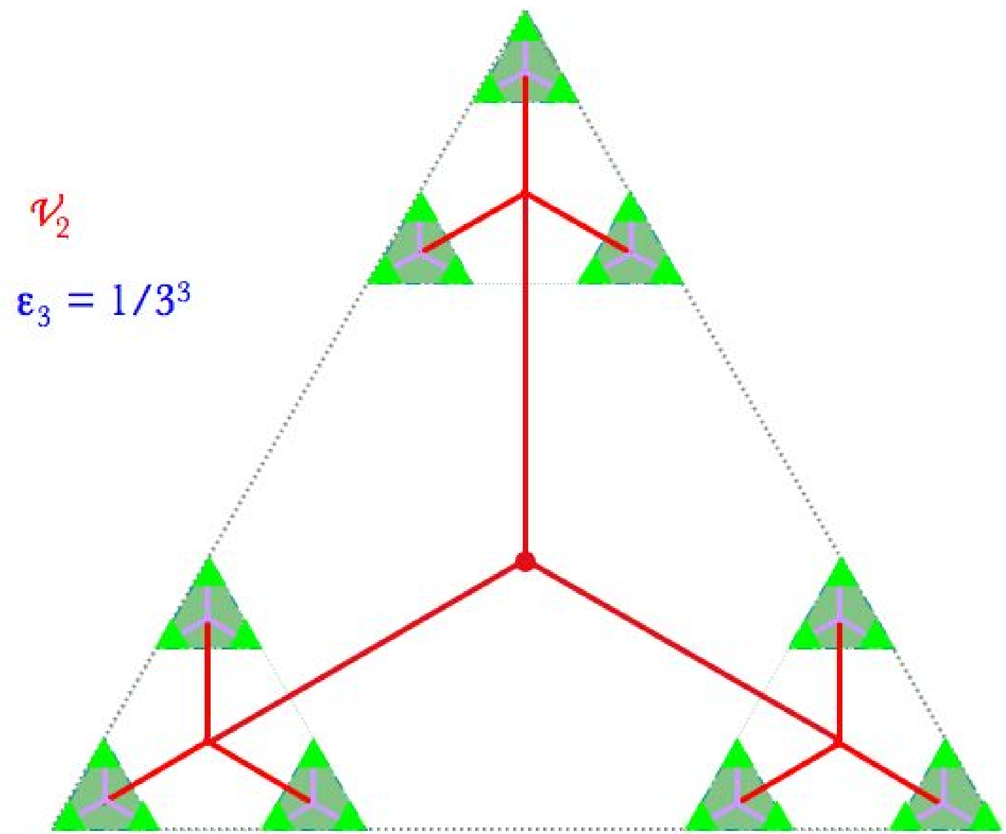
The Michon tree for the triadic ring $\mathbb{Z}(3)$

Michon's Tree



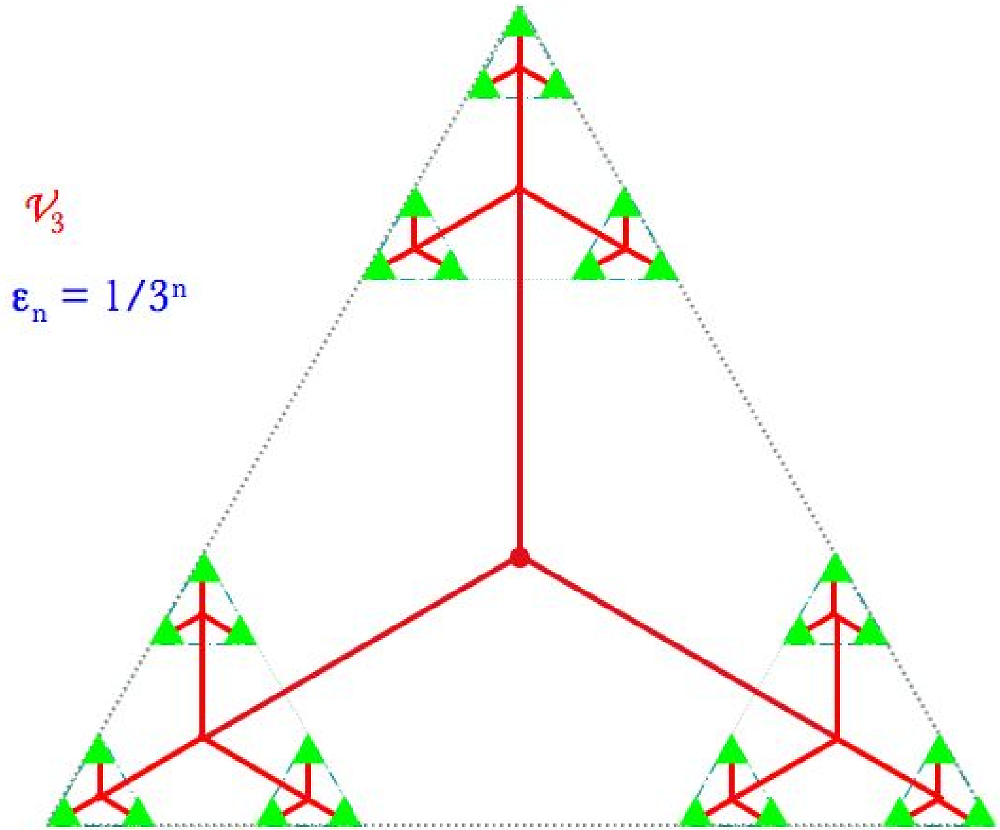
The Michon tree for the triadic ring $\mathbb{Z}(3)$

