The Topology of Tiling Spaces

Jean BELLISSARD

Science at the (Cutting) Edge

Local business results for cutting edge near East Lansing, MI - Change location

The Topology of Tiling Spaces

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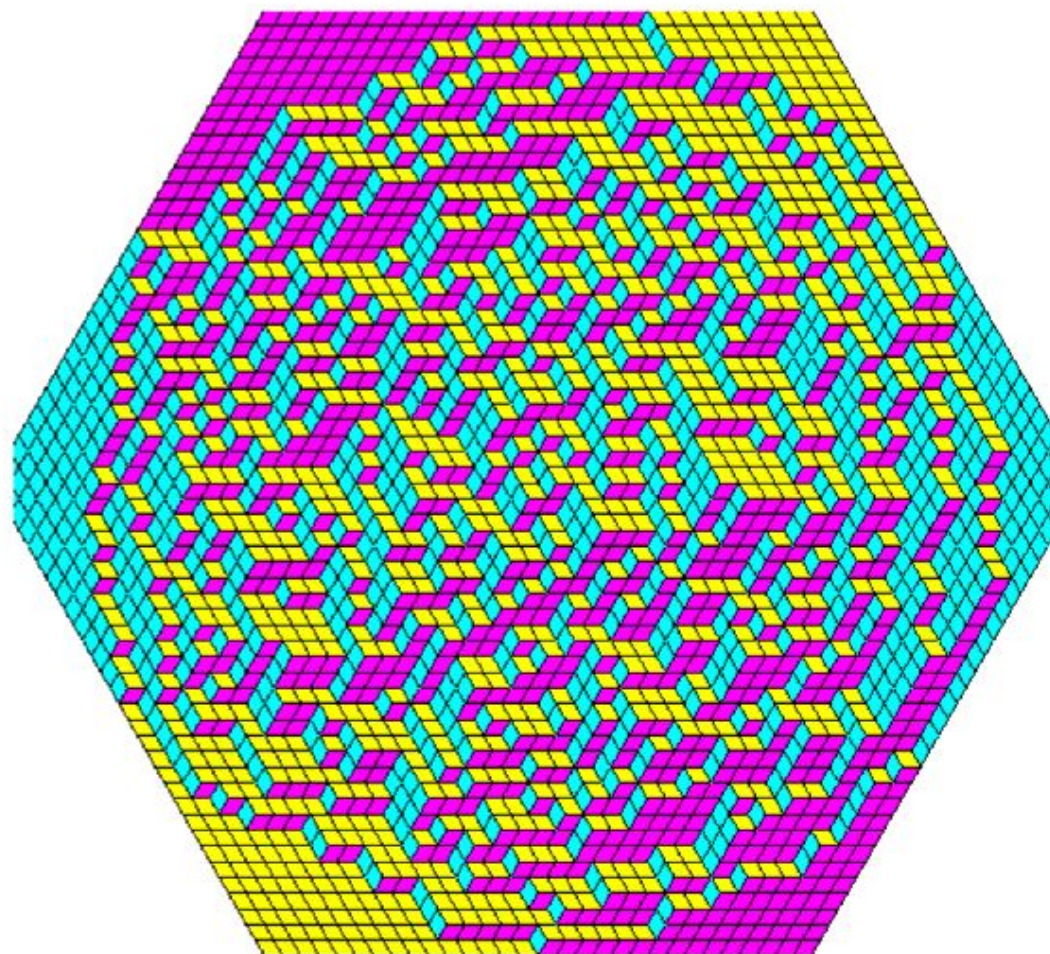arXiv:0705.2483, *submitted to* Ergod. Th. Dyn. Syst., 2007.

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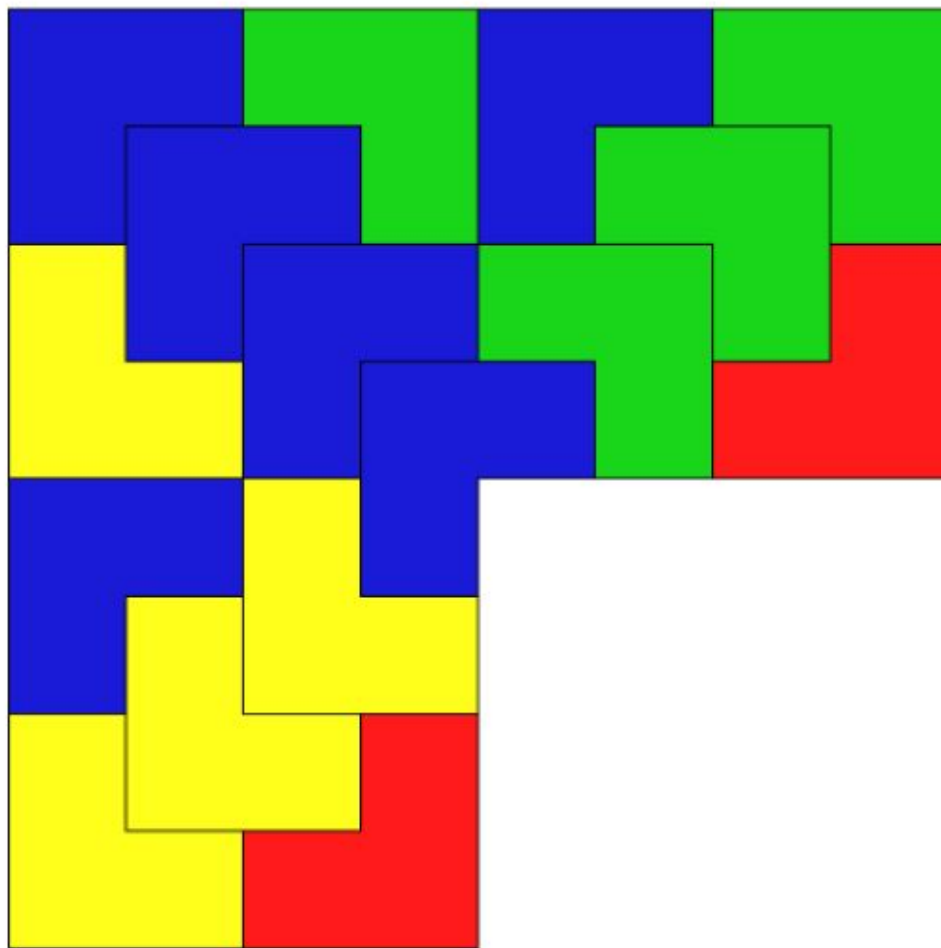
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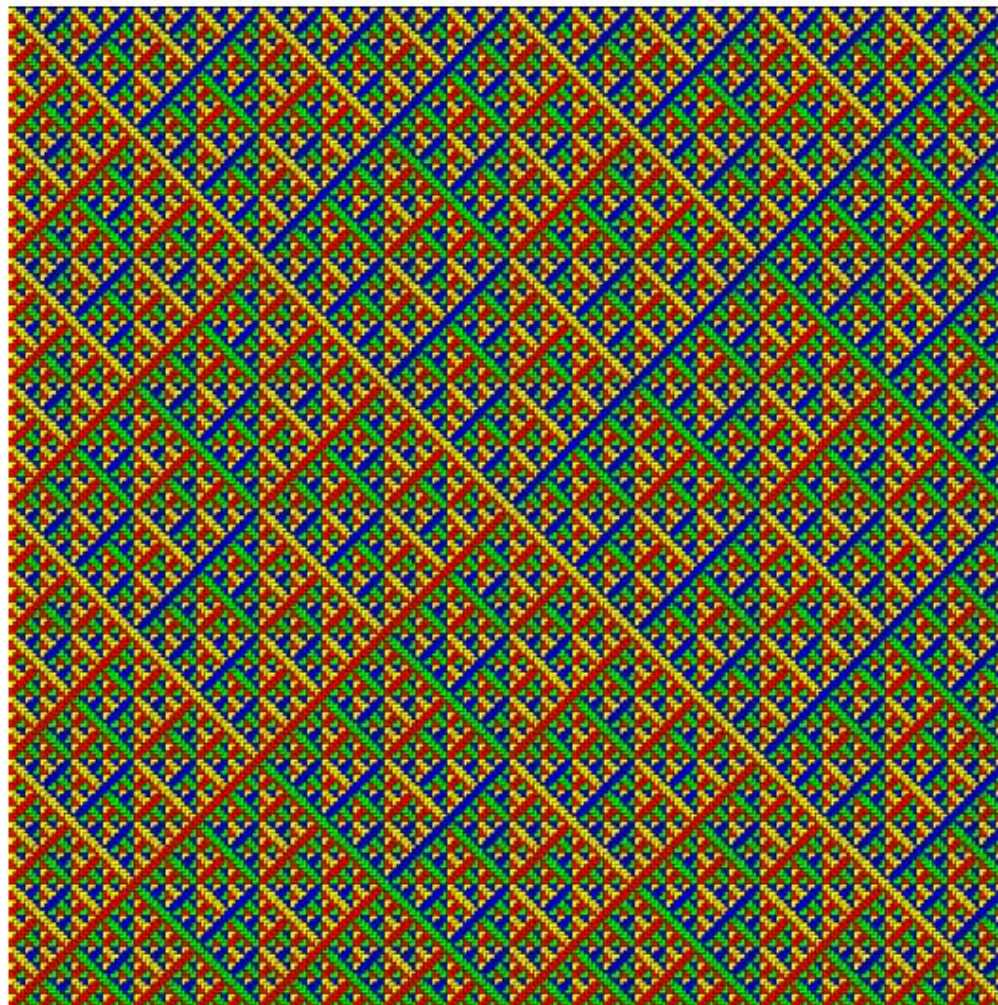
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I - Tilings, Tilings,...
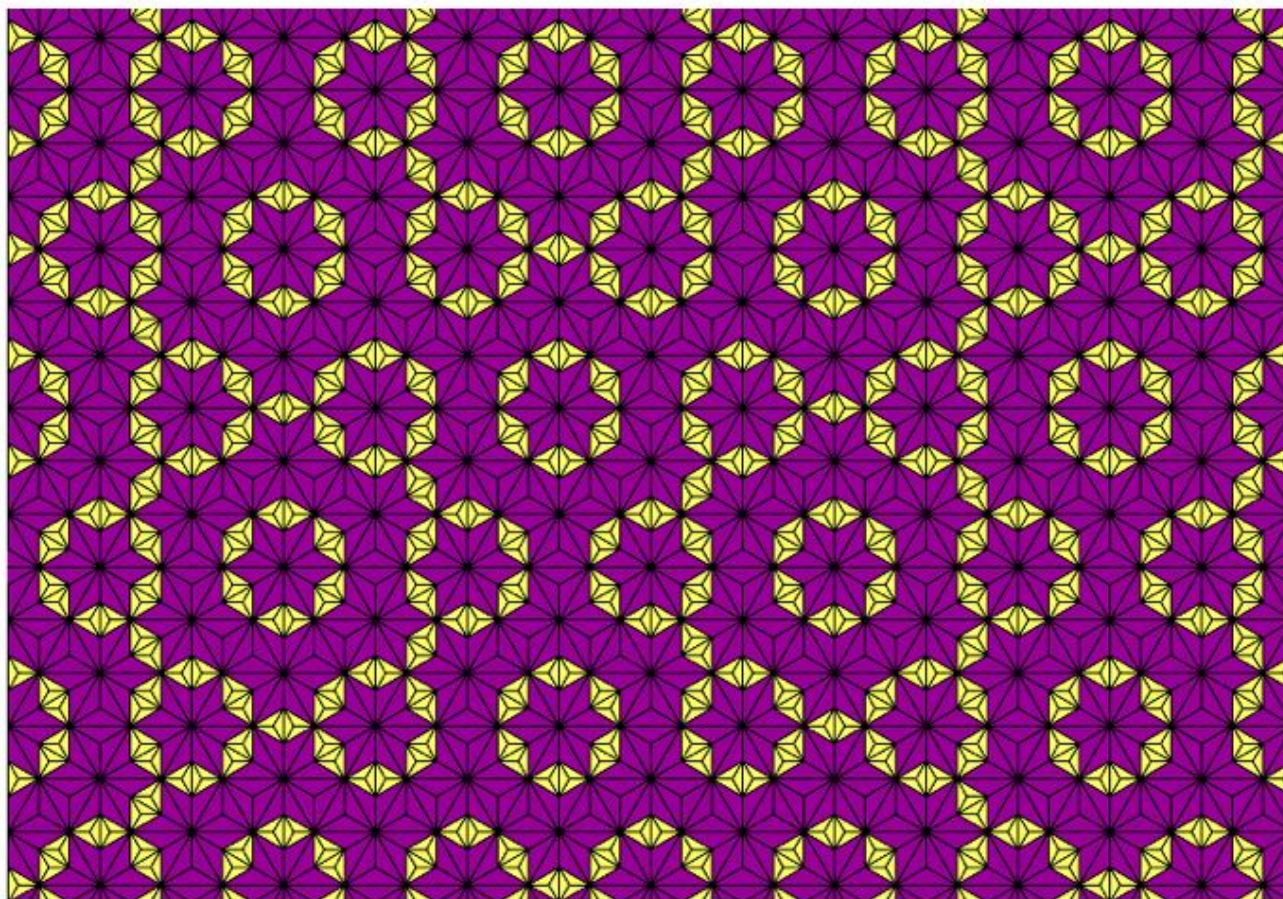
- A triangle tiling -

