NONCOMMUTATIVE GEOMETRY of FRACTALS

Sponsoring

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Content

I - Spectral Triples

Spectral Triples

A *spectral triple* is a family $(\mathcal{H}, \mathcal{A}, D)$, such that

- $\mathcal{H}$ is a Hilbert space
- D is a self-adjoint operator on $\mathcal{H}$ with *compact resolvent.*
- $\mathcal{A}$ is a *unital* C^*-algebra with a *faithful* representation π into $\mathcal{H}$
- There is a *core* $\mathcal{D}$ in the domain of D, and a *dense* $*$-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ such that if $a \in \mathcal{A}_0$ then $\pi(a)\mathcal{D} \subset \mathcal{D}$ and $\|[D, \pi(a)\| < \infty\}$.

- $(\mathcal{H}, \mathcal{A}, D)$ is called *even* if there is $G \in \mathcal{B}(\mathcal{H})$ such that
  - $G = G^* = G^{-1}$
  - $[G, \pi(f)] = 0$ for $f \in \mathcal{A}$
  - $GD = -DG$

Example

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the *torus*, $\mathcal{A} = C(\mathbb{T})$, $\mathcal{H} = L^2(\mathbb{T})$, where $\mathcal{A}$ acts by pointwise multiplication and $D = -\imath d/dx$

Then, if $\|x - y\| = \inf_{l \in \mathbb{Z}} |x - y + l|$,

$$\|[D, f]\| = \text{ess-sup} \left| \frac{df}{dx} \right| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} = \|f\|_{Lip}$$

and

$$\|x - y\| = \sup \left\{ \left| f(x) - f(y) \right| \; ; \; \|f\|_{Lip} \leq 1 \right\}$$

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

$$d(x, y) = \sup\{|f(x) - f(y)| \,;\, f \in \mathcal{A},\ \|[D, f]\| \leq 1\}$$

Actually $\|[D, f]\| = \|\nabla f\|_{L^\infty} = \|f\|_{C_{\mathrm{Lip}}}$ *and* $C^1(X) = \mathrm{Lip}(M)$.

Hence the algebra $\mathcal{A}$ encodes the *space*, the Dirac operator D encodes the *metric*. $\mathcal{H}$ is needed to define D.

Spectral Metric Spaces

Definition *A* spectral metric space *is a spectral triple* $(\mathcal{H}, \mathcal{A}, D)$ *such that*
(i) the commutant $\mathcal{A}' = \{a \in \mathcal{A}\,;\, [D, \pi(a)] = 0\}$ *is trivial,* $\mathcal{A}' = \mathbb{C}\mathbf{1}$
(ii) the Lipshitz ball $B_{Lip} = \{a \in \mathcal{A}\,;\, \|[D, \pi(a)]\| \leq 1\}$ *is precompact in* $\mathcal{A}/\mathcal{A}'$

Theorem [Rieffel '99] *A spectral triple* $(\mathcal{H}, \mathcal{A}, D)$ *is a spectral metric space if and only if the Connes metric, defined on the state space of* $\mathcal{A}$ *by*

$$d_C(\omega, \omega') = \sup\{|\omega(a) - \omega'(a)|\,;\, \|[D, \pi(a)]\| \leq 1\}$$

is well defined and equivalent to the weak-topology*

ζ-function and Spectral Dimension

Definition *A spectral metric space* $(\mathcal{H}, \mathcal{A}, D)$ *is called* summable *is there is* $p > 0$ *such that* $\mathrm{Tr}\,(|D|^{-p}) < \infty$. *Then, the* ζ*-function is defined as*

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

The *spectral dimension* is

$$s_D = \inf\left\{s > 0\,;\ \mathrm{Tr}\left(\frac{1}{|D|^s}\right) < \infty\right\}$$

Then ζ is *holomorphic* in $\mathfrak{Re}(s) > s_D$

Remark *For a Riemannian manifolds* $s_D = \dim(M)$

Connes trace & Volume Form

The spectral metric space is *spectrally regular* if the following limit is unique

$$\omega_{D}(a) = \lim_{s \downarrow s_{D}} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^{s}} \pi(a)\right) \qquad\qquad a \in \mathcal{A}$$

Then ω_{D} is a *trace* called the *Connes trace*.