Michon's Tree



The Michon tree for the triadic ring $\mathbb{Z}(3)$

Michon's Tree



The Michon tree for the triadic ring $\mathbb{Z}(3)$

The Boundary of a Tree

Let $\mathcal{T} = (0, \mathcal{V}, \mathcal{E})$ be a rooted tree. It will be called *Cantorian* if

- *Each vertex admits one descendant with more than one child*
- *Each vertex has only a finite number of children.*

Then $\partial\mathcal{T}$ is the set of infinite path starting form the root. If $v \in \mathcal{V}$ then $[v]$ will denote the set of such paths passing through v

Theorem *The Michon tree of a metric Cantor set is Cantorian. Conversely, if $\mathcal{T} = (0, \mathcal{V}, \mathcal{E})$ is a Cantorian tree, the family $\{[v]\,;\, v \in \mathcal{V}\}$ is the basis of a topology making $\partial\mathcal{T}$ a Cantor set.*

Ultrametric on $\partial\mathcal{T}$

A *weight* on $\mathcal{T}$ is a map $\delta : \mathcal{V} \mapsto \mathbb{R}_+$ such that

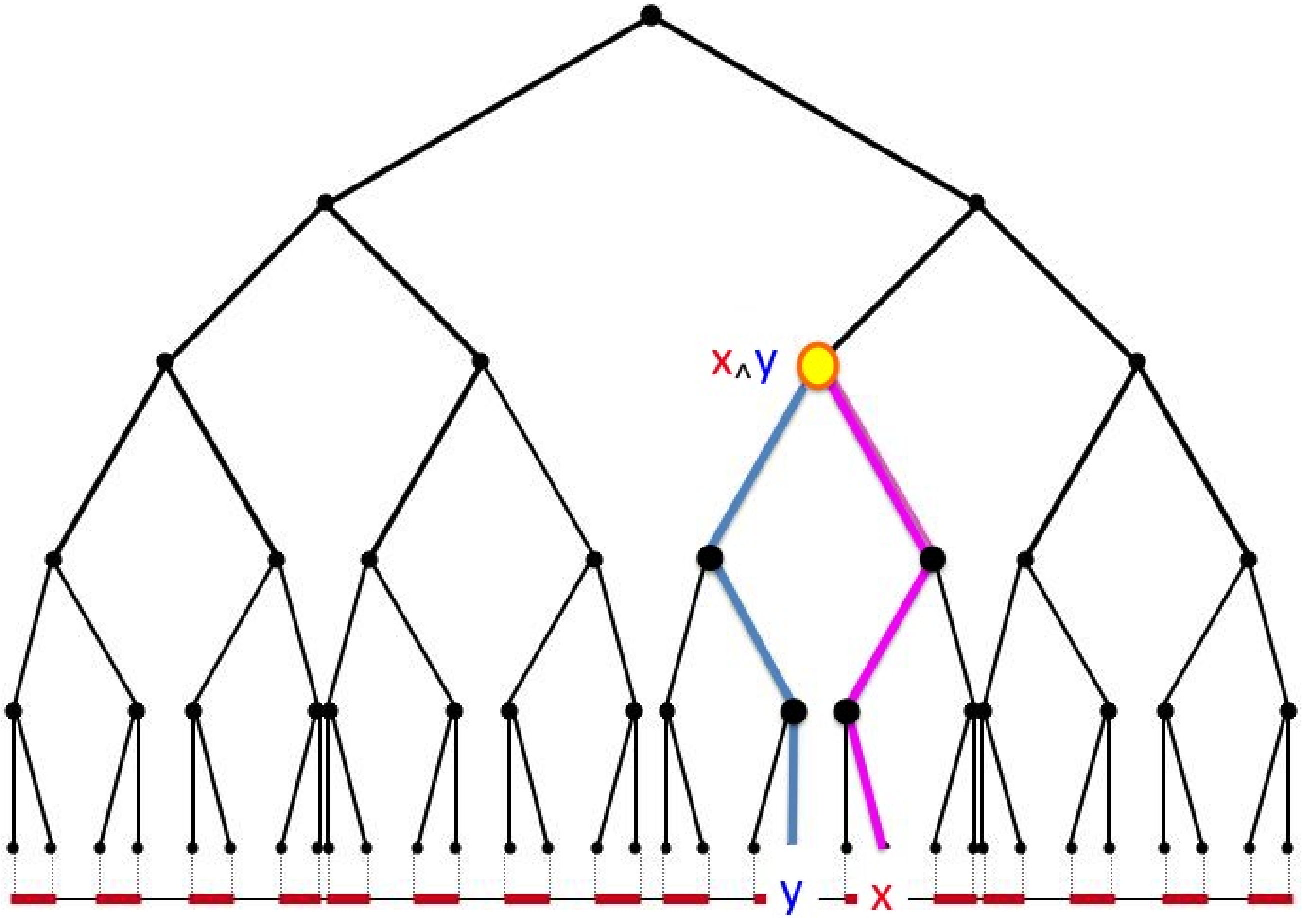
- If $w \in \mathcal{V}$ is a child of v then $\delta(v) \geq \delta(w)$,
- If $v \in \mathcal{V}$ has only one child w then $\delta(v) = \delta(w)$,
- If v_n is the decreasing sequence of vertices along an infinite path $x \in \partial\mathcal{T}$ then $\lim_{n\to\infty} \delta(v_n) = 0$.