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- The chair tiling -

- The Penrose tiling -

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$\varphi^2 = \varphi + 1$$

$$\varphi = 1 + 1/\varphi$$

- Kites and Darts -

Golden Triangle

Golden Gnomon
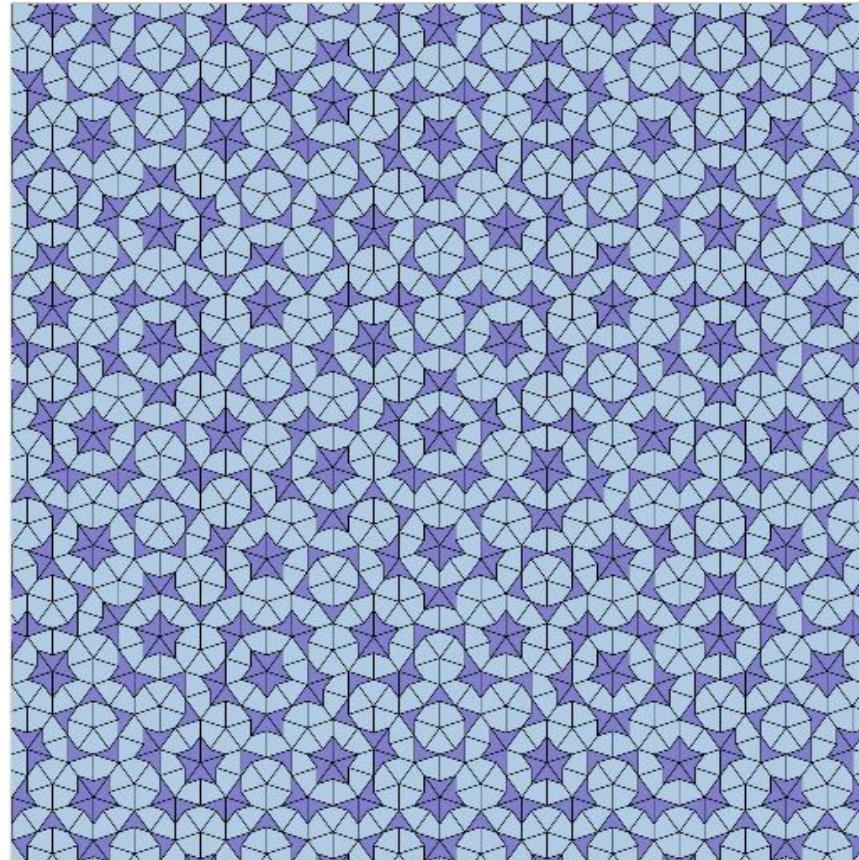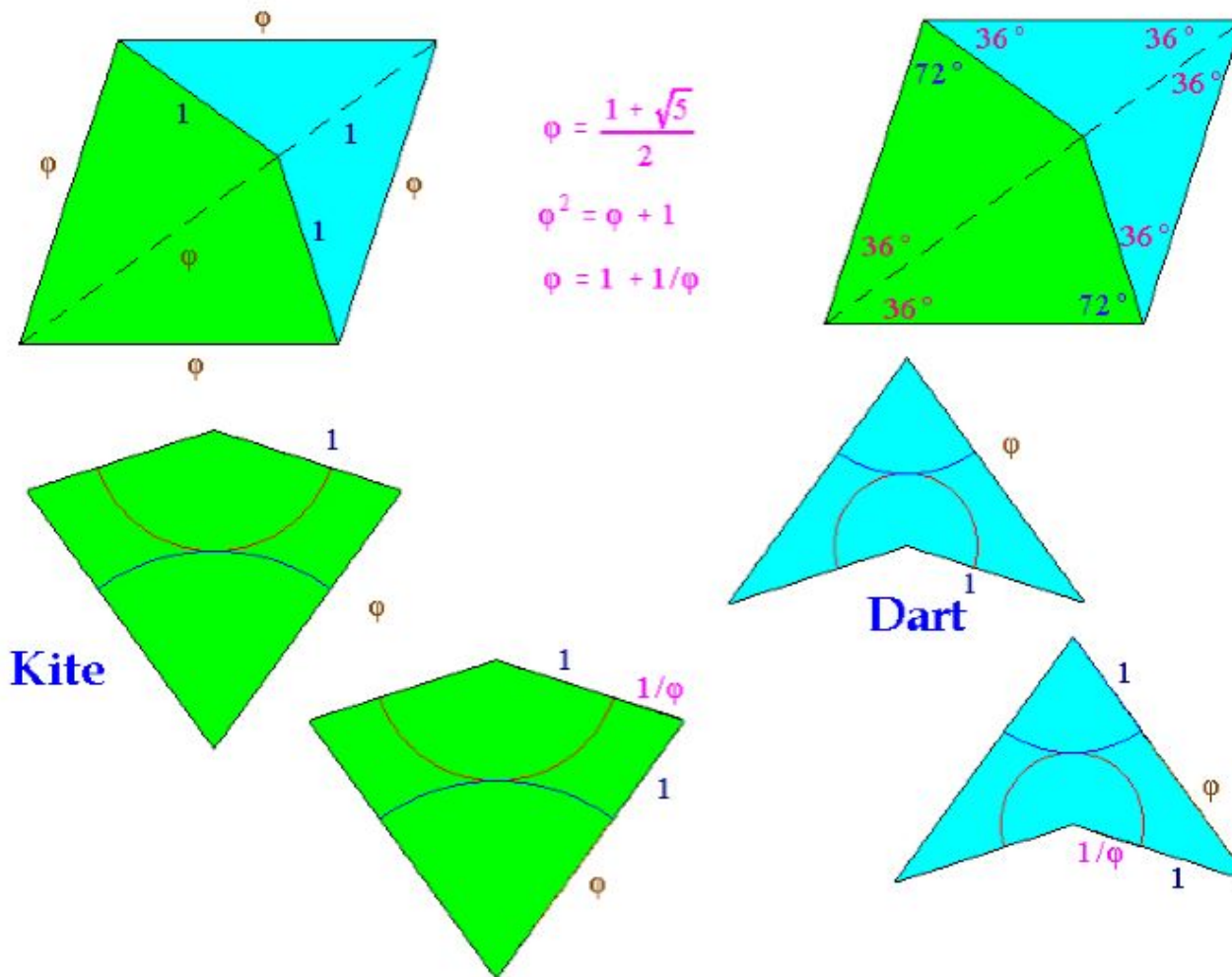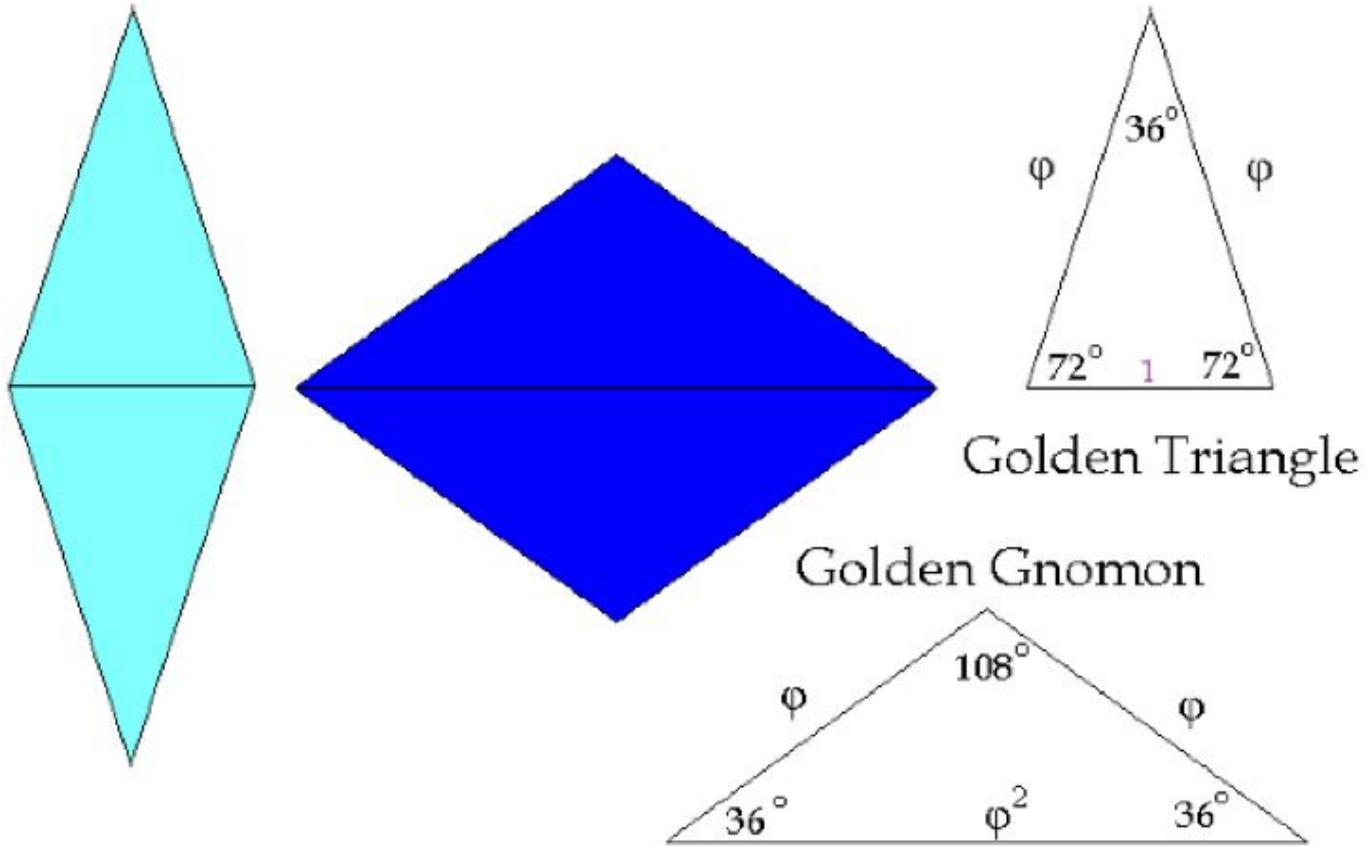
