

TOPOLOGICAL INSULATORS PHYSICS and DISORDER

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Update

This talk was given in a series of lectures on the use of Noncommutative Geometry in Solid State Physics, at the UC Santa Barbara in 2013. Most of the slides are the same. A list of recent works have been added at the end of the Part III.

Main References

M. KÖNIG, S. WIEDMANN, C. BRÜNE, A. ROTH, H. BUHMANN, L. W. MOLENKAMP, X. L. QI, S. C. ZHANG
Science, **318**, (2007), 766-770

D. HSIEH, D. QIAN, L. WRAY, Y. XIA, Y. S. HOR, R. J. CAVA, M. Z. HASAN, *Nature*, **452**, (2008), 970-975

M. Z. HASAN, C. L. KANE, Topological Insulators, *Rev. Mod. Phys.*, **82**, (2010), 3045-3067

H. SCHULZ-BALDES, S. TEUFEL, *Comm. Math. Phys.*, **319**, (2013), 649-681

E. PRODAN, B. LEUNG, J. BELLISSARD, "The non-commutative n th-Chern number ($n \geq 1$)", (*in preparation*), (2013)

Content

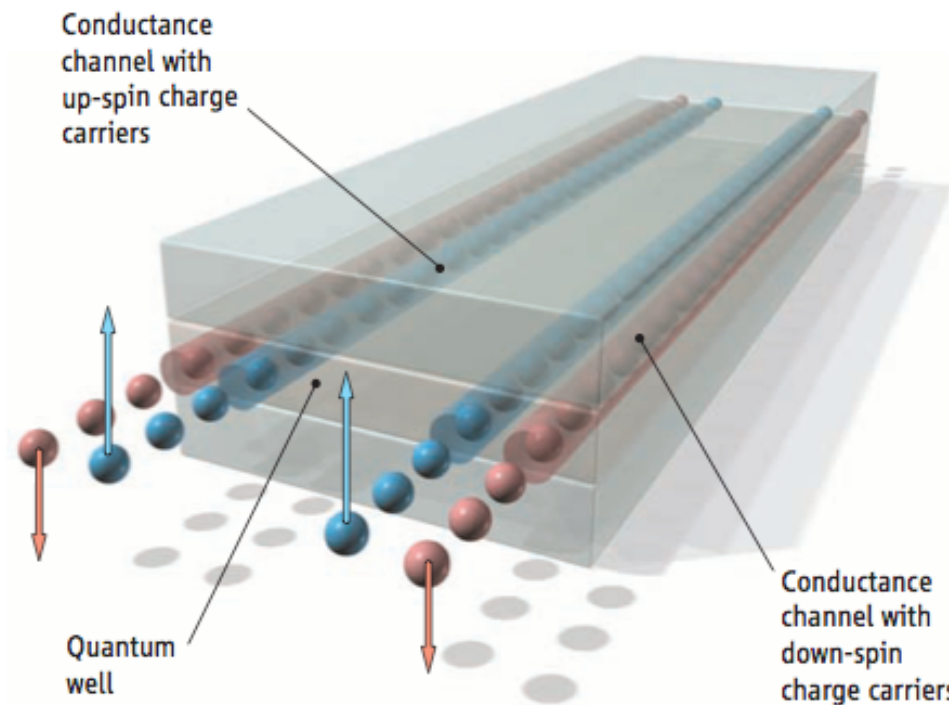
1. Topological Insulators
2. Semiconductors
3. The \mathbb{Z}_2 -Invariant
4. Magneto-Electric Response

I - TOPOLOGICAL INSULATORS

M. Z. HAZAN, C. L. KANE, Topological Insulators, *Rev. Mod. Phys.*, **82**, (2010), 3045-3067

Two Dimensional Compounds

M. KÖNIG, S. WIEDMANN, C. BRÜNE, A. ROTH, H. BUHMANN, L. W. MOLENKAMP, X. L. QI, S. C. ZHANG
Science, **318**, (2007), 766-770

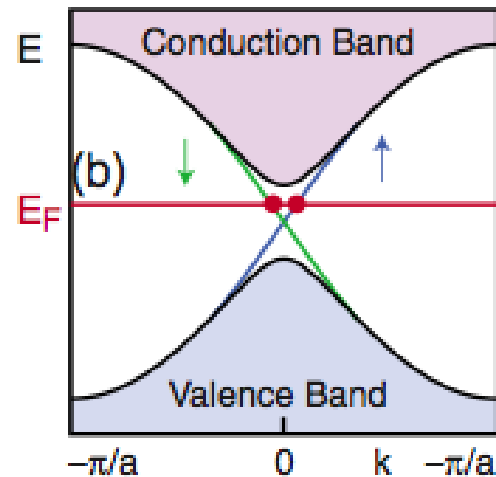
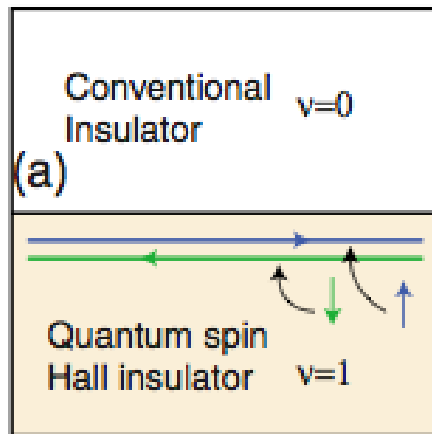


Schematic of the spin-polarized edge channels in a quantum spin Hall insulator.