Theorem *If $\mathcal{T}$ is a Cantorian rooted tree with a weight δ, then $\partial\mathcal{T}$ admits a canonical* ultrametric *d_δ defined by.*

$$d_\delta(x, y) = \delta([x \wedge y])$$

where $[x \wedge y]$ is the least common ancestor of x and y.

Ultrametric on $\partial\mathcal{T}$



The least common ancestor of x and y

Ultrametric on $\partial\mathcal{T}$

Theorem *Let $\mathcal{T}$ be a Cantorian rooted tree with weight δ. Then if $v \in \mathcal{V}$, $\delta(v)$ coincides with the diameter of $[v]$ for the canonical metric.*

Conversely, if $\mathcal{T}$ is the Michon tree of a metric Cantor set (C, d), with weight $\delta(v) = \mathrm{diam}(v)$, then there is a contracting homeomorphism from (C, d) onto $(\partial\mathcal{T}, d_\delta)$ and d_δ is the smallest ultrametric dominating d.

In particular, if d is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.

This gives a representation of all ultrametric Cantor sets together with a parametrization of the space of ultrametrics.

II - Spectral Triples

I. C. Palmer
Noncommutative Geometry and Compact Metric Spaces,
PhD Thesis, Georgia Institute of Technology, May 10, 2010.

Assumptions

The space (X, d) will satisfy

- (X, d) is compact, with *no isolated point*.
- $s_0 = \dim_H(X) < \infty$.
- Its *Hausdorff measure* $\mathcal{H}^{s_0}(X)$ at dimension s_0 is *positive* and *finite*.

Resolving Sequences

Let (X, d) be a *compact metric* space and let $F \subset X$ be a subset. For a *finite* open cover $\mathcal{U}$ of F let $\mathrm{diam}(\mathcal{U}) = \max\{\mathrm{diam}(U)\,;\, U \in \mathcal{U}\}$.

- A sequence $\mathfrak{U} = (\mathcal{U}_n)_{n\in\mathbb{N}}$ of finite open covers of F is *resolving* if $\lim_{n\to\infty} \mathrm{diam}(\mathcal{U}_n) = 0$.
- By convention $\mathcal{U}_0 = \{U_0\}$ for some open set $U_0 \supset F$.
- A resolving sequence is *strict* if $\mathrm{diam}(\mathcal{U}_{n+1}) < \min\{\mathrm{diam}(U)\,;\, U \in \mathcal{U}_n\}$.
- The *ζ-function* is $\zeta_{\mathfrak{U}}(s) = \sum_n \sum_{U\in\mathcal{U}_n} \mathrm{diam}(U)^s$.
- The *abscissa of convergence (a.o.c.)* $s_{\mathfrak{U}}$ is the minimum $s \in \mathbb{R}$ such that $\zeta_{\mathfrak{U}}(z)$ converges in $Re(z) > s$.