- Rhombi in Penrose's tiling -
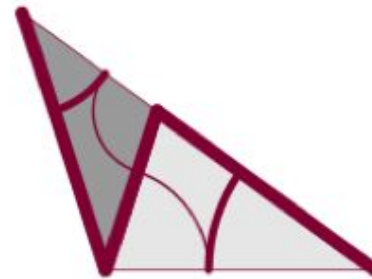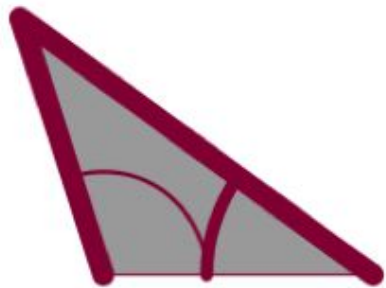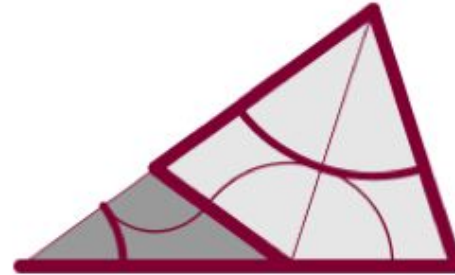
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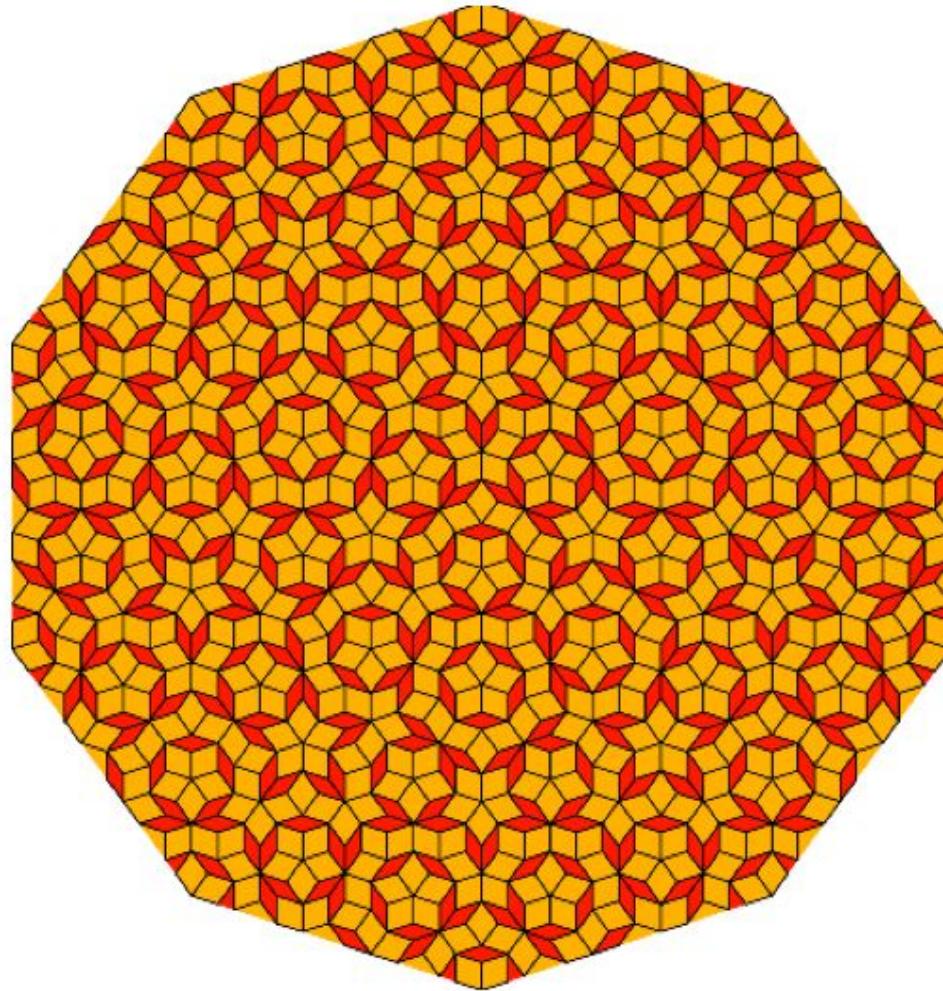
- The Penrose tiling -
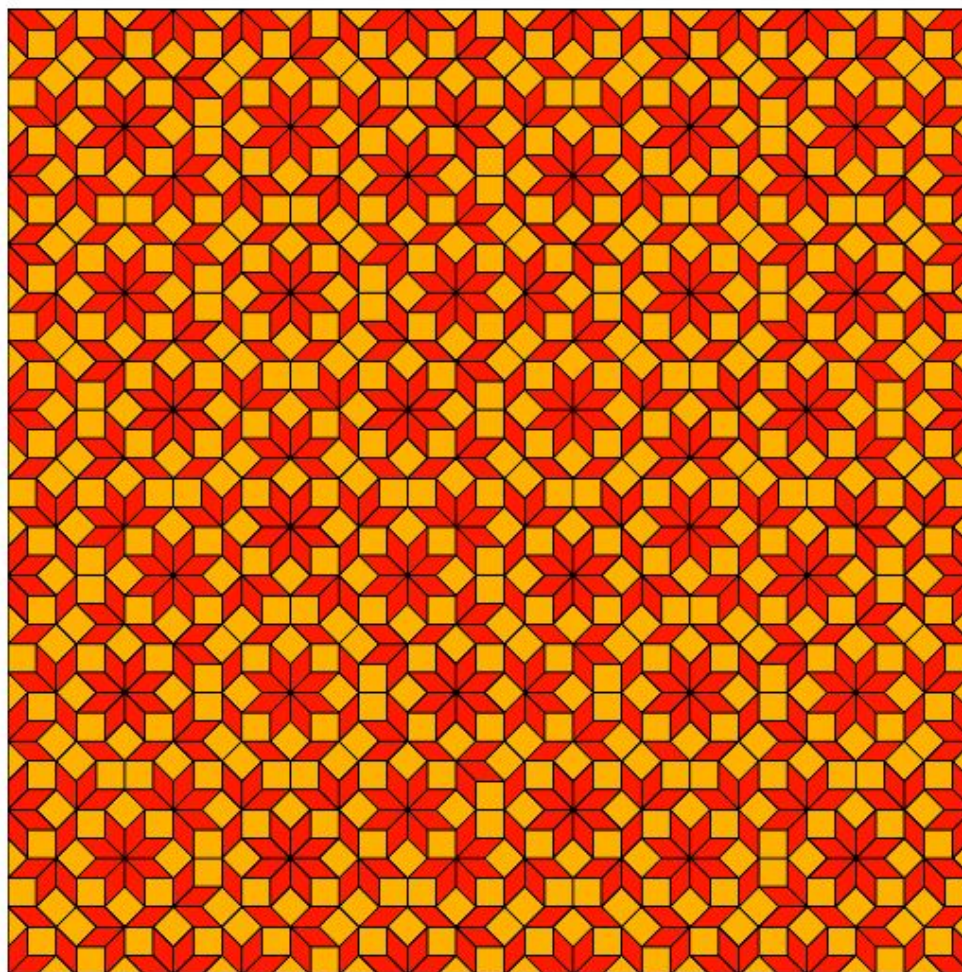
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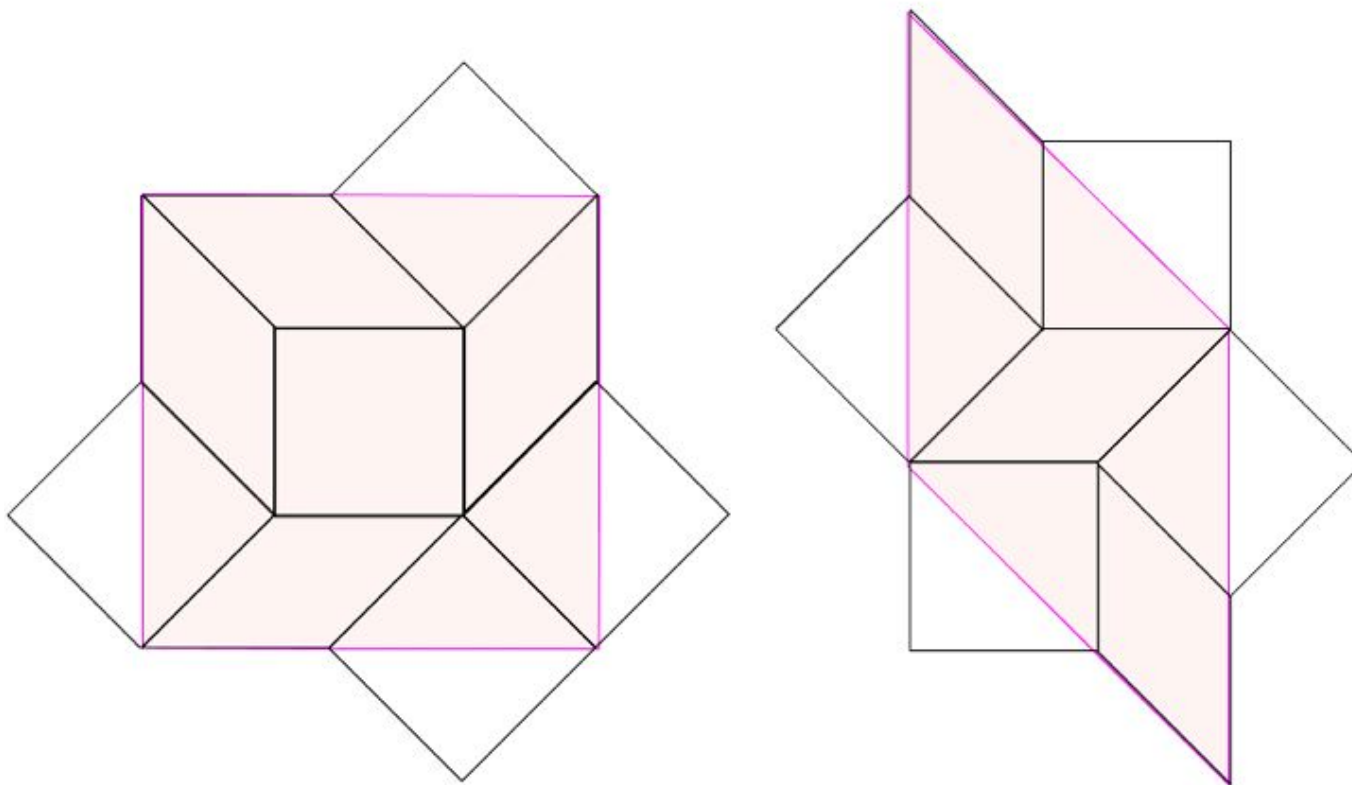