2D-*HgTe* semi-conductor with inverted band structure provide a way to create a spin polarized channel of electronic current, protected by topological invariant

Two Dimensional Compounds

M. KÖNIG, S. WIEDMANN, C. BRÜNE, A. ROTH, H. BUHMANN, L. W. MOLENKAMP, X. L. QI, S. C. ZHANG
Science, **318**, (2007), 766-770

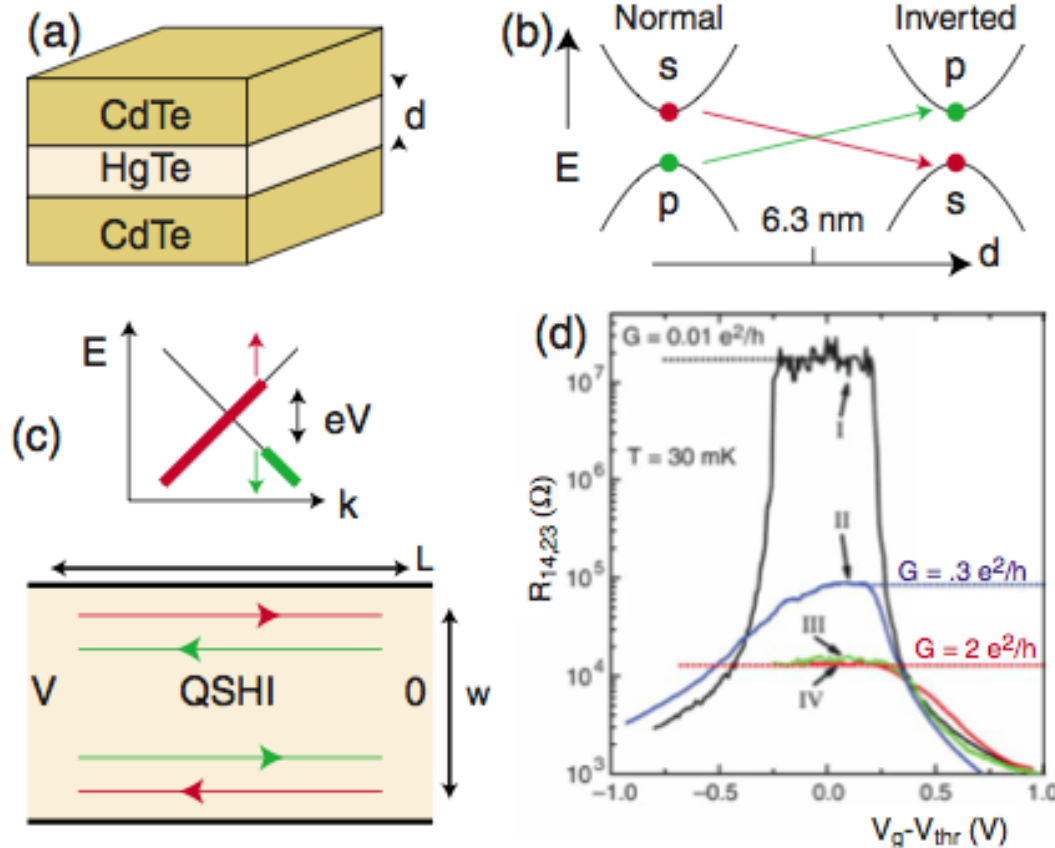


Edge states:
Right edge states
interpolating between
valence and conduction
bands
Colors show the spin

Slope = velocity

Two Dimensional Compounds

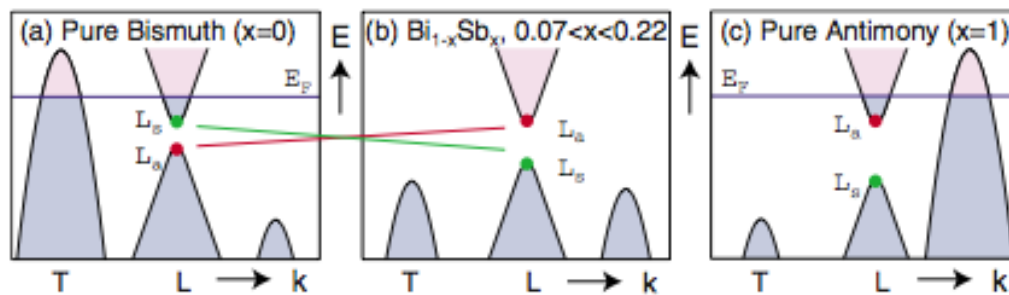
M. KÖNIG, S. WIEDMANN, C. BRÜNE, A. ROTH, H. BUHMANN, L. W. MOLENKAMP, X. L. QI, S. C. ZHANG
Science, **318**, (2007), 766-770