Hausdorff Dimension

The *Hausdorff dimension* of F is denoted by $\dim_H(F)$.

Theorem *(i) Given any resolving sequence* $\mathfrak{U}$ *of finite covers of* F*, its* a.o.c. *satisfies* $\dim_H(F) \leq s_{\mathfrak{U}}$.

(ii) There is always a resolving sequence $\mathfrak{U}$ *of finite covers of* F*, such that* $\dim_H(F) = s_{\mathfrak{U}}$.

Choice Functions

Given a resolving sequence $\mathfrak{U} = (\mathcal{U}_n)_{n\in\mathbb{N}}$, a *choice function* is a map $\tau : \coprod_{n=0}^{\infty} \mathcal{U}_n \to X \times X$ such that for any $n \geq 1$ and any open set $U \in \mathcal{U}_n$

- $\tau(U) = (x_U, y_U) \in \overline{U} \times \overline{U}$
- $\operatorname{diam}(U) \geq \operatorname{dist}(x_U, y_U) \geq \frac{\operatorname{diam}(U)}{1+\operatorname{diam}(U)}$
- The previous condition defines a closed subset $W_U \subset \overline{U} \times \overline{U}$

The *set of all choices* over $\mathfrak{U}$ will be denoted by $\Upsilon(\mathfrak{U})$; it is compact when identified with a closed subset of the Cartesian product

$$\Upsilon(\mathfrak{U}) \subset \prod_{n\geq 1} \prod_{U\in\mathcal{U}_n} W_U \,.$$

Spectral Triple

- Let $\mathcal{A} = C(X)$ be the unital Abelian *C^*-algebra* of continuous functions on X
- if $\mathfrak{U}$ is a resolving sequence, let $\ell^2(\mathfrak{U})$ denote the *Hilbert space* $\bigoplus_{n\geq 0} \ell^2(\mathcal{U}_n)$ and let $\mathcal{H}_{\mathfrak{U}} = \ell^2(\mathfrak{U}) \otimes \mathbb{C}^2$
- if $\tau \in \Upsilon(\mathfrak{U})$ a choice function, let π_τ be the *$*$-representation* of $\mathcal{A}$ defined by

$$\pi_\tau(f)\psi_U = \begin{bmatrix} f(x_U) & 0 \\ 0 & f(y_U) \end{bmatrix} \psi_U$$

where $\psi = ((\psi_U)_{U\in\mathcal{U}_n})_{n\geq 0} \in \mathcal{H}_{\mathfrak{U}}$.

Proposition *π_τ is faithfull.*

Spectral Triple

Given $\tau \in \Upsilon(\mathfrak{U})$, a *Dirac operator* is defined by

$$D_\tau \psi_U = \frac{1}{d(x_U, y_U)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_U$$

Theorem *(i) For any choice τ, the operator D_τ is self-adjoint with compact resolvent.*

(ii) For any Lipshitz continuous function f on X, the commutator $[D_\tau, \pi_\tau(f)]$ is bounded.

(iii) $\mathfrak{T}_{U,\tau} = (\mathcal{A}, \mathcal{H}_{\mathfrak{U}}, \pi_\tau, D_\tau)$ defines a family of spectral triples the Connes metric of which coincides with d on the set of pure states.

Spectral Triple

Computing $|D_\tau| = (D_\tau^* D_\tau)^{1/2}$ gives

$$|D_\tau|\psi_U = \frac{1}{d(x_U, y_U)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \psi_U$$

Theorem *If $\mathfrak{U}$ is a resolving sequence, then the original metric on (X, d) coincides with the Connes-Kantorovich metric, namely*

$$d(x, y) = \sup \left\{ |f(x) - f(y)| \, ; \, \sup_{\tau \in \Upsilon(\mathfrak{U})} \|[D_\tau, \pi_\tau(f)]\| \leq 1 \right\}$$

Spectral Dimension

In addition

$$\mathrm{Tr}\left(\frac{1}{|D_\tau|^s}\right) = 2\sum_{n\in\mathbb{N}}\sum_{U\in\mathfrak{U}_n} d(x_U, y_U)^s = 2\,\zeta_{\mathfrak{U}}(s)$$

Definition *(Connes '87-'94)* *The spectral dimension of a spectral triple* $(\mathcal{A}, \mathcal{H}, \pi, D)$ *is the infimum of real numbers s such that* $\mathrm{Tr}(|D|^{-s}) < \infty$.