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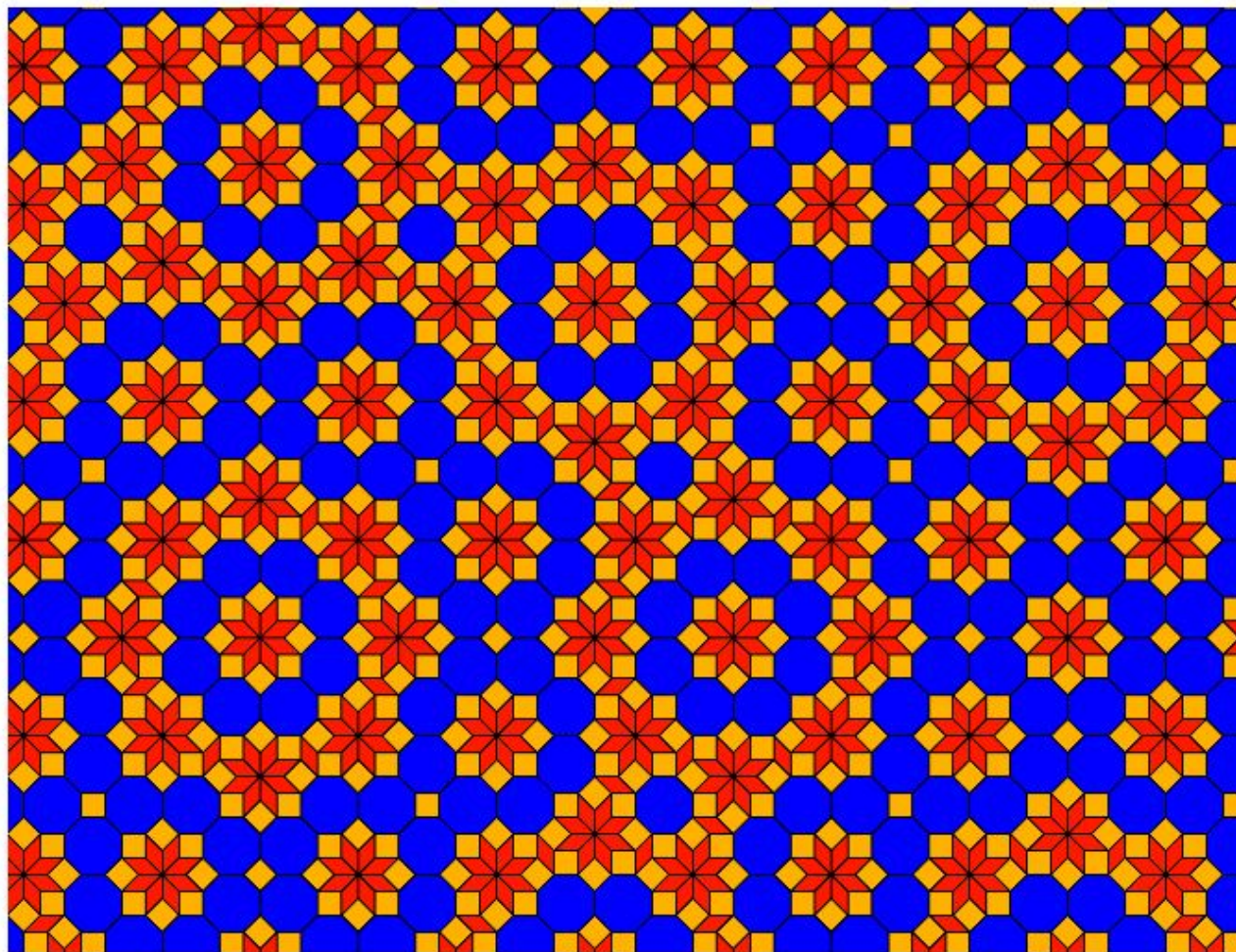
- Another octagonal tiling -
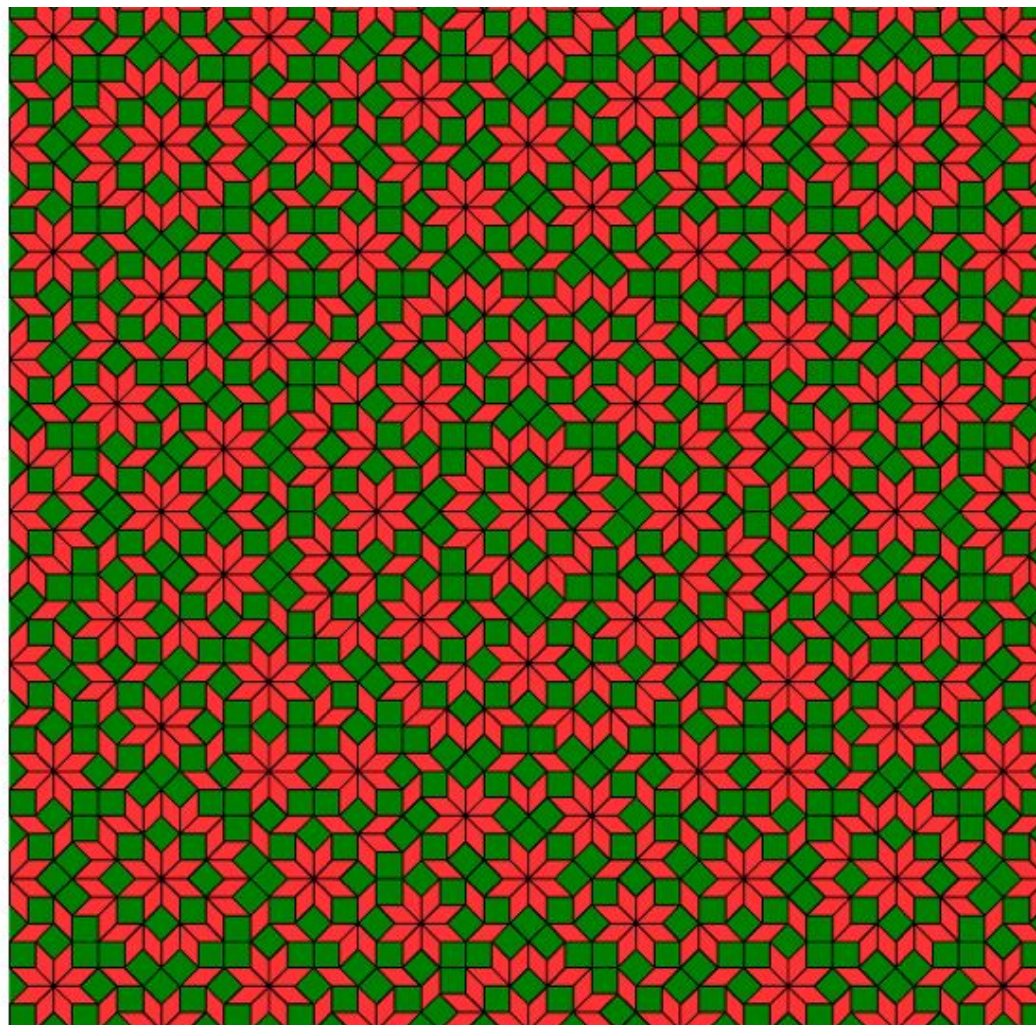
- Another octagonal tiling -
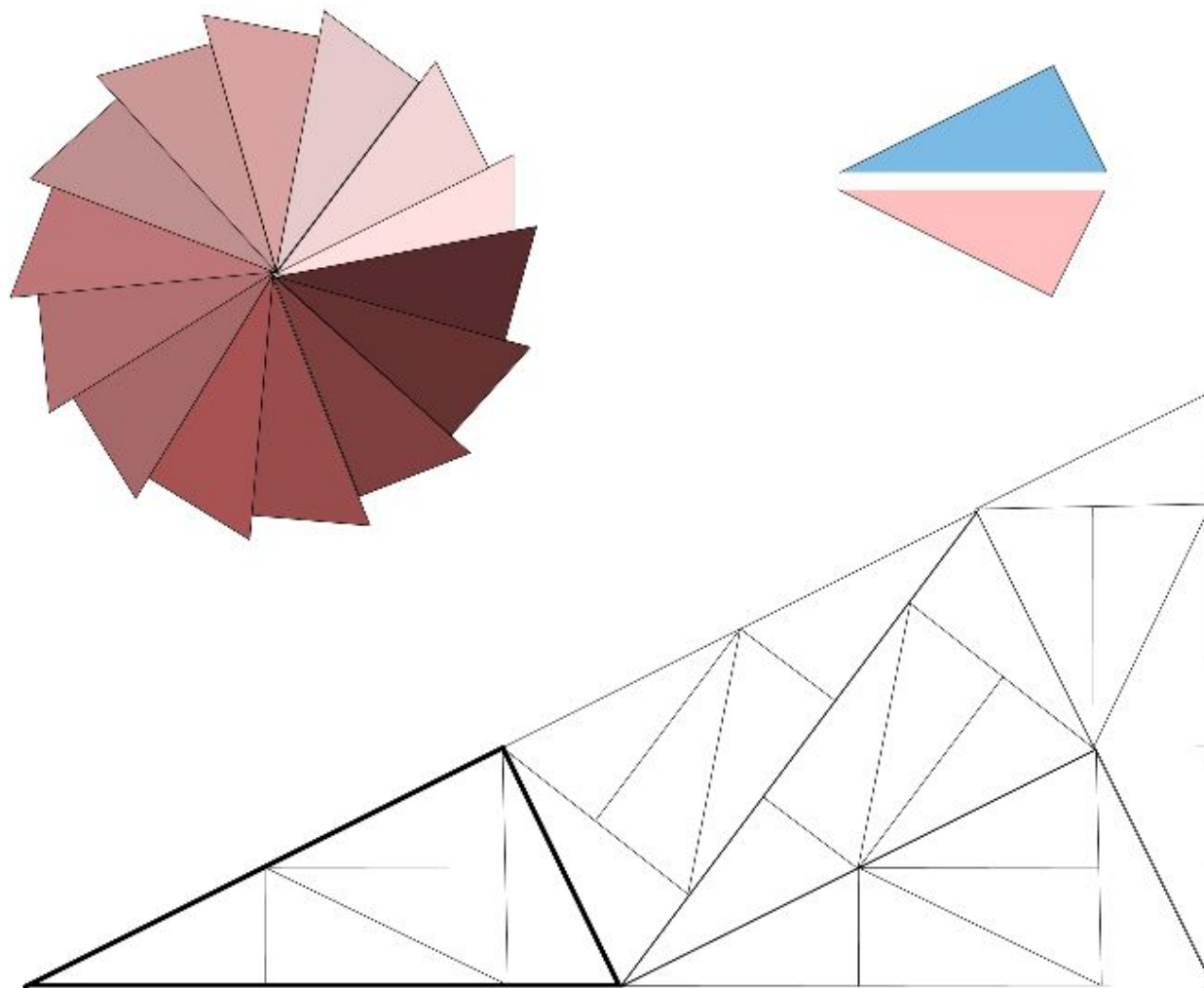
- Building the Pinwheel Tiling -
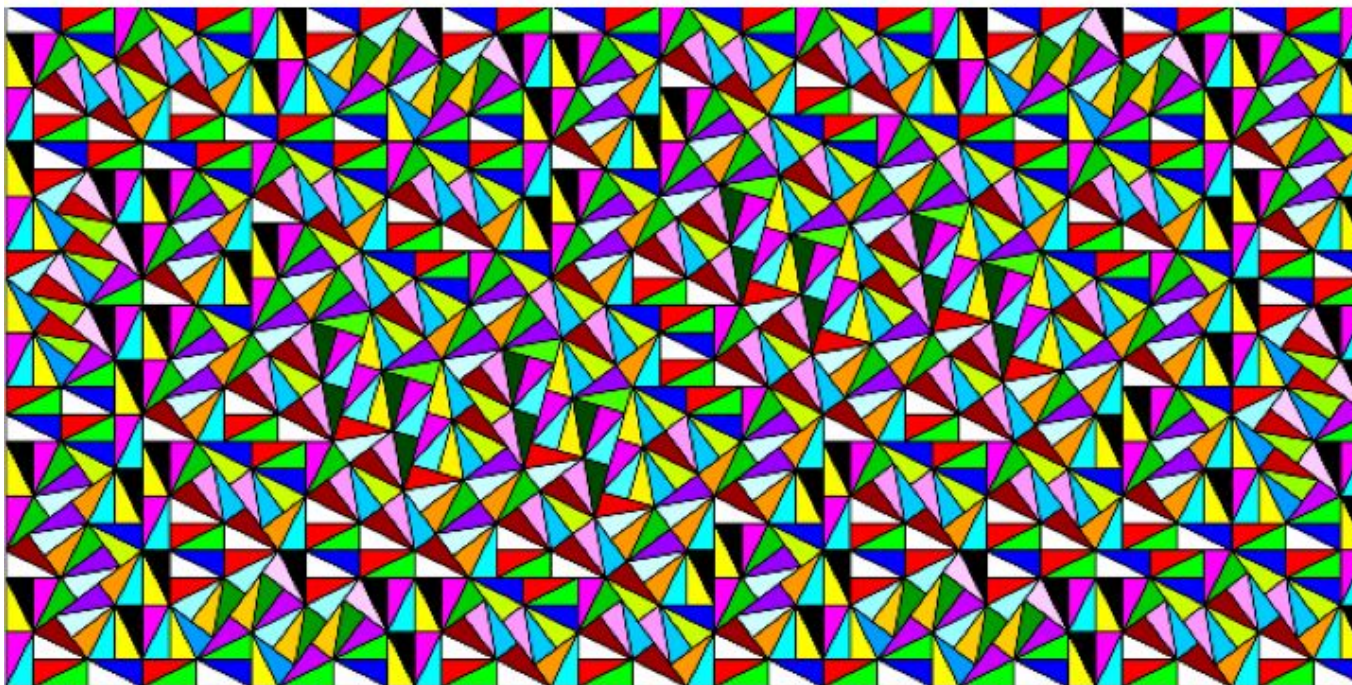
- The Pinwheel Tiling -