Edge states have
 quantized conductance

Three Dimensional Compounds

D. HsIEH, D. QIAN, L. WRAY, Y. XIA, Y. S. HOR, R. J. CAVA & M. Z. HASAN,
Nature, **452**, (2008), 970-975

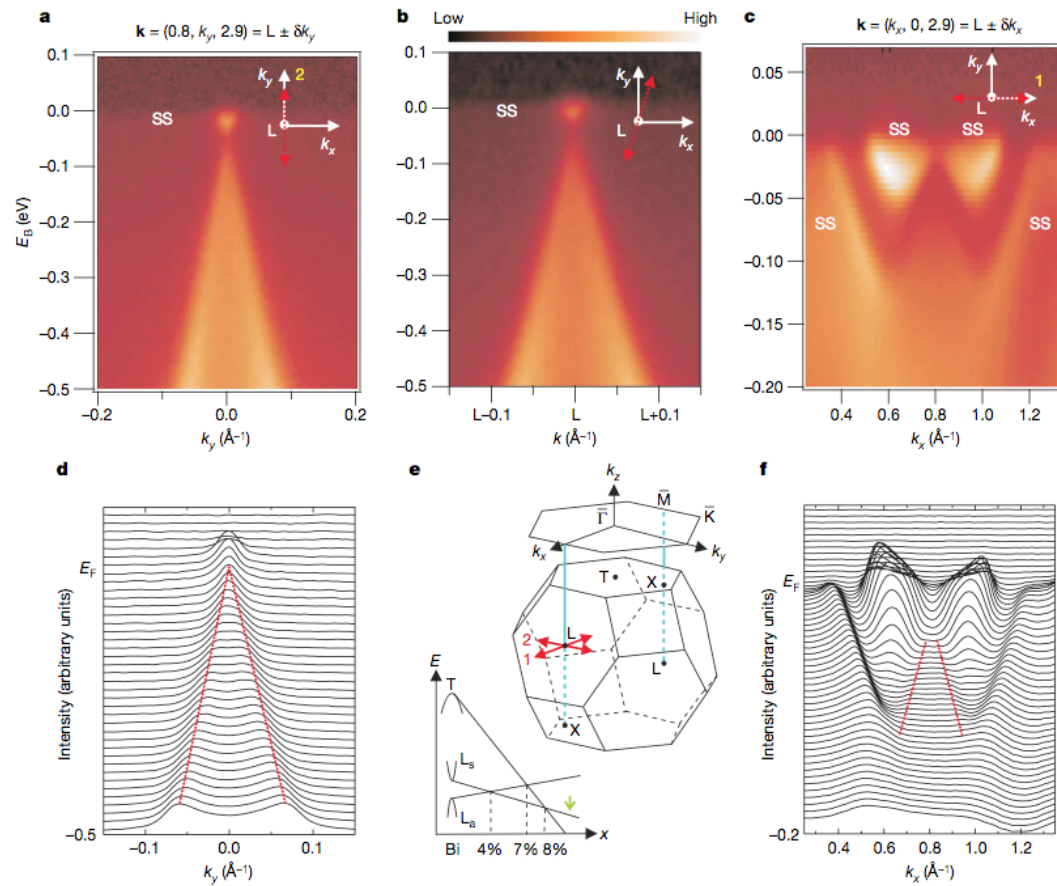


3D-Bi_{0.9}Sb_{0.1}
Inverted band structure

FIG. 8. (Color online) Schematic representation of the band structure of $\text{Bi}_{1-x}\text{Sb}_x$, which evolves from semimetallic behavior for $x < 0.07$ to semiconducting behavior for $0.07 < x < 0.22$ and back to semimetallic behavior for $x > 0.18$. The conduction and valence bands $L_{s,a}$ invert at $x \sim 0.04$.

Three Dimensional Compounds

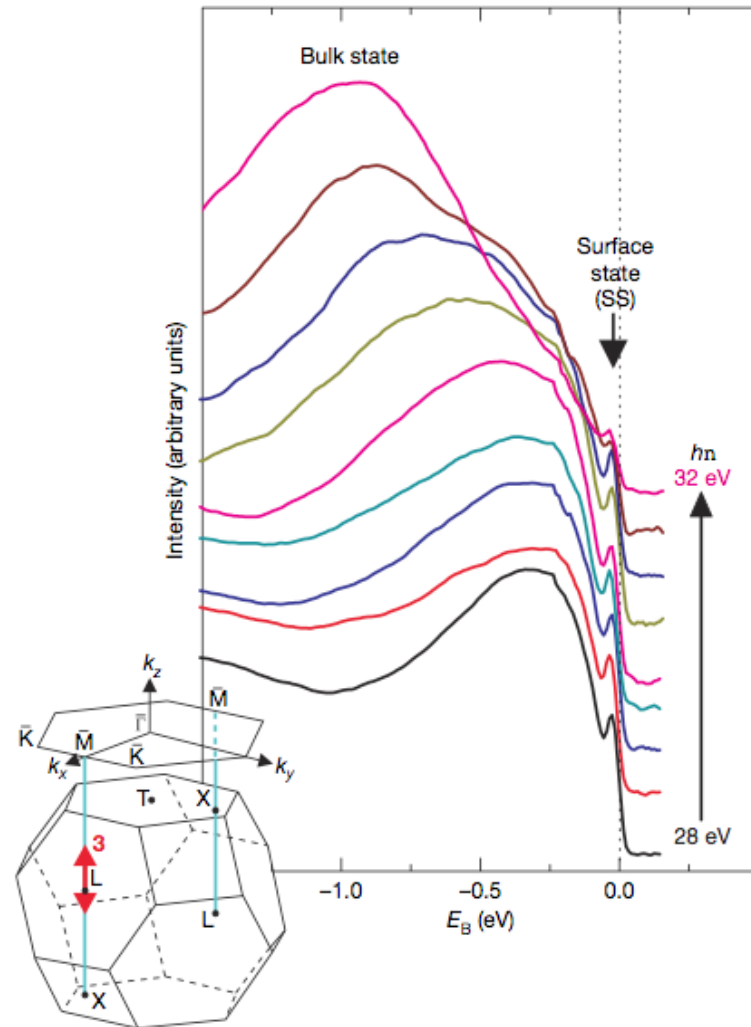
D. HSIEH, D. QIAN, L. WRAY, Y. XIA, Y. S. HOR, R. J. CAVA & M. Z. HASAN,
Nature, **452**, (2008), 970-975



3D- $\text{Bi}_{0.9}\text{Sb}_{0.1}$
Dirac dispersion cone

Three Dimensional Compounds

D. HsIEH, D. QIAN, L. WRAY, Y. XIA, Y. S. HOR, R. J. CAVA & M. Z. HASAN,
Nature, **452**, (2008), 970-975



$3\text{D-Bi}_{0.9}\text{Sb}_{0.1}$
Surface states evidence

II - Semiconductors

N. W. ASHCROFT, N. D. MERMIN, *Solid State Physics*, Holt, Rinehart and Winston Eds., (1976)

B. I. SHKLOVSKII, A. L. EFROS, *Electronic Properties of Doped Semiconductors*, Springer, (1984).

The Columns II-VI

	3A	4A	5A	6A
	5 B $1s^2 2s^2 p^1$	6 C $1s^2 2s^2 p^2$	7 N $1s^2 2s^2 p^3$	8 O $1s^2 2s^2 p^4$
	13 Al $[\text{Ne}] 3s^2 p^1$	14 Si $[\text{Ne}] 3s^2 p^2$	15 P $[\text{Ne}] 3s^2 p^3$	16 S $[\text{Ne}] 3s^2 p^4$
2B				
30 Zn $[\text{Ar}] 3d^{10} 4s^2$	31 Ga $[\text{Ar}] 3d^{10} 4s^2 p^1$	32 Ge $[\text{Ar}] 3d^{10} 4s^2 p^2$	33 As $[\text{Ar}] 3d^{10} 4s^2 p^3$	34 Se $[\text{Ar}] 3d^{10} 4s^2 p^4$
48 Cd $[\text{Kr}] 4d^{10} 5s^2$	49 In $[\text{Kr}] 4d^{10} 5s^2 p^1$	50 Sn $[\text{Kr}] 4d^{10} 5s^2 p^2$	51 Sb $[\text{Kr}] 4d^{10} 5s^2 p^3$	52 Te $[\text{Kr}] 4d^{10} 5s^2 p^4$
80 Hg $[\text{Xe}] 4f^{14} 5d^{10} 6s^2$	81 Tl $[\text{Xe}] 4f^{14} 5d^{10} 6s^2 p^1$	82 Pb $[\text{Xe}] 4f^{14} 5d^{10} 6s^2 p^2$	83 Bi $[\text{Xe}] 4f^{14} 5d^{10} 6s^2 p^3$	84 Po $[\text{Xe}] 4f^{14} 5d^{10} 6s^2 p^4$