Spectral Dimension

Theorem *Given a resolving sequence* $\mathfrak{U}$ *and a choice function* τ*, the spectral dimension of* $\mathfrak{T}_{\mathfrak{U},\tau}$ *with the* a.o.c. *of* $\mathfrak{U}$*. In particular it is independent of the choice function* τ *and*

$$\dim_S(\mathfrak{T}_{\mathfrak{U},\tau}) \geq \dim_H(X,d)$$

In addition, for some resolving sequence $\mathfrak{U}$, $\dim_S(\mathfrak{T}_{\mathfrak{U},\tau}) = \dim_H(X,d)$

Hence the Palmer spectral triple permits to *recover* the *Hausdorff dimension* of the space (X,d).

Connes State

A. Connes, *Noncommutative Geometry*, Acad. Press., San Diego (1994).

In a spectral triple $\mathfrak{T} = (\mathcal{A}, \mathcal{H}, \pi, D)$ with finite spectral dimension s_0, a *Connes state* is defined as a weak* limit state of the family of states $\omega_s \in \mathcal{S}(\mathcal{A})$ as $s \to s_0$ from above, where

$$\omega_s(a) = \frac{1}{\zeta_{\mathfrak{T}}(s)} \mathrm{Tr}\left(\frac{1}{|D|^s} \pi(a) \right) \qquad a \in \mathcal{A}$$

- If $\mathcal{A}$ is *unital*, its state space $\mathcal{S}(\mathcal{A})$ is weak* compact, so such limiting states always exist. Such limits can be obtained using *ultrafilters* on the interval (s_0, ∞).
- When it is *unique* it coincides with the *Dixmier trace* at $s = s_0$.

Connes State

Let now $\mathfrak{U}$ be a resolving sequence and τ be a choice function such that $\dim_S(\mathfrak{T}_{\mathfrak{U},\tau}) = \dim_H(X,d)$. Then

Theorem *(Palmer, '10)* *A Connes states, defined by an ultrafilter, is independent of the choice function τ chosen in taking the limit.*

There is a class of resolving sequences $\mathfrak{U}$ for which the Connes states is unique if and only if the Hausdorff measure of X at dimension $s_0 = \dim_H(X)$ is nonzero *and* finite. *In such a case, if $\mathcal{H}^{s_0}_{1,X}$ denotes the corresponding* normalized *measure, then*

$$\lim_{s\downarrow s_0} \frac{1}{\zeta_{\mathfrak{U}}(s)} \mathrm{Tr}\left(\frac{1}{|D_\tau|^s}\,\pi_\tau(f)\right) = \int_X f(x)\,d\mathcal{H}^{s_0}_{1,X}(x)$$

Connes State

Fact: Any probability measure μ on X induces a probability measure ν on the space of choices $\Upsilon(\mathfrak{U})$. This is because

$$\Upsilon(\mathfrak{U}) = \prod_{n \geq 1} \prod_{U \in \mathcal{U}_n} W_U$$

Hence on each W_U, the restriction of the measure $\mu \otimes \mu$ is bounded and Borel. By normalizing it on each coordinate, this defines a probability ν_U

$$\nu = \bigotimes_{n \geq 1} \bigotimes_{U \in \mathcal{U}_n} \nu_U$$

III - The Pearson Laplacian

J. Pearson, J. Bellissard
Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets,
J. Noncommut. Geo., **3**, (2009), 447-480.

Goal

Question: Can an analog of the *Laplace-Beltrami Operator* be built on a compact metric space, in particular on an ultrametric Cantor set ?