Aperiodic Materials

1. *Periodic Crystals* in d-dimensions:
   translation and crystal symmetries.
   Translation group $\mathcal{T} \simeq \mathbb{Z}^d$.

2. *Periodic Crystals in a Uniform Magnetic Field*;
   magnetic oscillations, Shubnikov-de Haas, de Haas-van Alfen.
   The magnetic field breaks the translation invariance to give some quasiperiodicity.

3. *Quasicrystals*: no translation symmetry, but icosahedral symmetry. Ex.:

   (a) $\mathbf{Al_{62.5}Cu_{25}Fe_{12.5}}$;

   (b) $\mathbf{Al_{70}Pd_{22}Mn_8}$;

   (c) $\mathbf{Al_{70}Pd_{22}Re_8}$;

4. *Disordered Media*: random atomic positions

   (a) Normal metals (with defects or impurities);

   (b) Alloys

   (c) Doped semiconductors ($\mathbf{Si}$, $\mathbf{AsGa}$, …);
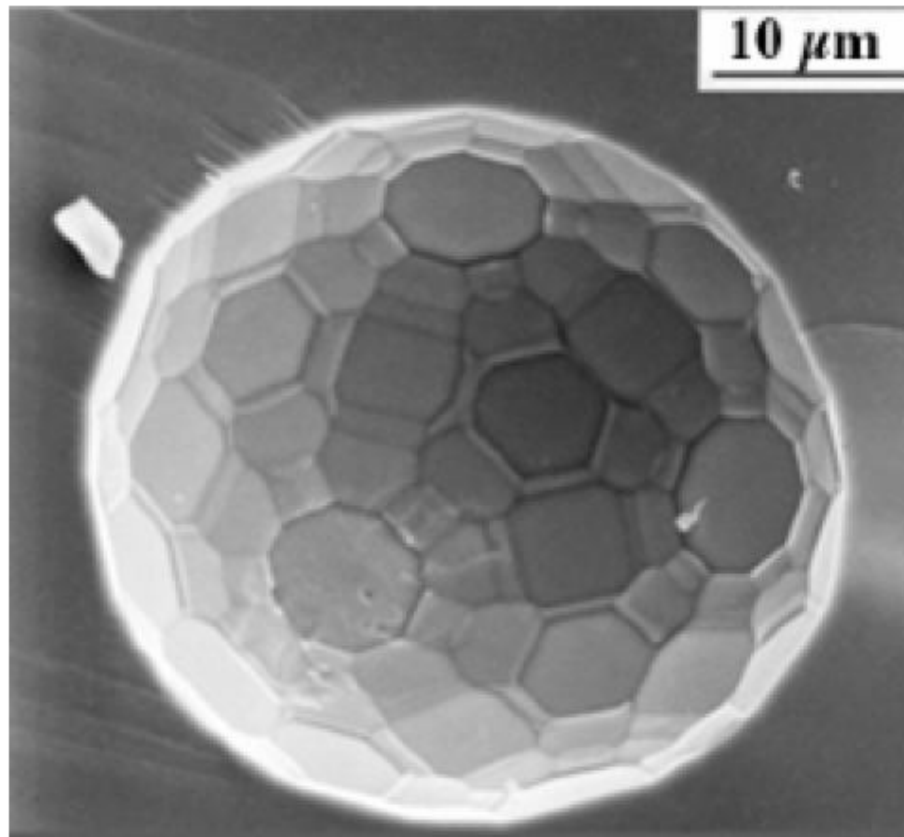
- The icosahedral quasicrystal *AlPdMn* -
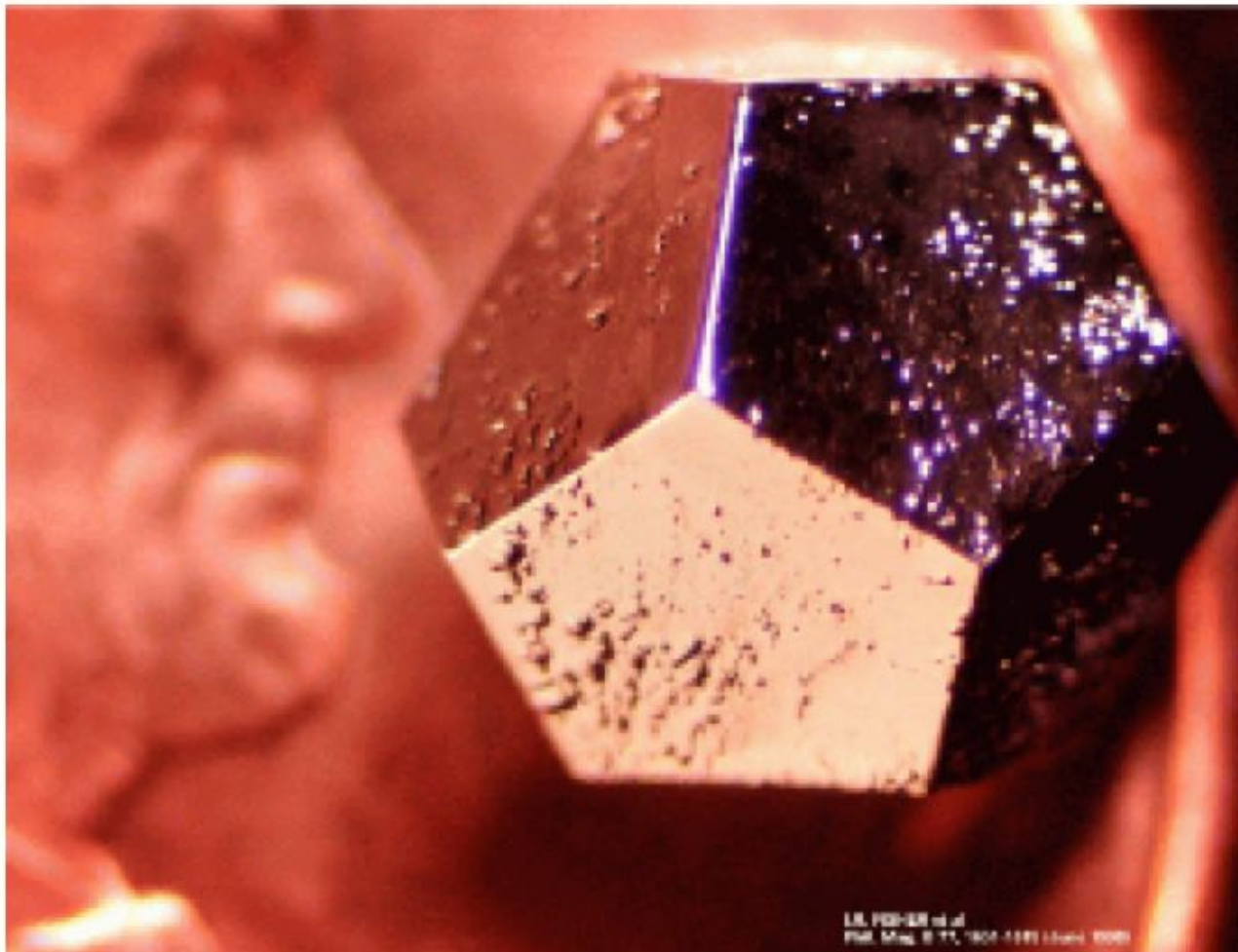
- The icosahedral quasicrystal *HoMgZn*-

II - The Hull as a Dynamical System

Point Sets

A subset $\mathcal{L} \subset \mathbb{R}^d$ may be:

1. *Discrete*.

2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.

3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.

4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.

5. *Finite Local Complexity (FLC)*: $\mathcal{L} - \mathcal{L}$ is discrete and closed.

6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

Point Sets and Point Measures

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $C_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x - y) \quad \in \mathfrak{M}(\mathbb{R}^d) \, .$$

Point Sets and Tilings

Given a tiling with finitely many tiles *(modulo translations)*, a De-lone set is obtained by defining a point in the interior of each *(translation equivalence class of)* tile.

Conversely, given a Delone set, a tiling is built through the *Voronoi cells*

$$V(x) \;=\; \{a \in \mathbb{R}^d \,;\, |a - x| < |a - y| \,, \forall y \mathcal{L} \setminus \{x\}\}$$

1. $V(x)$ is an *open convex polyhedron* containing $B(x;r)$ and contained into $\overline{B(x;R)}$.

2. Two Voronoi cells touch face-to-face.

3. If $\mathcal{L}$ is *FLC*, then the Voronoi tiling has finitely many tiles modulo transla-tions.
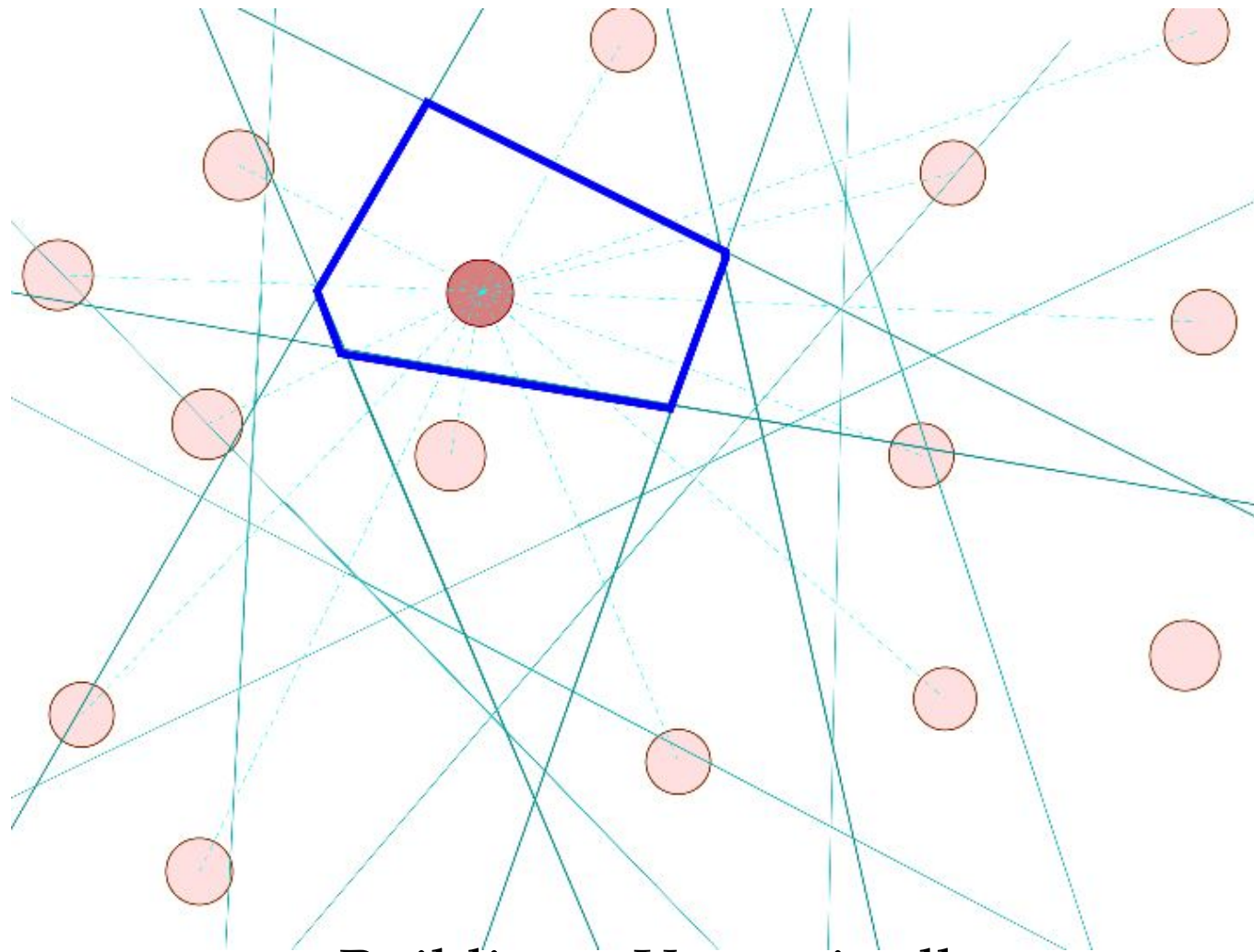
- Building a Voronoi cell-
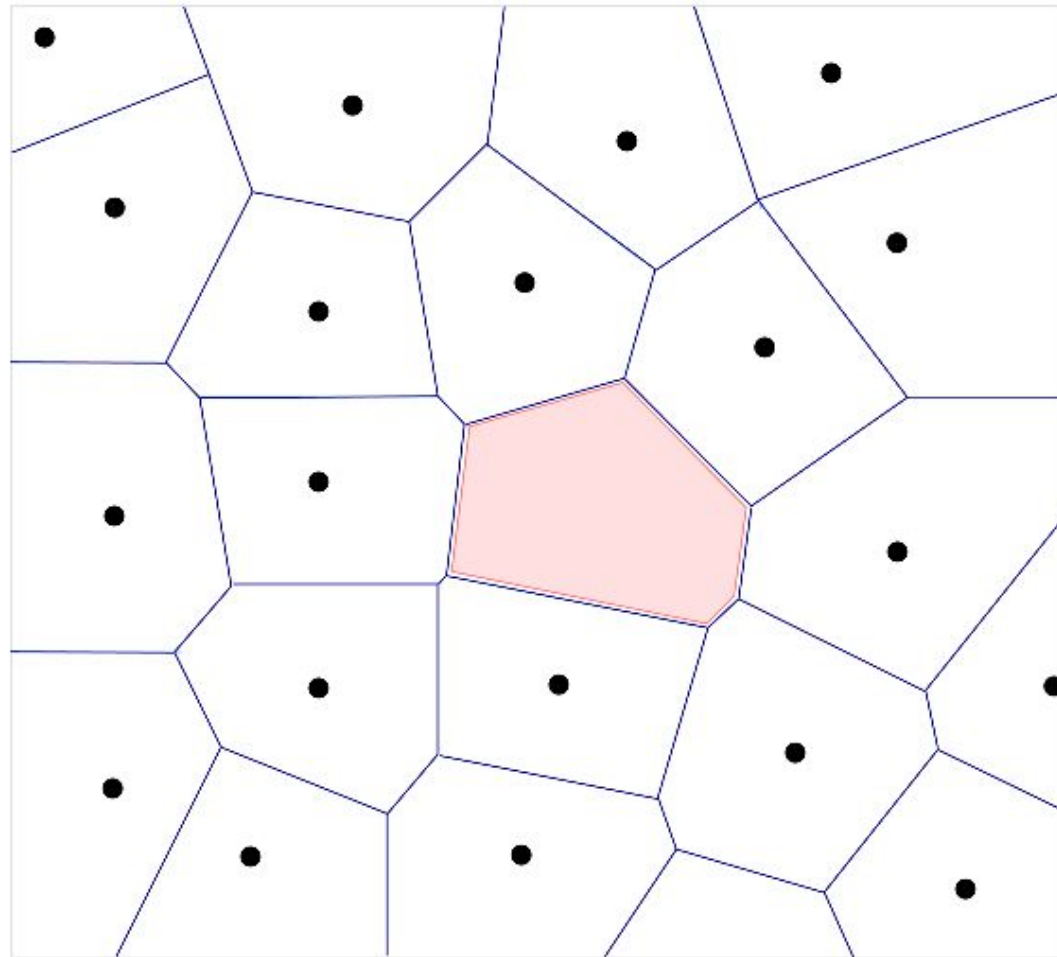
- A Delone set and its Voronoi Tiling-

The Hull

A point measure is $\mu \in \mathfrak{M}(\mathbb{R}^d)$ such that $\mu(B) \in \mathbb{N}$ for any ball $B \subset \mathbb{R}^d$. Its support is

1. *Discrete*.