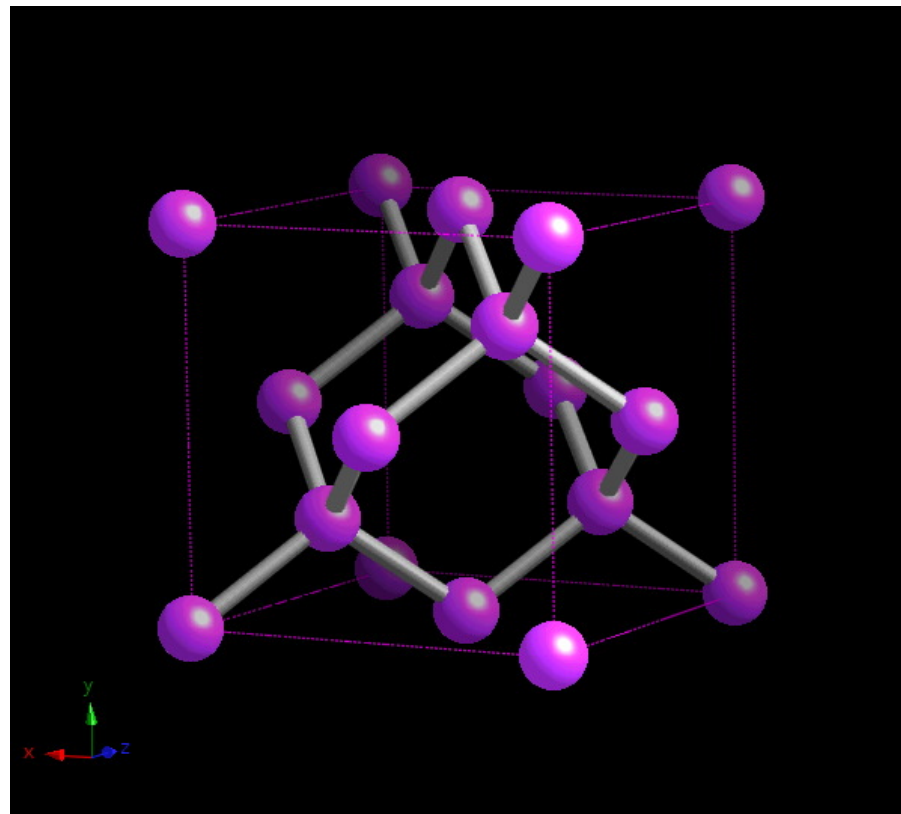
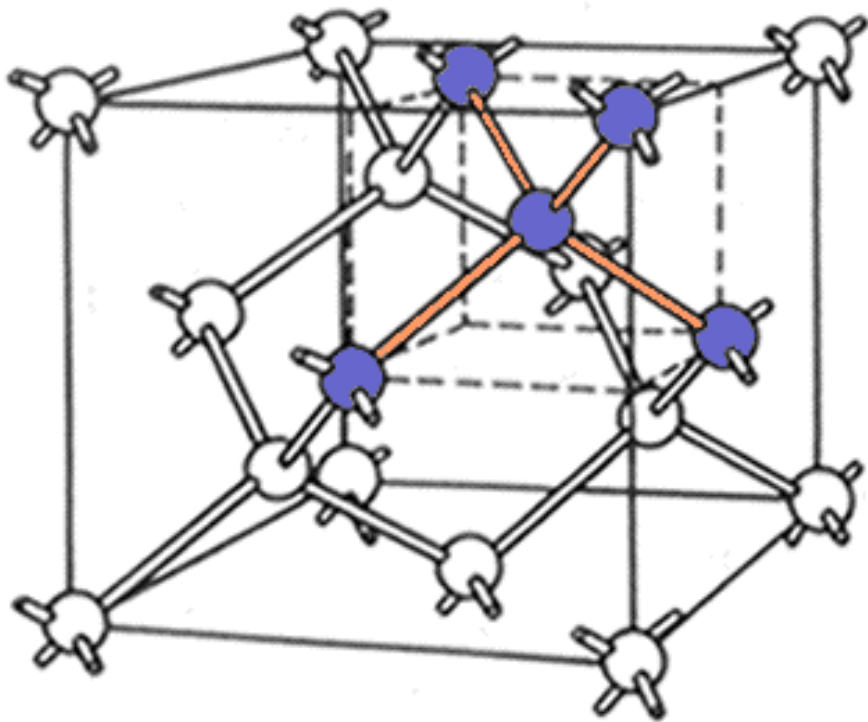
Column IV: *Si, Ge*
basic semiconductors,

III-V compound:
like *Ga-As*

II-VI compounds:
like *Hg-Te, Cd-Te,*

3D-compounds:
Bi_{1-x}Sb_x

The Diamond Lattice



The Diamond Lattice

G. LEMAN, J. FRIEDEL, *J. Appl. Phys.*, **33**, (1962), 281-285

- Projecting \mathbb{Z}^4 onto \mathbb{R}^3 perpendicular to $f = 1/2(1, 1, 1, 1)$:
each patch $\{m, m + e_1, m + e_2, m + e_3, m + e_4\}$ projects as a

regular tetrahedron

- If $i \in \{0, 1, 2, 3\}$ then

$$\mathcal{E}_i = \left\{ m \in \mathbb{Z}^4; \sum_{\alpha=1}^4 m_\alpha = i \right\}$$

Then keep only the points in $\mathcal{E}_0 \cup \mathcal{E}_1$

- $m \in \mathcal{E}_0 \Leftrightarrow m + e_\alpha \in \mathcal{E}_1$ gives the *staggering* between tetrahedra.

Chemical Bonds

G. LEMAN, J. FRIEDEL, *J. Appl. Phys.*, **33**, (1962), 281-285

- Each atom has *4-valence electrons* in orbitals s, p_x, p_y, p_z with same energy.
- Each such orbital is a *vector* in a Hilbert space $\Rightarrow \mathbb{C}^4$.
- Using the \mathbb{Z}_4 Fourier transform gives four linear combinations with the symmetry of the *tetrahedron*. The *electron density* points in space in the four directions of the tetrahedron.
- Each valence state creates a band in the lattice. The spins double their number, thus *8-bands*
- Adding the *e – e interaction* onsite leads to *splitting* of levels \Rightarrow creates a gap between *4-valence* and *4-conduction* bands.