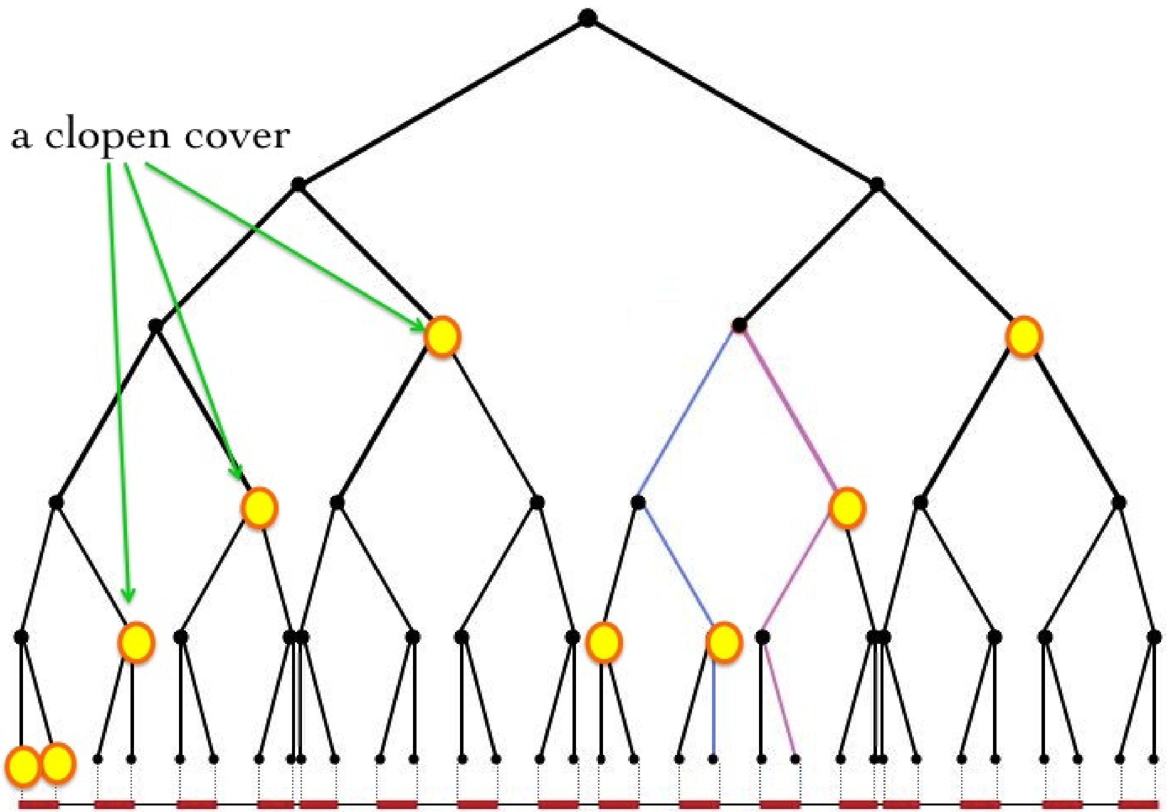
Goal

- Connes proposed to see *noncommutative Riemannian* manifolds as a *spectral triple*. The previous Sections show that it also includes compact metric spaces, even if *not smooth*.
- The Dirac operator on a smooth Riemannian manifold permits to recover the geodesic distance *(Connes metric)*, in a way analogous to the *Kantorovich metric* *(Kantorovich '40, '57, '58)* on the space of probability measures (also called *Wasserstein distance* *(Wasserstein '69, Dobrushin '70)*).
- A Laplacian is the generator of a *Markov semigroup* (Brownian motion). Using the *Leibniz* formula and the *Lindblad* construction, it defines a generalized Dirac operator, allowing, in principle, to *recover the metric*.

Clopen Partitions

- On an ultrametric Cantor set, the Michon tree encodes the *topology* and the weight encodes its *metric*.
- **Fact:** finite open covers can always be replaced by *finite open partitions*.
- Palmer's construction applies on such Cantor sets: *there are resolving sequences* $\mathfrak{U}$ *of* clopen partitions *giving spectral triples with spectral dimension equal to the Hausdorff dimension and Connes state equal to the corresponding Hausdorff measure*.

Clopen Partitions



Graphic representation of a clopen partition

Choice: an interpretation

- A *choice* $\tau \in \Upsilon(\mathfrak{U})$ can be seen as a *discrete* version of a *tangent vector field* on a space which is not a manifold and has no tangent space.
- In this context, the commutator $\nabla_\tau f = [D_\tau, \pi_\tau(f)]$ can be interpreted as the *directional derivative* in the direction of the corresponding tangent vector.
- To construct the *Laplacian*, seen as a quadratic form, one ought to compute $|\nabla_\tau f|^2$ and to integrate over the directions (here the *choice*) and over the space (given here by the *Dixmier trace*).