2. *r-Uniformly discrete*: iff $\forall B$ ball of radius r, $\mu(B) \leq 1$.

3. *R-Relatively dense*: iff for each ball B of radius R, $\mu(B) \geq 1$.

$\mathbb{R}^d$ acts on $\mathfrak{M}(\mathbb{R}^d)$ by translation.

Theorem 1 *The set of r-uniformly discrete point measures is compact and $\mathbb{R}^d$-invariant.*
Its subset of R-relatively dense measures is compact and $\mathbb{R}^d$-invariant.

Definition 1 *Given $\mathcal{L}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{L}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $v^{\mathcal{L}}$.*

Hence the Hull is a *compact metrizable space* on which $\mathbb{R}^d$ *acts by homeomorphisms*.

Properties of the Hull

If $\mathcal{L} \subset \mathbb{R}^d$ is r-uniformly discrete with Hull Ω then using compactness

1. each point *$\omega \in \Omega$ is an r-uniformly discrete* point measure with support $\mathcal{L}_\omega$.

2. if $\mathcal{L}$ is *(r, R)-Delone*, so are all $\mathcal{L}_\omega$'s.

3. if, in addition, $\mathcal{L}$ is *FLC*, so are all the $\mathcal{L}_\omega$'s.
   Moreover then $\mathcal{L} - \mathcal{L} = \mathcal{L}_\omega - \mathcal{L}_\omega \ \forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{L}_\omega$.*

Theorem 2 *If $\mathcal{L}$ is FLC, then its transversal is completely discontinuous.*

Local Isomorphism Classes and Tiling Space

A *patch* is a finite subset of $\mathcal{L}$ of the form

$$p = (\mathcal{L} - x) \cap \overline{B(0, r_1)} \qquad x \in \mathcal{L}, \ r_1 \geq 0$$

Given $\mathcal{L}$ a repetitive, FLC, Delone set let $\mathcal{W}$ be its set of finite patches: it is called the *the $\mathcal{L}$-dictionary*.

A Delone set (or a Tiling) $\mathcal{L}'$ is *locally isomorphic* to $\mathcal{L}$ if it has the same dictionary. The *Tiling Space* of $\mathcal{L}$ is the set of *Local Isomorphism Classes* of $\mathcal{L}$.

Theorem 3 *The Tiling Space of $\mathcal{L}$ coincides with its Hull.*

Minimality

$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

Theorem 4 $\mathbb{R}^d$ *acts minimaly on* Ω *if and only if* $\mathcal{L}$ *is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for *Si* atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.
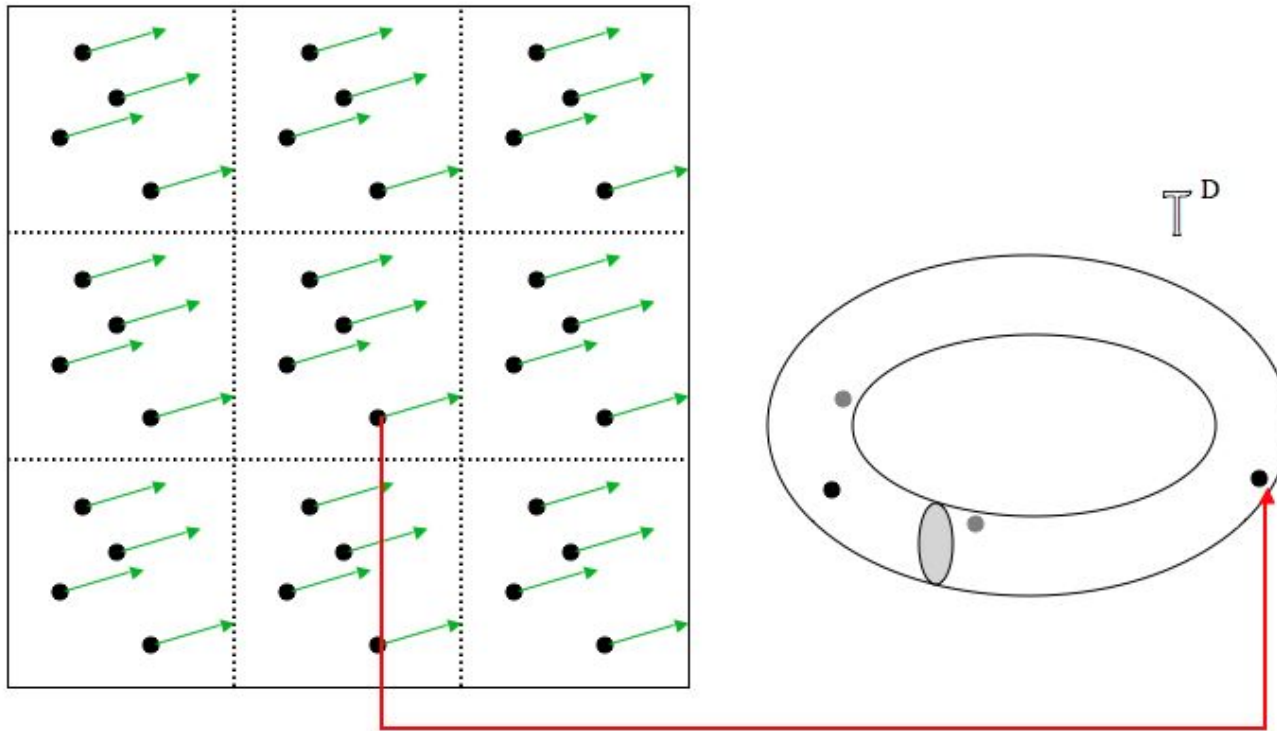
- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

$$\mathbb{R}^d \simeq \mathcal{E}_{\parallel} \xleftarrow{\pi_{\parallel}} \mathbb{R}^n \xrightarrow{\pi_{\perp}} \mathcal{E}_{\perp} \simeq \mathbb{R}^{n-d}$$

$$\mathcal{L} \xleftarrow{\pi_{\parallel}} \tilde{\mathcal{L}} \xrightarrow{\pi_{\perp}} W \ ,$$

Here

1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,

2. the *window* W is a compact polytope.

3. $\mathcal{L}$ is the *quasilattice* in $\mathcal{E}_{\parallel}$ defined as

$$\mathcal{L} \ = \ \{\pi_{\parallel}(m) \in \mathcal{E}_{\parallel} \, ; \, m \in \tilde{\mathcal{L}} \, , \, \pi_{\perp}(m) \in W\}$$
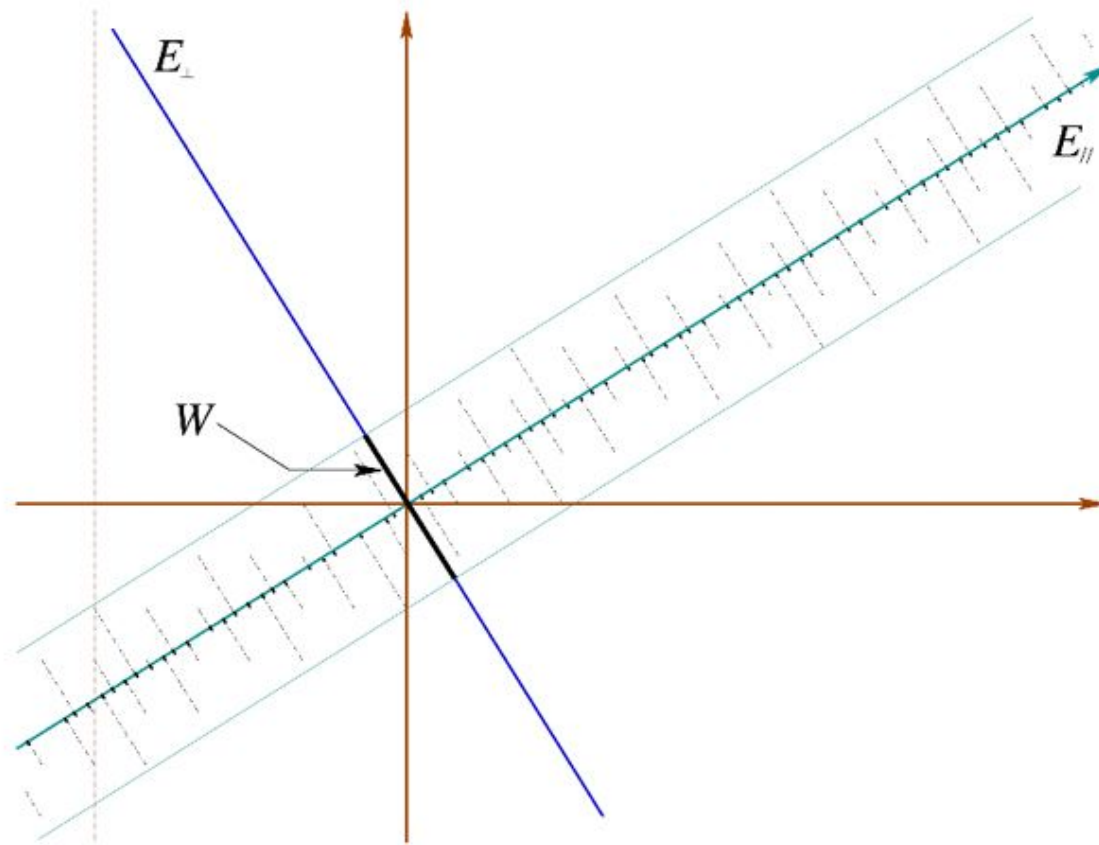
– The cut–and–project construction –
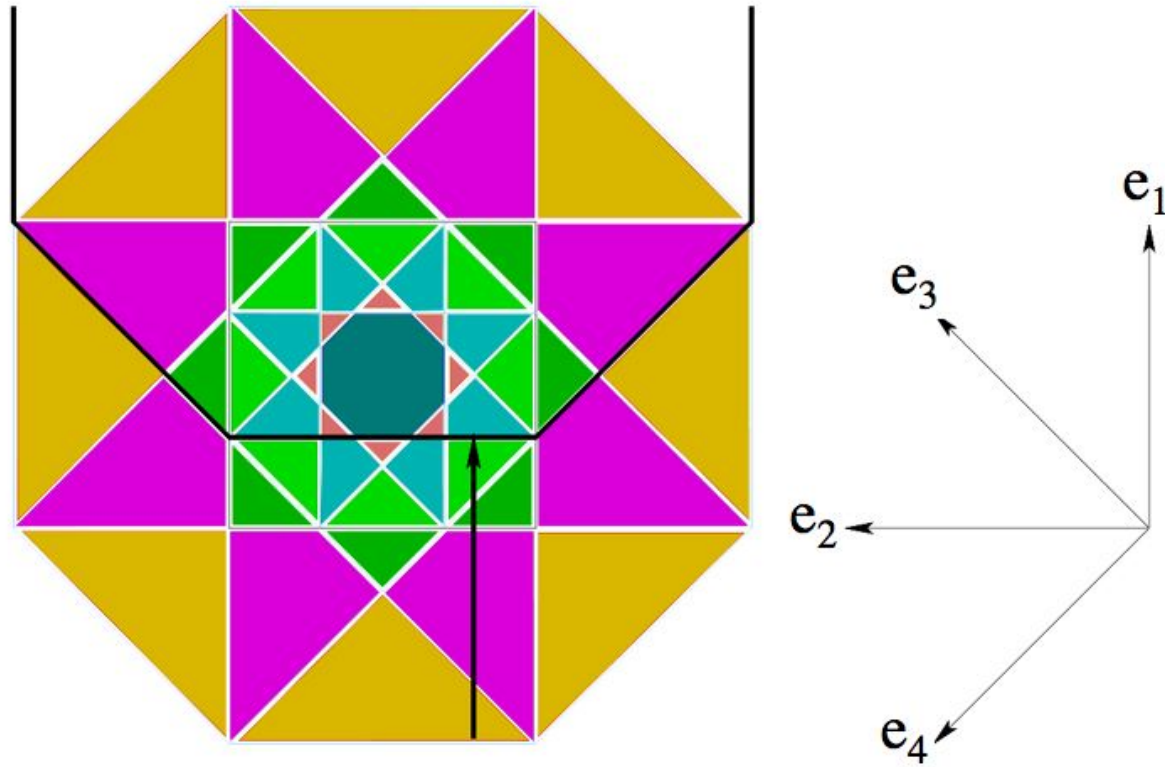
- The transversal of the Octagonal Tiling is completely disconnected -

III - The Gap Labeling Theorem

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J. Kaminker, I. Putnam, Michigan Math. J., **51**, (2003), 537-546.