Spin-Orbit Coupling

E. U. CONDON, G. H. SHORTLEY, *The Theory of Atomic Spectra*, Cambridge University Press (1935)

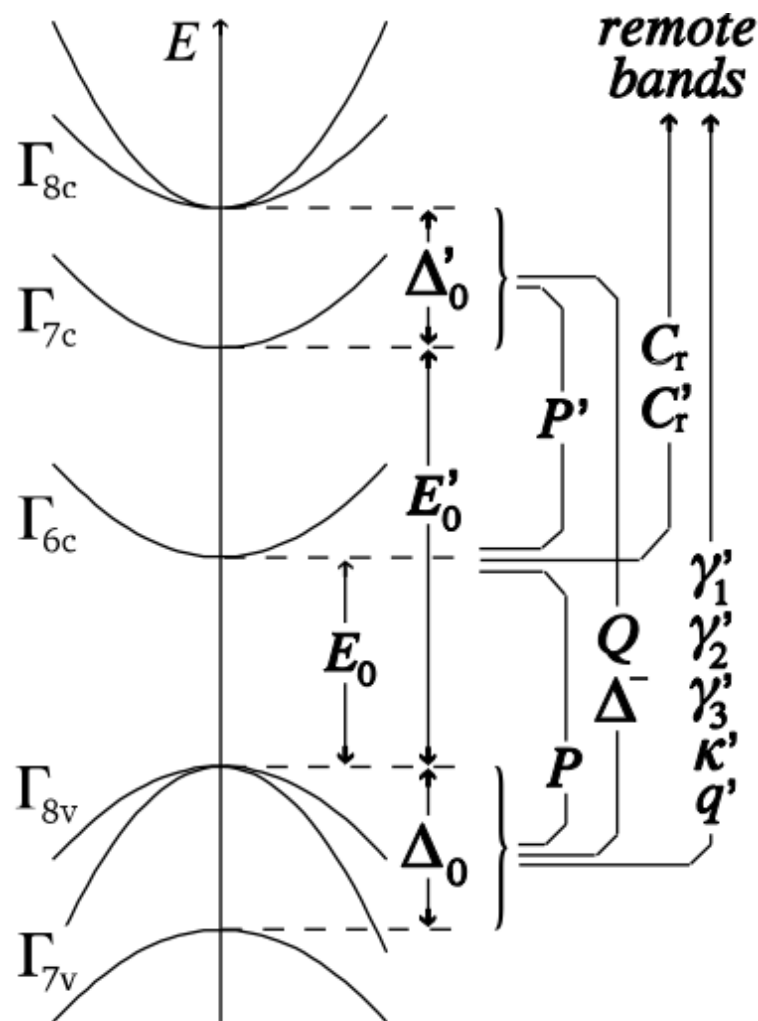
- The electron rotation around a nucleus creates a magnetic field which interact with its own spin (spin-orbit coupling)
- Relativistic corrections gives the spin-orbit energy

$$H_{SO} = \frac{\mu_B}{\hbar m_e c^2} \frac{1}{r} \frac{\partial U(r)}{\partial r} \vec{L} \cdot \vec{S}$$

- \vec{L} is the *angular momentum*, \vec{S} is the *spin*
- $U(r)$ is the radial potential seen by the electron.
- μ_B =Bohr magneton, m_e =electron mass, e =electron charge, c =speed of light.

Spin-Orbit Coupling

R. WINKLER, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems*, Springer, (2003)



If the diamond lattice is shared by two atomic species, the point symmetry between the two sublattices is broken. Then the spin-orbit interaction creates a band splitting

III - THE \mathbb{Z}_2 -INVARIANT

M. Z. HAZAN, C. L. KANE, Topological Insulators, *Rev. Mod. Phys.*, **82**, (2010), 3045-3067

M. ATIYAH, I. M. SINGER, Index Theory for Skew-Adjoint Fredholm Operators,
Inst. Hautes Études Sci. Publ. Math., **37**, (1969), 5-26

Real Hilbert Spaces

- A Hilbert space can be seen as a *real Hilbert space* \mathcal{H} equipped with a real linear operator J such that

$$J^{-1} = J^T = -J \quad \langle f|g \rangle_{\mathbb{C}} = \langle f|g \rangle_{\mathbb{R}} + i\langle Jf|g \rangle_{\mathbb{R}}$$

- Then \mathbb{C} acts through $z = x + iy \mapsto x + Jy$
- A \mathbb{C} -linear operator A is an \mathbb{R} -linear map such that $AJ = JA$.
- *Complex conjugacy* is given by an \mathbb{R} -linear map C such that $C = C^{-1} = C^T$, $CJ + JC = 0$. Then any $f \in \mathcal{H}$ can be uniquely written as $f = f_0 + Jf_1$ with $Cf_i = f_i$

Time Reversal Symmetry

- For the time dependent *Schrödinger equation* the time-reversal symmetry is given by C:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V\psi \quad \Rightarrow \quad -i\hbar \frac{\partial \bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m} \Delta \bar{\psi} + V\bar{\psi}$$

- For the relativistic *Dirac equation* the time-reversal symmetry is given by an operator Θ such that $\Theta^2 = -1$. The same occurs for spin-orbit coupling.
- In the real version of Hilbert spaces this gives two \mathbb{R} -linear map J, Θ such that (*Clifford Algebra*)

$$J^{-1} = J^T = -J \quad \Theta^{-1} = \Theta^T = -\Theta \quad \Theta J + J\Theta = 0$$

Kramers Degeneracy

- Let $H = H^*$ be a *time-reversal symmetric* selfadjoint operator. In the real version then H is an \mathbb{R} -linear map on \mathcal{H} commuting with both J, Θ .
- If $f \in \mathcal{H}$ is an *eigenstate* of H , namely $Hf = Ef$, then $H\Theta f = E\Theta f$ is another one.
- However $\langle \Theta f | f \rangle_{\mathbb{C}} = 0$. Since $\|f\| = \|\Theta f\|$, it follows that every eigenvalue is *twice degenerate*.