Laplacian construction

Guess: use the following quadratic form (for $f, g \in \mathcal{A}$)

$$Q_{\mathfrak{U},\tau}(f,g) = \lim_{s\downarrow s_0} \frac{1}{\zeta_{\mathfrak{U}}(s)} \int_{\Upsilon(\mathfrak{U})} \mathrm{Tr}\left(\frac{1}{|D_\tau|^s} [D_\tau^\beta, \pi_\tau(f)]^* [D_\tau^\beta, \pi_\tau(g)] \right) d\nu(\tau)$$

for some $\beta \in \mathbb{R}_+$ and then

$$D_\tau^\beta \overset{def}{=} D_\tau |D_\tau|^{\beta - 1}$$

Laplacian construction

Facts:

- If (X, d) is a *Riemannian manifold*, this gives the *Laplace-Beltrami* operator indeed with $\beta = 1$.
- For an *ultrametric Cantor* set (C, d) the form *vanishes* ! This can be understood since locally constant function are dense in $\mathcal{C}(C)$.
- For *fractals*, like the Sierpisky gasket, the diffusion is given by such a formula provided with $\beta \notin \mathbb{Q}$, where β is computable or is shown to exist in each example.

The Pearson Laplacian

Theorem *Let $\mathfrak{U}$ be a resolving sequence of clopen partitions on the ultrametric Cantor set (C, d), such that $s_0 = \dim_S(\mathfrak{T}_{\mathfrak{U},\tau}) = \dim_H(C, d)$ and that the Connes state coincides with the Hausdorff measure $\mu = \mathcal{H}^{s_0}_{1,X}$. Let ν denotes the probability measure on $\Upsilon(\mathfrak{U})$ induced by μ. Then the following quadratic form defines a one-parameter family of densely defined self-adjoint operators Δ_s on the Hilbert space $L^2(C, \mu)$*

$$Q_s(f, g) = \frac{1}{2} \int_{\Upsilon(\mathfrak{U})} \mathrm{Tr} \left(\frac{1}{|D_\tau|^s} [D_\tau, \pi_\tau(f)]^* [D_\tau, \pi_\tau(g)] \right) d\nu(\tau)$$

Then Δ_s has discrete spectrum with either an accumulation point at infinity or at finite distance, depending on the value of s.

The Pearson Laplacian

- There is a *dense set* of continuous functions $f \in C(X)$ such that the commutator $[D_\tau, \pi_\tau(f)]$ is *finite rank*. In addition, the absence of the normalizing factor $\zeta_{\mathfrak{U}}(s)$, which diverges at $s = s_0$ permits this integral to *make sense* at any value of $s \in \mathbb{R}$.

- The *eigenstates* are common to all the Δ_s's and can be *computed explicitly* *(Julien, Savinien,'10)*.

The Pearson Laplacian

In the examples computed explicitly *(Pearson, JB, '09, Julien, Savinien,'10, Julien, Kellendonk, Savinien, '15)*, the case $s = s_0$ is more interesting

- the eigenvalue distribution asymptotics follows a *Weyl law* with the dimension played by s_0,
- the *random process* induced by this operator, which plays the role of a *Brownien motion* on the Cantor set, is *almost diffusive*

$$\mathbb{E}\{d(X_{t_0}, X_{t_0+t})^2\} \overset{t\downarrow 0}{=} O(t\ln(1/t))$$

IV - The Dever Exponent

J. Dever, *A Localized Diffusive Time Exponent for Compact Metric Spaces*
`arXiv:1710.07872`, November 1, 2017

J. Dever, Dissertation (April 30, 2018)
Local Space and Time Scaling Exponents for Diffusion on Compact Metric Spaces,
`arXiv:1806.04269`, June 11, 2018

Goal

- In models the exponent β can be computed explicitly. But no conceptual definition was provided so far.
  However, several authors have already realized the importance of the concept of *diffusion time exponent*, without providing a universal definition.
  John Dever proposed such a definition in his Dissertation.

- John Dever also revised the concept of Hausdorff dimension, by introducing the *local Hausdorff dimension* and by giving *examples* of spaces with variable Hausdorff dimension.

Local Dimension

Let (X, d) be a compact metric space

- Given $x \in X$ let $\mathcal{N}(x)$ denote the set of open neighborhoods of x. Le *local dimension* is defined by

$$\alpha(x) = \inf\{\dim_H(U)\,;\, U \in \mathcal{N}(x)\} = \inf\{\dim_H(B(x;r))\,;\, r > 0\}$$

- The local dimension is *upper semicontinuous* and nonnegative. In particular it is Borel and bounded.

ϵ-nets

- For $\epsilon > 0$. An *ϵ-net* is a subset $N \subset X$ such that

$$\bigcup_{x \in N} B(x;\epsilon) = X\,, \qquad\qquad B(x;\epsilon) \cap N = \{x\}$$

- For $2\epsilon < \rho$, the *proximity graph*, denoted by $\mathcal{G}(N,\rho) = (\mathcal{V},\mathcal{E})$, admits $\mathcal{V} = N$ as *vertex* set, and a pair $(x,y) \in N \times N$ is an *edge* if $0 < d(x,y) < \rho$.
- $\mathcal{G}(N,\rho) = (\mathcal{V},\mathcal{E})$ is *finite, simple, without loops*.