Remark

(i) By compactness, limit states always exist, but the limit may not be unique.
(ii) Even if unique this state might be trivial.
(iii) In the example of compact Riemannian manifold the Connes state exists and defines the volume form.

Hilbert Space

If the Connes trace is well defined, it induces a *GNS-representation* as follows

- The Hilbert space $L^2(\mathcal{A}, \omega_D)$ is defined from $\mathcal{A}$ through the inner product

$$\langle a|b\rangle = \omega_D(a^* b)$$

- The algebra $\mathcal{A}$ acts by *left multiplication*.

Laplacian ?

How could one define a Laplacian ? The following might be a way. If the quadratic form

$$Q(a,b) = \lim_{s \downarrow s_D} \frac{1}{\zeta(s)} \operatorname{Tr}\left(\frac{1}{|D|^s} [D,\pi(a)]^*[D,\pi(b)]\right)$$

extends to $L^2(\mathcal{A},\omega_D)$ as a *closable quadratic form*, then, it defines a positive operator which generates a *Markov semi-group* and is a candidate for being the analog of the *Laplace-Beltrami operator*.

Laplacian ?

The example of the *Sierpinsky gasket* shows that it should be rather something like

$$Q(a,b) = \lim_{s \downarrow s_D} \frac{1}{\zeta(s)} \operatorname{Tr}\left(\frac{1}{|D|^s}[D^\alpha, \pi(a)]^*[D^\alpha, \pi(b)]\right)$$

for a suitable value of the exponent α.

Problem: *Can one define geometrically the exponent* α *for a compact metric space ?*

II - Compact Metric Spaces

I. Palmer, *Noncommutative Geometry of compact metric spaces*, PhD Thesis, May 3rd, 2010.

Open Covers

Let (X, d) be a *compact metric space* with an infinite number of points. Let $\mathcal{A} = C(X)$.

- An *open cover* $\mathcal{U}$ is a family of open sets of X with union equal to X. Then $\operatorname{diam}\mathcal{U} = sup\{\operatorname{diam}(U)\,;\, U \in \mathcal{U}\}$. All open covers used here will be at most *countable*
- A *resolving sequence* is a family $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \operatorname{diam}(\mathcal{U}_n) = 0$$

- A resolving sequence is *strict* if all $\mathcal{U}_n$'s are finite and if

$$\operatorname{diam}(\mathcal{U}_n) < \inf\{\operatorname{diam}(U)\,;\, U \in \mathcal{U}_{n-1}\} \quad \forall n$$

Choice Functions

Given a resolving sequence $\xi = (\mathcal{U}_n)_{n\in\mathbb{N}}$ a *choice function* is a map $\tau : \mathcal{U}(\xi) = \coprod_n \mathcal{U}_n \mapsto X \times X$ such that

- $\tau(U) = (x_U, y_U) \in U \times U$
- there is $C > 0$ such that

$$\mathrm{diam}(U) \geq d(x_U, y_U) \geq \frac{\mathrm{diam}(U)}{1 + C\,\mathrm{diam}(U)} \qquad \forall U \in \mathcal{U}(\xi)$$

The *set* of such choice functions is denoted by $\Upsilon(\xi)$.

A Family of Spectral Triples

- Given a *resolving sequence* ξ, let $\mathcal{H}_\xi = \ell^2(\mathcal{U}(\xi)) \otimes \mathbb{C}^2$

- For τ a *choice* let $D_{\xi,\tau}$ be the *Dirac operator* defined by

$$D_{\xi,\tau}\psi\,(U) = \frac{1}{d(x_U, y_U)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

- For $f \in C(X)$ let $\pi_{\xi,\tau}$ be the *representation* of $\mathcal{A} = C(X)$ given by

$$\pi_{\xi,\tau}(f)\psi\,(U) = \begin{bmatrix} f(x_U) & 0 \\ 0 & f(y_U) \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