Momentum Space

- If the Hamiltonian is *periodic* on the diamond lattice, Bloch theory implies that it can be seen as a k -dependent 8×8 -matrix with $k \in \mathbb{B} \simeq \mathbb{T}^3$. Matrix *indices* label the *8-orbitals* close to the Fermi level.
- *Time reversal* symmetry implies

$$\Theta H(k) \Theta^{-1} = H(-k)$$

- Let Λ be the set of points in the Brillouin zone such that $k = -k$.
- At each point $k \in \Lambda$, the matrix $H(k)$ commutes with Θ and its eigenstates are *Kramers-degenerate*.

The \mathbb{Z}_2 -index

- Each *occupied band*, labelled by m , is given by a Bloch function $u_m(k)$ with values in the Hilbert space \mathcal{H} . Let

$$W_{m,n}(k) = \langle u_m(k) | \Theta u_n(-k) \rangle$$

- $W(k)$ is unitary and $W^T(k) = -W(-k)$. In particular it is *antisymmetric* at points of Λ .

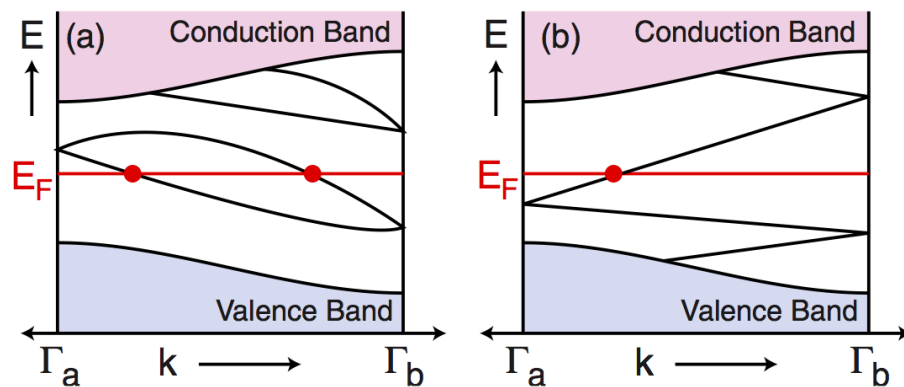
(Smoothness is required ! The $u_m(k)$ must be time-reversal "compatible" !)

- Then the \mathbb{Z}_2 -index $\nu \in \mathbb{Z}_2$ is defined by

$$(-1)^\nu \stackrel{\text{def}}{=} \prod_{k \in \Lambda} \frac{\text{Pf}[W(k)]}{\sqrt{\det(W(k))}}$$

Edge States

- In experiments, a *spin-polarized current* is protected on the *edges* of the quantum well. This edge is $1D$.
- Truncated space along the edge leads to new *eigenstates in the Fermi gap*. The corresponding eigenfunction are *localized* on the edges. If the edge is a straight line, the periodicity along the edge allows to use Bloch theory with $k \in \mathbb{T}$.



Edge states are doubly degenerate on Λ . The parity of the number of states with energy E_F is given by ν

Disordered Case: Numerical Results

E. PRODAN, B. LEUNG, *Phys. Rev. B*, **85**, 205136, (2012)

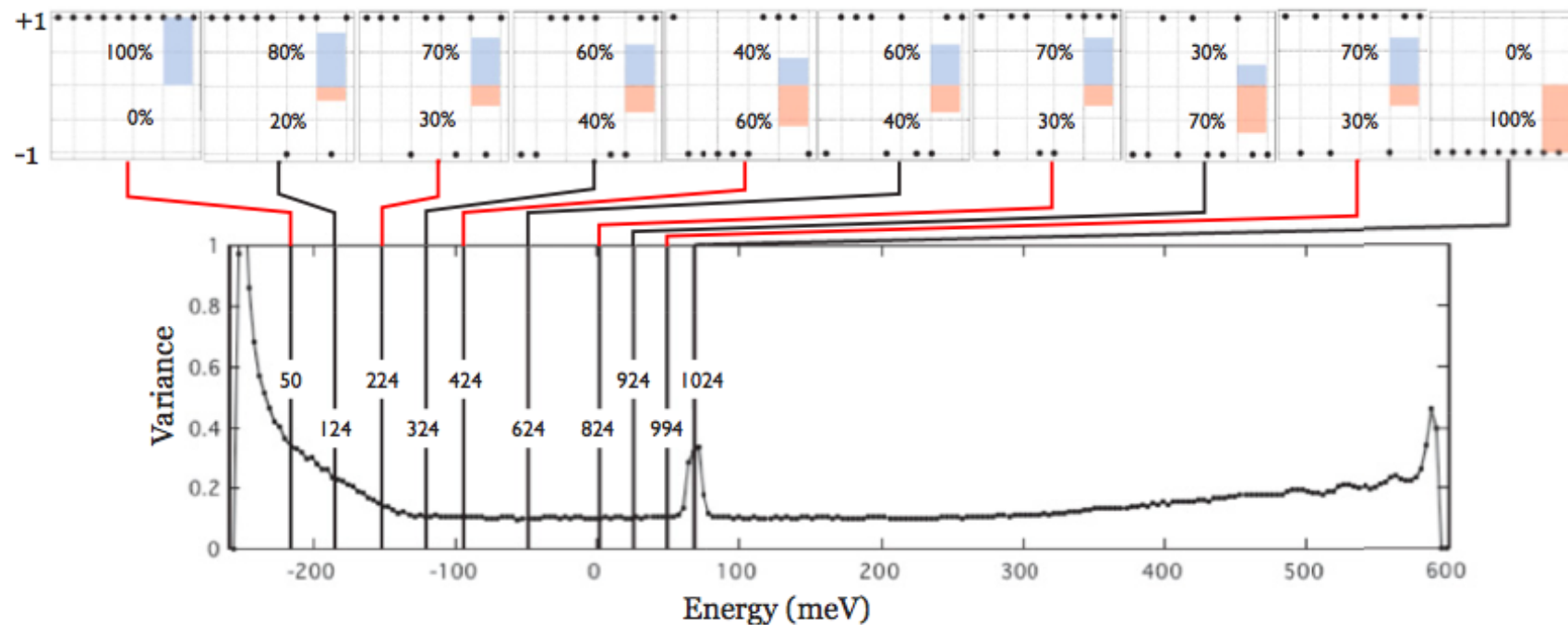


FIG. 5. (Color online) The upper panels show the \mathbb{Z}_2 invariant computed along the path (1) of Fig. 4 on an $8 \times 8 \times 8$ unit-cell lattice via twisted boundary conditions. The dimension of the occupied space was slowly reduced from 1024 to 124, as indicated in the figure. For each \mathbb{Z}_2 , the calculation was repeated for 10 random disorder configurations and the output is shown by the full dots, exactly how it occurs in the actual calculation. The percentages of the $\mathbb{Z}_2 = \pm 1$ occurrences are displayed in each panel. The lower panel shows the variance of the level spacings for $W = 300$ meV, and the averaged Fermi levels (see the vertical lines) corresponding to each \mathbb{Z}_2 calculation.