Graph Markov Process

- On a *finite simple* graph with *no loop*, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the simplest random walk $(X_n)_{n\geq 0}$ is a discrete time stationary Markov process, such that at every vertex $x \in \mathcal{V}$ the walker choose to jump through one of the edges attached to x with *equal probability* for each of these edges. Let then P be the corresponding stochastic matrix with matrix elements

$$P_{xy} = \mathrm{Prob}\{X_{n+1} = y | X_n = x\} = \frac{1}{\deg(x)} \qquad (x, y) \in \mathcal{E}$$

$$\deg(x) = \#\{z \in \mathcal{V}\,;\,(x,z) \in \mathcal{E}\}$$

- The corresponding stationary state μ is given by

$$\mu(x) = \frac{\deg(x)}{\#\mathcal{V}}$$

Exit Time

- Given an ϵ-net on (X, d), $\rho \geq 2\epsilon$ and the corresponding proximity graph $\mathcal{G}(N, \rho)$, let then $(X_n)_{n \geq 0}$ denote the corresponding *random walk* on this graph.
- Let $U \subset X$. Then let $\tau_{N,U}(x)$ denotes the *first exit time* of the walker from $U \cap N$. Then let $\tau^+_{N,U} = \sup\{\tau_{N,U}(x)\,;\, x \in U \cap N\}$
- For $\delta > 0$ and $\beta \geq 0$, let

$$\omega_\beta(U) = \sup_{\delta > 0} \inf\{\tau^+_{N,U}\, \epsilon^\beta\,;\, N \text{ an } \epsilon\text{-net}\,,\, 0 < \epsilon < \delta\}$$

Exit Time Exponent

Proposition *There is a unique* $\beta(U) \geq 0$ *such that* $\omega_\beta(U) = 0$ *for* $\beta > \beta(U)$ *and* $\omega_\beta(U) = \infty$ *for* $\beta < \beta(U)$.

If $U \subset V$ *then* $\beta(U) \leq \beta(V)$.

In particular $\beta(x) = \inf\{\beta(U)\,;\, U \in \mathcal{N}(x)\} = \lim_{r\downarrow 0} \beta(B(x;r))$ *exists.*

Remark: this definition applies to *any compact metric space* (X, d).

Building a Laplacian: a Program

- Build a *Palmer spectral triple*, using an *appropriate* resolving sequence of finite open coverings $\mathfrak{U}$.
- If $\tau \in \Upsilon(\mathfrak{U})$ is a choice and if γ is either the *local Hausdorff* exponent or the *local exit time* exponent, the modified Dirac operator $D_{\gamma,\tau}$ is

$$D_{\gamma,\tau}\psi_U = \frac{1}{d(x_U, y_U)^{\gamma(U)}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_U$$

Building a Laplacian: a Program

- Investigate again whether the *local Hausdorff* dimension α can be *recovered* as in the case of the global one, through a ζ-function.
- If $\zeta_1(s) = \mathrm{Tr}(|D_\alpha|^s)$ is the *abscissa of convergent* equal to $s_0 = 1$, as one can expect ?
- If yes, the corresponding Connes states is defined by

$$\mu_\alpha(f) = \lim_{s \downarrow 1} \mathrm{Tr}\left\{ \frac{1}{|D_{\alpha,\tau}|^s} \, \pi_\tau(f) \right\}$$

Does it coincide with the *local Hausdorff measure* like in Dever's Dissertation ? When is it a *locally Ahlfors* regular measure ?

Building a Laplacian: a Program

- Can we make sense of the *quadratic form* (closability, Dirichlet), which should define the *Laplacian*,

$$Q_s(f,g) = \lim_{s\to 1} \frac{1}{2\zeta_1(s)} \int_{\Upsilon(\mathfrak{U})} \mathrm{Tr}\left(\frac{1}{|D_{\alpha,\tau}|^s} [D_{\beta/2\tau}, \pi_\tau(f)]^* [D_{\beta/2,\tau}, \pi_\tau(g)] \right) d\nu(\tau) \ ?$$

- If yes, this should provide some *geometrical constraint* on the space (X,d).
- Is the corresponding positive operator *local*, or *strongly local* ?

Thanks for Listening!!