M. Benameur, H. Oyono-Oyono, C. R. Math. Acad. Sci. Paris, **334**, (2002), 667-670.

Schrödinger's Operator

Ignoring electrons-electrons interactions, the one-electron Hamiltonian is given by

$$H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y\in\mathcal{L}_\omega} v(\cdot - y)$$

Its *integrated density of states (IDS)* is defined by

$$\mathcal{N}(E) = \lim_{\Lambda\uparrow\mathbb{R}^d} \frac{1}{|\Lambda|} \#\{\text{eigenvalues of } H_\omega \upharpoonright_\Lambda \leq E\}$$

For any $\mathbb{R}^d$-*invariant probability measure* $\mathbb{P}$ on Ω the limit exists a.e. and is independent of ω. It defines a nondecreasing function of E constant on the spectral gaps of H_ω. It is asymptotic at large E's to the IDS of the free Hamiltonian.
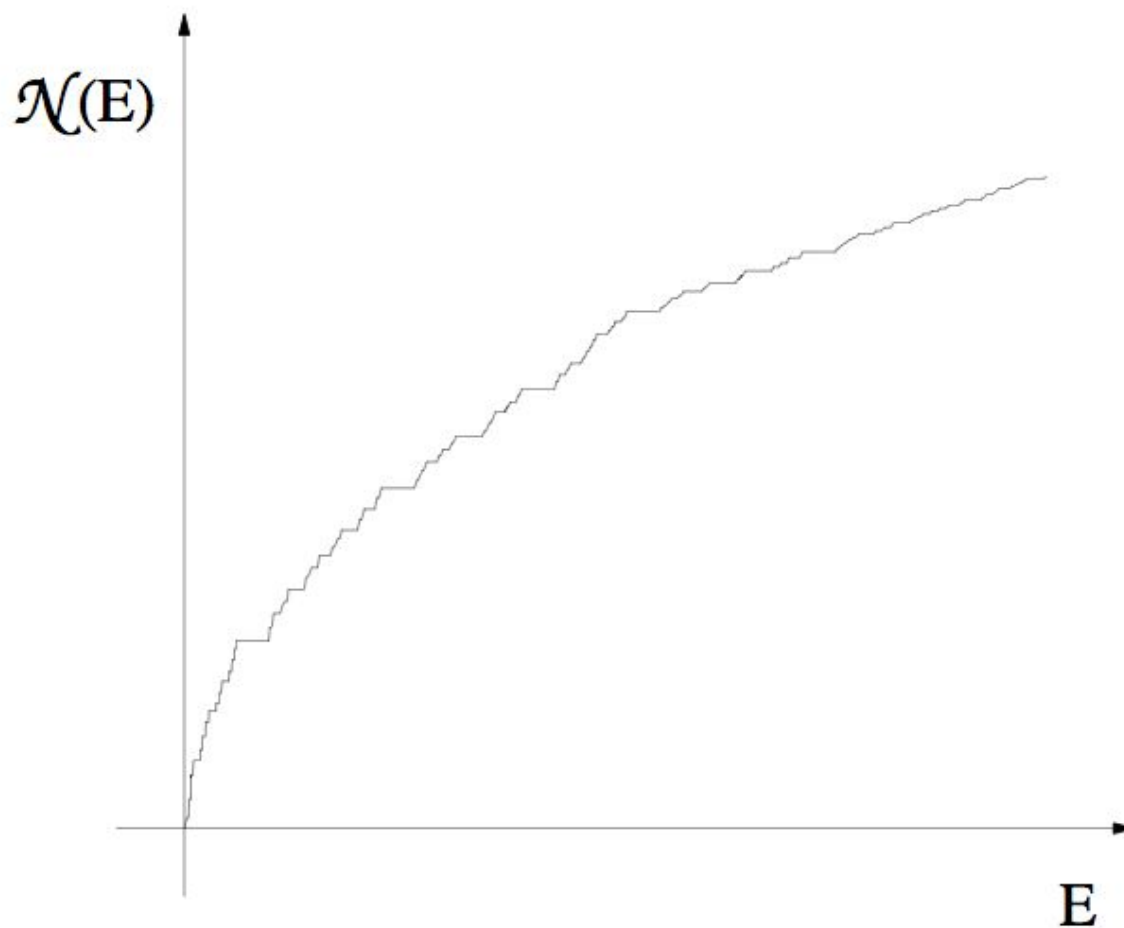
- An example of IDS -

Phonons, Vibrational Modes

Atom vibrations in the *harmonic approximation* are solution of

$$M \, \frac{d^2 \vec{u}_{(\omega,x)}}{dt^2} = \sum_{y \in \mathcal{L}_\omega; y \neq x} K_\omega(x,y) \left(\vec{u}_{(\omega,x)} - \vec{u}_{(\omega,y)} \right)$$

- M= atomic mass,

- $\vec{u}_{(\omega,x)}$ displacement vector of the atom located at $x \in \mathcal{L}_\omega$,

- $K_\omega(x,y)$ is the matrix of *spring constants*.

The *density of vibrational modes (IDVM)* is the IDS of $K_\omega^{1/2}$.

Gap Labels

Theorem 5 *The value of the IDS or of the IDVM on gaps is a linear combination of the occurrence probabilities of finite patches with integer coefficients.*

The proof goes through the group of K-theory of the hull. The result is model independent.

The abstract result goes back to 1982 (J.B). In 1D, proved in 1993 (JB). Recent proof in any dimension

for aperiodic, repetitive, aperiodic tilings by KAMINKER-PUTNAM, BENAMEUR & OYONO-OYONO, JB-

BENDETTI-GAMBAUDO *in 2001.*

IV - Branched Oriented Flat Riemannian Manifolds

Laminations and Boxes

A *lamination* is a foliated manifold with C^∞-structure along the leaves but only finite C^0-structure transversally. The *Hull of a Delone set is a lamination* with $\mathbb{R}^d$-orbits as leaves.

If $\mathcal{L}$ is a *FLC, repetitive, Delone* set, with Hull Ω a *box* is the home-omorphic image of

$$\phi : (\omega, x) \in F \times U \mapsto \tau^{-x}\omega \in \Omega$$

if F is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and τ denotes the $\mathbb{R}^d$-action on Ω.

A *quasi-partition* is a family $(B_i)_{i=1}^n$ of boxes such that $\bigcup_i \overline{B_i} = \Omega$ and $B_i \cap B_j = \emptyset$.

Theorem 6 *The Hull of a FLC, repetitive, Delone set admits a finite quasi-partition. It is always possible to choose these boxes as $\phi(F \times U)$ with U a d-rectangle.*

Branched Oriented Flat Manifolds

Flattening a box decomposition along the transverse direction leads to a *Branched Oriented Flat manifold*. Such manifolds can be built from the tiling itself as follows

Step 1:

1. X is the disjoint union of all *prototiles*;

2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;

3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.
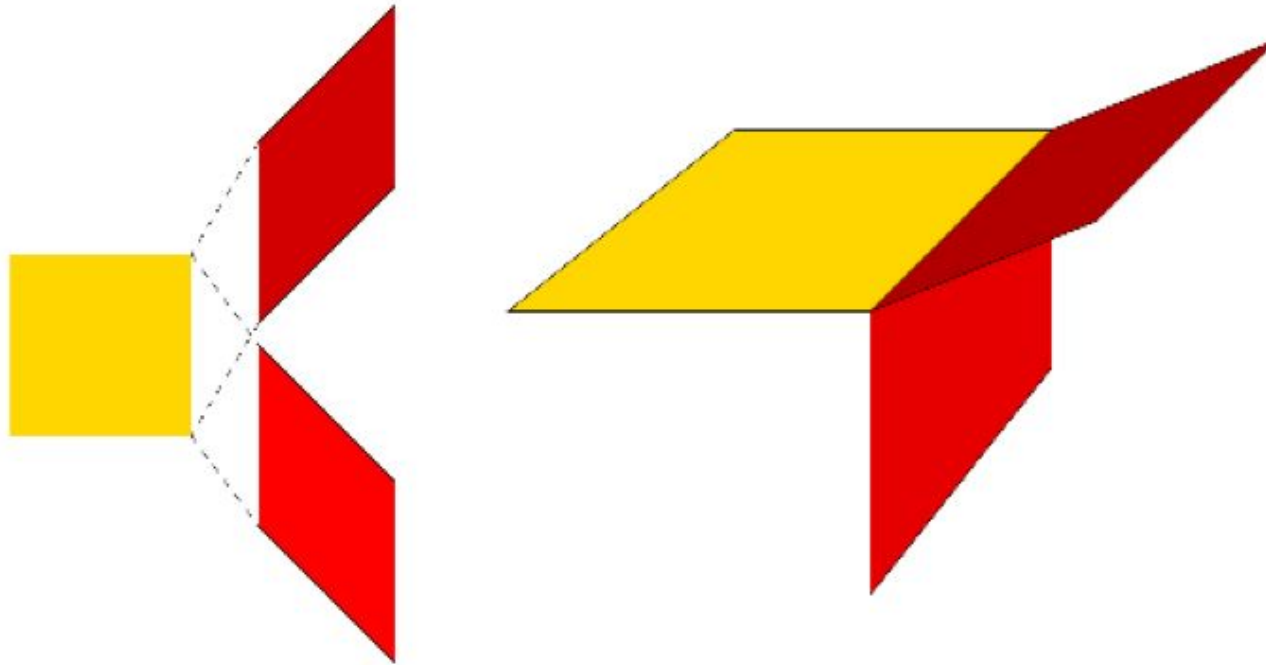
- Branching -
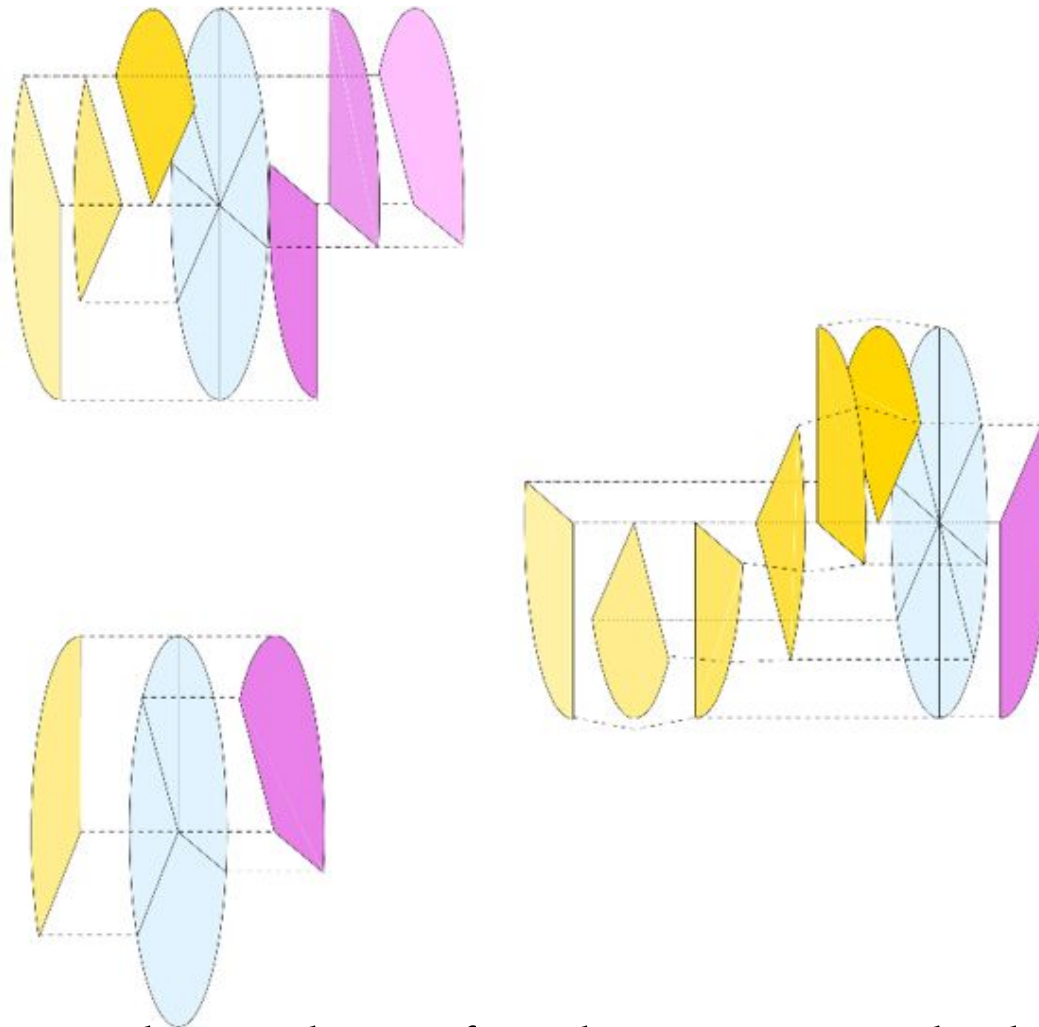
- Vertex branching for the octagonal tiling -