Classifications

E. PRODAN, H. SCHULZ-BALDES, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer 2016.

- Fundamental symmetries *time reversal symmetry (TRS)*, *particle-hole symmetry (PHS)*, and *chiral symmetry (CHS)*, generate a finite group with 10 inequivalent irreducible representations (*Zirnbauer '96, Atland-Zirnbauer '97*). They give a *classification* of 3D-topological insulators (*Schneider-Ryu-Furusaki-Ludwig '08*).
- *Real*, *Complex*, and *Quaternionic* K-theory give a topological classification into 10 classes (*Kitaev '09*), in any dimension.
- The *Bott Periodicity Theorem* on the stable homotopy of groups (*Bott '57,'59*), allows to identify K_i with K_{i+2} in complex K-theory, and K_i with K_{i+8} in real or quaternionic K-theory. It can also be expressed through the classification of *Clifford Modules* (*Atiyah-Bott-Shapiro '64*) liable to allow disorder.

Classifications

E. PRODAN, H. SCHULZ-BALDES, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer 2016.

j	TRS	PHS	CHS	CAZ	$d = 0, 8$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
0	0	0	0	A	\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}	
1	0	0	1	AIII		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}
0	+1	0	0	AI	\mathbb{Z}				$2\mathbb{Z}$		\mathbb{Z}_2	\mathbb{Z}_2
1	+1	+1	1	BDI	\mathbb{Z}_2	\mathbb{Z}				$2\mathbb{Z}$		\mathbb{Z}_2
2	0	+1	0	D	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}				$2\mathbb{Z}$	
3	-1	+1	1	DIII		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}				$2\mathbb{Z}$
4	-1	0	0	AII	$2\mathbb{Z}$		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}			
5	-1	-1	1	CII		$2\mathbb{Z}$		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}		
6	0	-1	0	C			$2\mathbb{Z}$		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	
7	+1	-1	1	CI				$2\mathbb{Z}$		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}

Table 2.1 Classification table of strong topological insulator and superconductors. Each row represents a universal symmetry class, defined by the presence (1 or ± 1) or absence (0) of the three symmetries: time-reversal (TRS), particle-hole (PHS) and chiral (CHS), and by how TRS and PHS transformations square to either +1 or -1. Each universality class is identified by a Cartan-Altland-Zirnbauer (CAZ) label. The strong topological phases are organized by their corresponding symmetry class and space dimension $d = 0, \dots, 8$. These phases are in one-to-one relation with the elements of the empty, \mathbb{Z}_2 , \mathbb{Z} or $2\mathbb{Z}$ groups. The table is further divided into the complex classes A and AIII (top two rows), which are the object of the present study, and the real classes AI, \dots , CI (the remaining 8 rows).

Edge States

J. KELLENDONK, *Ann. Henri Poincaré*, **20**(6), 2039-2070, (2019)

- The *bulk-boundary correspondence* has been established in the language of *K-theory* from 2000 onward for the Quantum Hall Effect (*Schulz-Baldes, Kellendonk & Richter '00, '02, Kellendonk & Schulz-Baldes '04*).
- A fascinating application to atomic physics is the *K-theoretic* interpretation of the *Levinson Theorem* (*Kellendonk & Richard '06, '08, '12*). It was later extended to the case of a *localized impurity* in a perfect crystal (*Bellissard & Schulz-Baldes '12*).
- At the recent work by *Kellendonk & Prodan* above, the bulk-boundary correspondence has been rigorously studied for an idealization of a *1D-quasicrystal*, the *Kohmoto model*. In particular the *existence of edge states* was established.

Open Problems

- Is the definition of ν amenable to an index theory valid also for *disordered systems* ?
- Atiyah-Singer developed a \mathbb{Z}_2 -index theory for *antisymmetric Fredholm* operators. Is ν equal to such an index ?
- Can one established the *persistence of edge states* under switching on the *disorder*? Even when localized states *fill the gap* at the Fermi level ?

IV - MAGNETO-ELECTRIC RESPONSE

Main References

Periodic Case:

R. D. KING-SMITH, D. VANDERBILT, *Phys. Rev. B*, **47**, (1993), 1651-1654

X. L. QI, T. L. HUGHES, S. C. ZHANG, *Phys. Rev. B*, **78**, (2008), 195424

A. M. ESSIN, J. E. MOORE, D. VANDERBILT, *Phys. Rev. Lett.*, **102**, (2009), 146805

A. M. ESSIN, A. M. TURNER, J. E. MOORE, D. VANDERBILT, *Phys. Rev. B*, **81**, (2010), 205104

A. MALASHEVICH, I. SOUZA, S. COH, D. VANDERBILT, *New J. Phys.*, **12**, (2010), 053032

Disordered Case:

B. LEUNG, E. PRODAN, *J. Phys. A: Math. and Theor.*, **46**, (2013), 085205

H. SCHULZ-BALDES, S. TEUFEL, *Comm. Math. Phys.*, **319**, (2013), 649-681

E. PRODAN, B. LEUNG, J. BELLISSARD, *J. Phys. A: Math. and Theor.*, **48**, (2013), 485202

Magneto-Electric Response Coefficient

- The *magnetization* \vec{M} induced by the electronic *orbital* motion induced by a small *electric field* \vec{E} is given to lowest order by the response coefficient at zero magnetic field \vec{B}

$$\alpha_{ij} = \left. \frac{\partial M_j}{\partial E_i} \right|_{\vec{B}=0}$$

- Equivalently the electric *polarization* \vec{P} of the orbital motion induced by a small *magnetic field* is also given by the same coefficient

$$\alpha_{ij} = \left. \frac{\partial P_i}{\partial B_j} \right|_{\vec{E}=0}$$

Magneto-Electric Response Coefficient

- For a periodic 3D-insulator, with Bloch functions $u_m(k)$ representing the *occupied bands* let \mathcal{A}_j^{mn} be the *Berry connection* defined by

$$\mathcal{A}_j^{mn}(k) = \left\langle u_m(k) \left| \frac{\partial}{\partial k_j} \right| u_n(k) \right\rangle$$

- Then the *tracial part* is given by (Qi, Hughes, Zhang '06)

$$\theta \stackrel{\text{def}}{=} \frac{1}{3} \sum_{i=1}^3 \alpha_{ii} = \frac{1}{2\pi} \int_{\mathbb{B}} d^3k \epsilon^{ijl} \text{Tr} \left[\mathcal{A}_i \partial_j \mathcal{A}_l - \frac{2i}{3} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_l \right]$$

- The *r.h.s.* is a topological invariant called the *Chern-Simons action*.