Regularity

Theorem *Each* $\mathfrak{T}_{\xi,\tau} = (\mathcal{H}_\xi, \mathcal{A}, D_{\xi,\tau}, \pi_{\xi,\tau})$ *defines a spectral metric space such that* $\mathcal{A}_0 = \mathrm{Lip}(X, d)$ *is the space of Lipshitz continuous functions on X. Such a triple is even when endowed with the grading operator*

$$G\psi(U) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi(U) \qquad \psi \in \mathcal{H}$$

In addition, the family $\{\mathfrak{T}_{\xi,\tau}\,;\, \tau \in \Upsilon(\xi)\}$ *is regular in that*

$$d(x,y) = \sup\{|f(x - f(y)|\,;\, \sup_{\tau \in \Upsilon(\xi)} \|[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]\| \leq 1\}$$

Summability

Theorem *There is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral metric spaces if and only if the Hausdorff dimension of* X *is finite.*

If so, the spectral dimension s_D *satisfies* $s_D \geq \dim_H(X)$.

If $\dim_H(X) < \infty$ *there is a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of summable spectral triples with spectral dimension* $s_D = \dim_H(X)$.

Hausdorff Measure

Theorem *There exist a resolving sequence leading to a family* $\mathfrak{T}_{\xi,\tau}$ *of spectrally regular spectral metric spaces if and only if the Hausdorff measure of X is positive and finite.*

In such a case the Connes state coincides with the normalized Hausdorff measure on X.

Then the Connes state is given by the following limit *independently* of the choice τ

$$\frac{\int_X f(x)\mathcal{H}^{s_D}(dx)}{\mathcal{H}^{s_D}(X)} = \lim_{s\downarrow s_D} \frac{1}{\zeta_{\xi,\tau}(s)} \operatorname{Tr}\left(\frac{1}{|D_{\xi,\tau}|^s}\, \pi_{\xi,\tau}(f)\right) \qquad f \in C(X)$$

III - The Pearson Laplacian

Directional Derivative, Tangent Space

If $\tau(U) = (x_U, y_U)$ then

$$[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]\,\psi\,(U) = \frac{f(x_U) - f(x_U)}{d(x_U, y_U)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(U)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(U)$ can be interpreted as a coarse grained version of a *unit tangent vector* at U.
- the set Υ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*.
- $[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]$ could be written as $\nabla_\tau f$.

Choice Averaging

- The *choice space* Υ is given by $\prod_U \Upsilon(U)$ where $\Upsilon(U)$ is a compact subset of $\overline{U} \times \overline{U}$.
- Let ν_U be the probability measure on $\Upsilon(U)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\overline{U} \times \overline{U}$.
- This leads to the probability

$$\nu = \bigotimes_U \nu_U$$

Hence ν_U can be interpreted as the average over the tangent unit sphere at U.

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(X)$)

$$Q_s(f,g) = \int_{\Upsilon} d\nu(\tau)\ \mathrm{Tr}\left(\frac{1}{|D_{\xi,\tau}|^s}[D_{\xi,\tau}, \pi_{\xi,\tau}(f)]^*\ [D_{\xi,\tau}, \pi_{\xi,\tau}(g)]\right)$$

Theorem: *If (X,d) is an ultrametric Cantor set, having a positive finite Hausdorff measure μ_H, for each $s \in \mathbb{R}$, the quadratic forms Q_s is densely defined, closable in $L^2(X,\mu_H)$ and is a Dirichlet form.*

The corresponding positive operator Δ_s has pure point spectrum. It is bounded if and only if $s > \dim_H(X) + 2$ and has compact resolvent otherwise.

The eigenspaces are common to all s's and can be explicitly computed.

Jump Process

Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in X. It can be shown to be a *jump process*.

In addition for most examples computed so far, like the triadic Cantor set *(Pearson, JB, '08)* of for the tiling space of a substitution tiling *(Julien, Savinien '11)*, the diffusion corresponding to $s = \dim_H(X) + 2$ satisfies

$$\mathbb{E}\left(d(X_{t+\delta}, X_t)^2\right) \stackrel{\delta\downarrow 0}{\to} const.\ \delta \ln(1/\delta)$$

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Thanks for Listening !