Step 2:

1. Having defined the patch p_n for $n \geq 0$, let $\mathcal{L}_n$ be the subset of $\mathcal{L}$ of points centered at a translated of p_n. By repetitivity this is a FLC repetitive Delone set too. Its prototiles are tiled by tiles of $\mathcal{L}$ and define a finite family $\mathfrak{P}_n$ of patches.

2. Each patch in $\mathcal{T} \in \mathfrak{P}_n$ will be collared by the patches of $\mathfrak{P}_{n-1}$ touching it from outside along its frontier. Call such a patch *modulo translation* a *a collared patch* and $\mathfrak{P}_n^c$ their set.

3. Proceed then as in Step 1 by replacing prototiles by collared patches to get the BOF-manifold B_n.

4. Then choose p_{n+1} to be the collared patch in $\mathfrak{P}_n^c$ containing p_n.
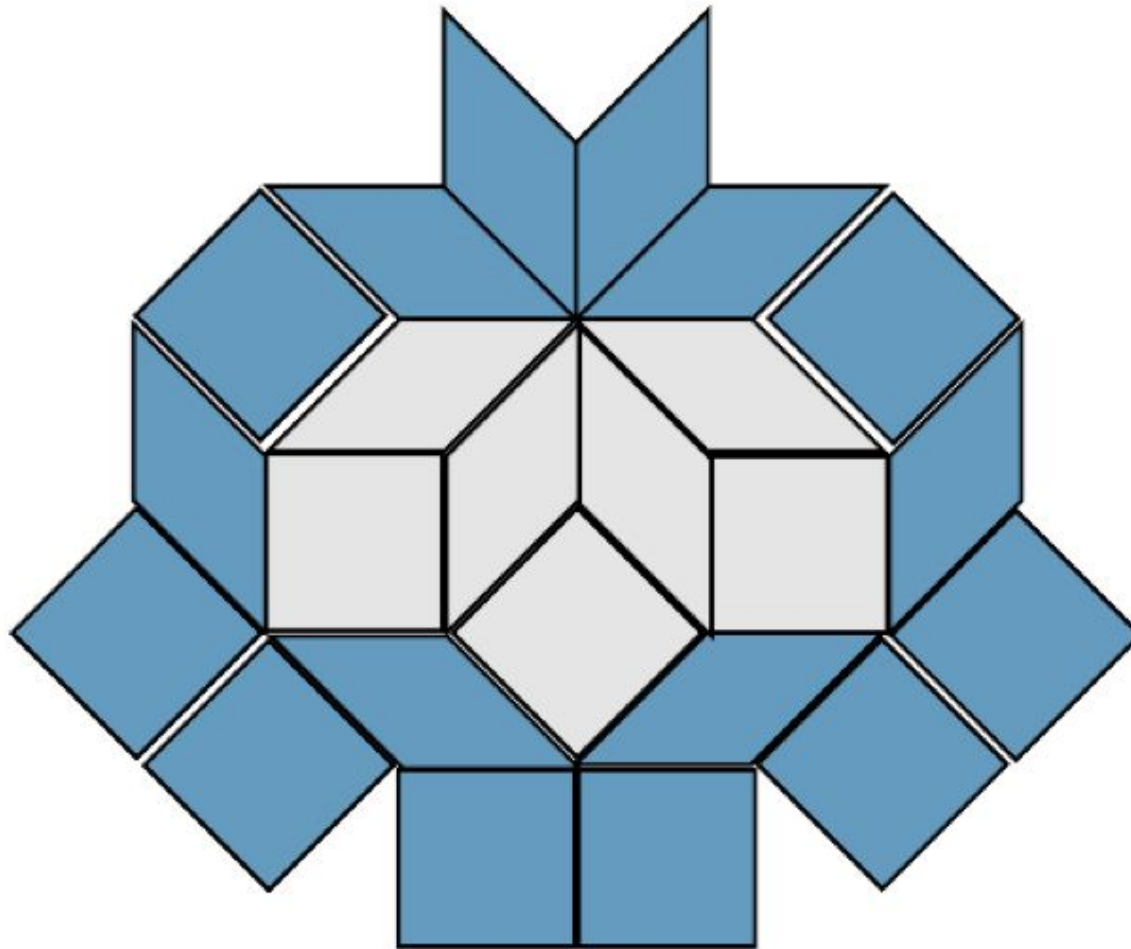
- A collared patch -

Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n. Note that $Df_n = \mathbf{1}$.

2. Call Ω the *projective limit* of the sequence

$$\ldots \overset{f_{n+1}}{\longrightarrow} B_{n+1} \overset{f_n}{\longrightarrow} B_n \overset{f_{n-1}}{\longrightarrow} \ldots$$

3. $X_1, \cdots X_d$ are the commuting constant vector fields on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action τ on Ω.

Theorem 7 *The dynamical system*

$$(\Omega, \mathbb{R}^d, \tau) = \varprojlim(B_n, f_n)$$

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

V - Cohomology and *K*-Theory

Čech Cohomology of the Hull

Let $\mathcal{U}$ be an *open covering* of the Hull. If $U \in \mathcal{U}$, $\mathcal{F}(U)$ is the space of integer valued locally constant function on U.

For $n \in \mathbb{N}$, the n-chains are the element of $C^n(\mathcal{U})$, namely the *free abelian group* generated by the elements of $\mathcal{F}(U_0 \cap \cdots \cap U_n)$ when the U_i varies in $\mathcal{U}$. A differential is defined by

$$d : C^n(\mathcal{U}) \mapsto C^{n+1}(\mathcal{U})$$

$$df\left(\bigcap_{i=0}^{n+1} U_i\right) = \sum_{j=0}^{n} (-1)^j f\left(\bigcap_{i:i\neq j} U_i\right)$$

This defines a *complex* with cohomology $\check{H}^n(\mathcal{U}, \mathbb{Z})$. The Čech cohomology group of the Hull Ω is defined as

$$\check{H}^n(\Omega, \mathbb{Z}) \;=\; \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathbb{Z})$$

with ordering given by *refinement* on the set of open covers. Thanks to properties of the cohomology, if f_n^* is the map induced by f_n on the cohomology

$$\check{H}^n(\Omega, \mathbb{Z}) \;=\; \varinjlim_{n} \left(\check{H}^n(B_n, \mathbb{Z}), f_n^* \right)$$

Examples

J. E. ANDERSON, I. PUTNAM, Ergodic Theory Dynam. Systems, **18**, (1998), 509-537.
L. SADUN, Topology of Tiling Spaces. AMS (2008)

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- Fibonacci: divides $\mathbb{R}$ into intervals a, b of length $1, \sigma = (\sqrt{5}-1)/2$ according to the substitution rule $a \mapsto ab$, $b \mapsto a$. Then $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^2$
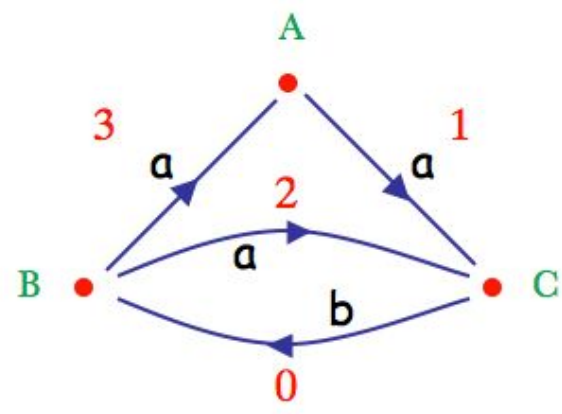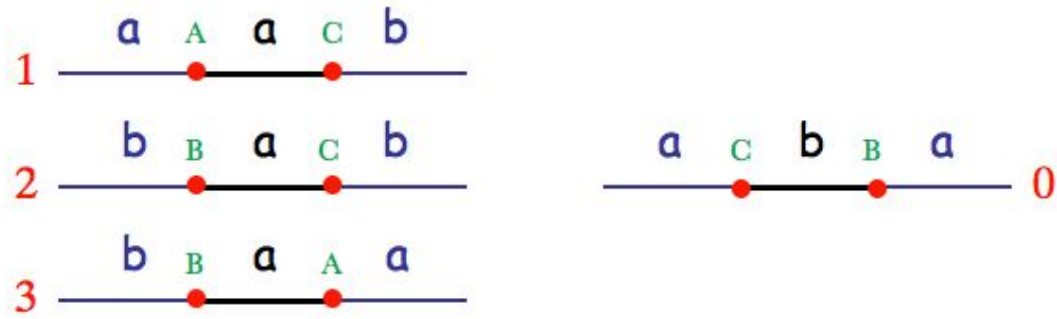
- The Anderson-Putnam complex for the Fibonacci tiling -

Examples

J. E. Anderson, I. Putnam, Ergodic Theory Dynam. Systems, **18**, (1998), 509-537.
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Examples

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- Chair tiling: $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$, $H^2 = \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$

Other Cohomologies

- Longitudinal Cohomology (CONNES, MOORE-SCHOCHET)

- Pattern-equivariant cohomology (KELLENDONK-PUTNAM, SADUN)

- PV-cohomology (SAVINIEN-BELLISSARD)

In maximal degree the Čech *Homology* does exists. It contains a natural *positive cone* isomorphic to the set of *positive $\mathbb{R}^d$-invariant measures* on the Hull (BELLISSARD-BENEDETTI-GAMBAUDO).

Cohomology and K-theory

The main topological property of the Hull (or tiling psace) is summarized in the following

Theorem 8 *(i) The various cohomologies, Čech, longitudinal, pattern-equivariant and PV, are isomorphic.*
(ii) There is a spectral sequence converging to the K-group of the Hull with page 2 given by the cohomology of the Hull.
(iii) In dimension $d \leq 3$ the K-group coincides with the cohomology.

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.

2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.

3. This dynamical system can be seen as a *lamination* or, equivalently, as the *inverse limit* of *Branched Oriented Flat Riemannian Manifolds*.

4. The Čech cohomology is equivalent to the longitudinal one, obtained by inverse limit, to the pattern-equivariant one or to the Pimsner cohomology are equivalent *Cohomology* of the Hull. The *K-group* of the Hull can be computed through a spectral sequence with the cohomology in page 2.

5. In maximum degree, the *Homology* gives the family of *invariant measures* and the *Gap Labelling Theorem*.