Polarization

- There is a problem in *defining unambiguously* the electric polarization of a solid.
- However the *change* of the polarization under an adiabatic evolution can be defined through *perturbation theory* (King-Smith & Vanderbilt '93). (Ex.: piezoelectric, magnetoelectric response)
- The *adiabatic* variation of the polarization is given by

$$\frac{\partial \vec{P}}{\partial \lambda} = \mathcal{T}_{\mathbb{P}} [\rho(\lambda) \vec{J}(\lambda)] \quad \vec{J}(\lambda) = i [H(\lambda), \vec{X}] \quad (\text{charge current})$$

where $\rho(\lambda)$ is the adiabatic evolution of the *density matrix* defining the quantum state of the system.

Polarization

- Let $\lambda \in [0, 1] \mapsto H(\lambda)$ be a *smooth* adiabatic evolution of the Hamiltonian. It will be assumed that a *spectral gap persists* along the way at the Fermi level and that $\rho(\lambda = 0) = P_F$.
- The adiabatic evolution is driven by

$$i\epsilon \frac{\partial \rho}{\partial \lambda} = [H(\lambda), \rho(\lambda)] \quad \epsilon \downarrow 0$$

- Then (*King-Smith & Vanderbilt '83, Schulz-Baldes & Teufel '13*)

$$\Delta \vec{P} = \int_0^1 d\lambda \frac{\partial \vec{P}}{\partial \lambda} = \int_0^1 d\lambda \mathcal{T}_{\mathbb{P}} \left(P_F(\lambda) \left[\partial_\lambda P_F(\lambda), \vec{\nabla} P_F(\lambda) \right] \right) + O(\epsilon^\infty)$$

Magnetization

- A similar formula can be established for the $3D$ -magnetization \vec{M} at zero temperature (*Schulz-Baldes & Teufel '13*)

$$M_i = \frac{i \epsilon^{ijk}}{2} \mathcal{T}_{\mathbb{P}} \left(|E_F - H| \left[\partial_j P_F, \partial_k P_F \right] \right)$$

- The previous formula holds also if the *Fermi level* belongs to a *mobility gap*, namely if $P_F \in \mathcal{S}$ is Sobolev (*Schulz-Baldes & Teufel '13*).

Magneto-Electric Response Coefficient

- Differentiating the polarization *w.r.t.* to the magnetic field can be done using a *Itô derivative* δ_B acting on the observable algebra

(JB '88)

- It gives (Leung & Prodan '13)

$$\theta = \frac{1}{2} \int_0^1 d\lambda \mathcal{T}_{\mathbb{P}} (P_F dP_F dP_F dP_F dP_F) + \frac{1}{3} \mathcal{T}_{\mathbb{P}} ((1 - 2P_F) dP_F \delta_B P_F) \Big|_{\lambda=0}^{\lambda=1}$$

- Here λ is introduced as a *fourth dimension* and then

$$df = \partial_{\lambda} f d\lambda + \sum_{i=1}^3 \partial_i f dk_i$$

Magneto-Electric Response Coefficient

- If γ is *adiabatic smooth paths* in the the Hamiltonian space, let $\Theta\gamma$ be the path obtained in the Hamiltonian space by *time-reversal symmetry*. Then $\gamma - \Theta\gamma$ is a *loop* in the Hamiltonian space.
- The magneto-electric response is then given by (*Qi, Hughes, Zhang '06, Leung & Prodan '13*)

$$\Delta\theta(\gamma - \Theta\gamma) = \frac{1}{2} \oint_{\gamma - \Theta\gamma} d\lambda \mathcal{T}_{\mathbb{P}}(P_F dP_F dP_F dP_F dP_F) = \frac{1}{2} \mathbf{Ch}_2(P_F)$$

- This formula is still valid provided P_F belongs to the Sobolev space $\mathcal{W}^{2,4}(\mathcal{A}, \mathcal{T}_{\mathbb{P}})$ (*Leung, Prodan, JB '13*)

$$\mathcal{W}^{2,4}(\mathcal{A}, \mathcal{T}_{\mathbb{P}}) = \left\{ A \in \mathcal{A}; \mathcal{T}_{\mathbb{P}}(|A|^2) + (|\vec{\nabla} A|^4)^{1/2} < \infty \right\}$$

Sobolev Norm & Localization

- The expression

$$\ell_4(E_F) = \left(|\vec{\nabla} P_F|^4 \right)^{1/4}$$

has the dimension of a *length* and can be seen as a milder version of the *localization length* at the Fermi level.

- **Theorem** *In the case of strong enough independent disorder at each lattice site, the Aizenman-Molčanov technique applies to prove that $\ell_4(E_F)$ is finite so that $P_F \in \mathcal{W}^{2,4}$*

Conclusion

- *the second Chern number $\mathbf{Ch}_2(P_F)$ is an integer. (Connes '83)*
- *The magneto-electric response $\Delta\theta(\gamma - \Theta\gamma)$ at zero magnetic field along an adiabatic loop of the Hamiltonian, is quantized and is given by half the second Chern number of the Fermi projection.*
- *$\Delta\theta(\gamma - \Theta\gamma)$ is either an integer or half an integer; both character survive in the limit of zero magnetic field, namely for topological insulators with non trivial \mathbb{Z}_2 invariant there are loops for which $\Delta\theta(\gamma - \Theta\gamma)$ is half an integer.*
- *This quantization survives if the solid is disordered whenever the Fermi level belongs to a mobility gap.*



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Thanks for listening !