The TRANSVERSE GEOMETRY of TILING SPACES

Sponsoring

Jean BELLISSARD

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics
e-mail: jeanbel@math.gatech.edu

Collaborations

J. Pearson, (*Gatech*, Atlanta, GA)

J. Savinien, (*U. Metz*, Metz, France)

A. Julien, (*U. Victoria*, Victoria, BC)

I. Palmer, (*NSA*, Washington DC)

R. Parada, (*Gatech*, Atlanta, GA)

Main References

A. Beurling, J. Deny, *Dirichlet spaces*, Proc. Nat. Acad. Sci. U.S.A., **45**, (1959), 208-215.

M. Fukushima, *Dirichlet forms and Markov processes*,
North-Holland Math. Lib., 23., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1980.

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

G. Michon, *Les Cantors réguliers*, C. R. Acad. Sci. Paris Sér. I Math., (19), **300**, (1985) 673-675.

J. Pearson, J. Bellissard, *Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets*,
J. Noncommutative Geometry, **3**, (2009), 447-480.

A. Julien, J. Savinien, *Transverse Laplacians for substitution tilings*,
Comm. Math. Phys., **301**, (2011), 285-318.

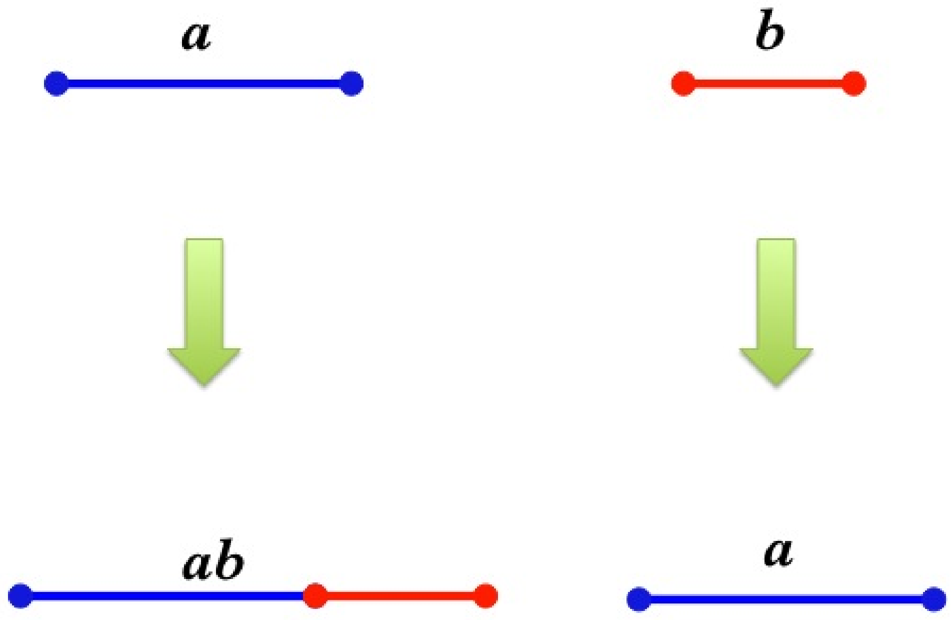
I. Palmer, *Noncommutative Geometry and Compact Metric Spaces*,
PhD Thesis, Georgia Institute of Technology, May 2010

Content

1. Tilings and their Transversal
2. Spectral Triple
3. The Pearson Laplacian
4. Open Problems

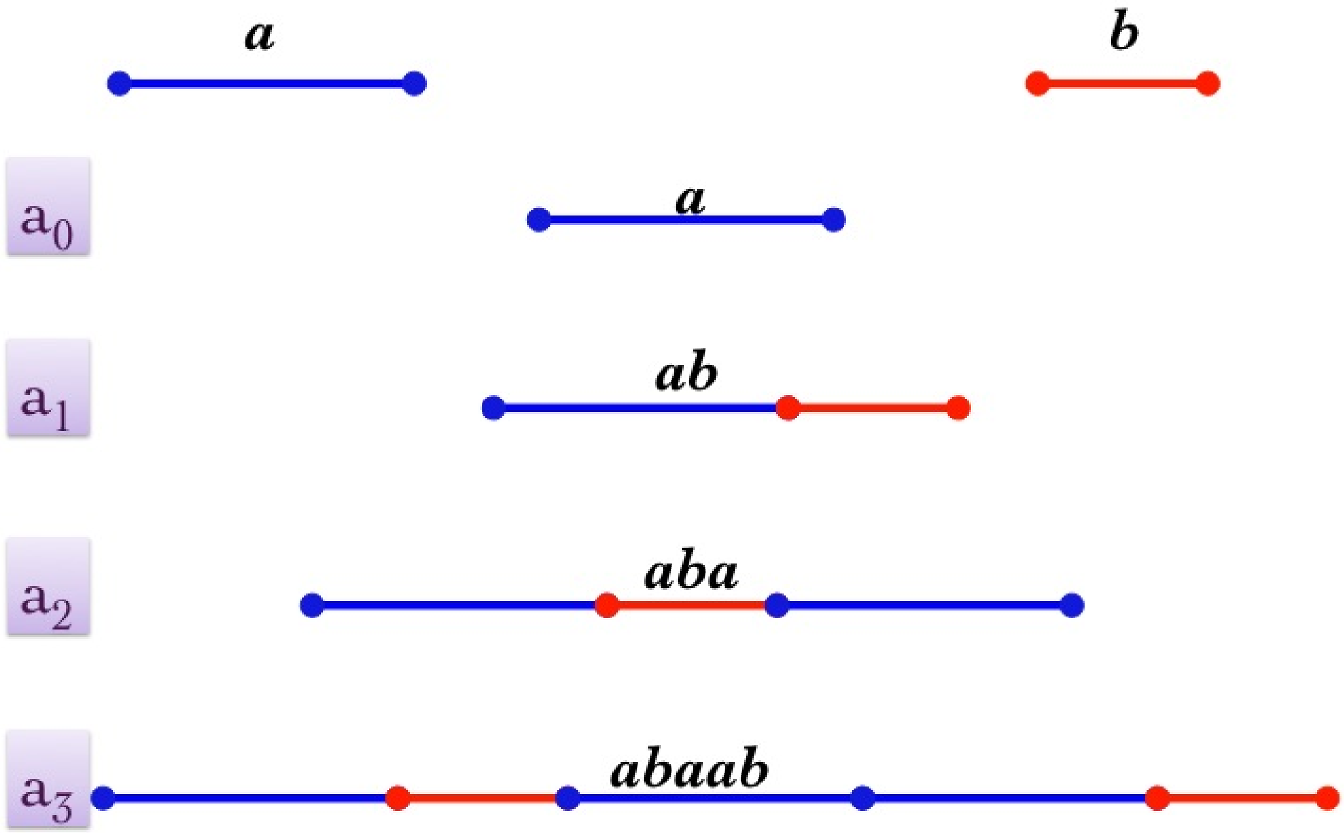
I - Tilings and their Transversal

The Fibonacci Tiling

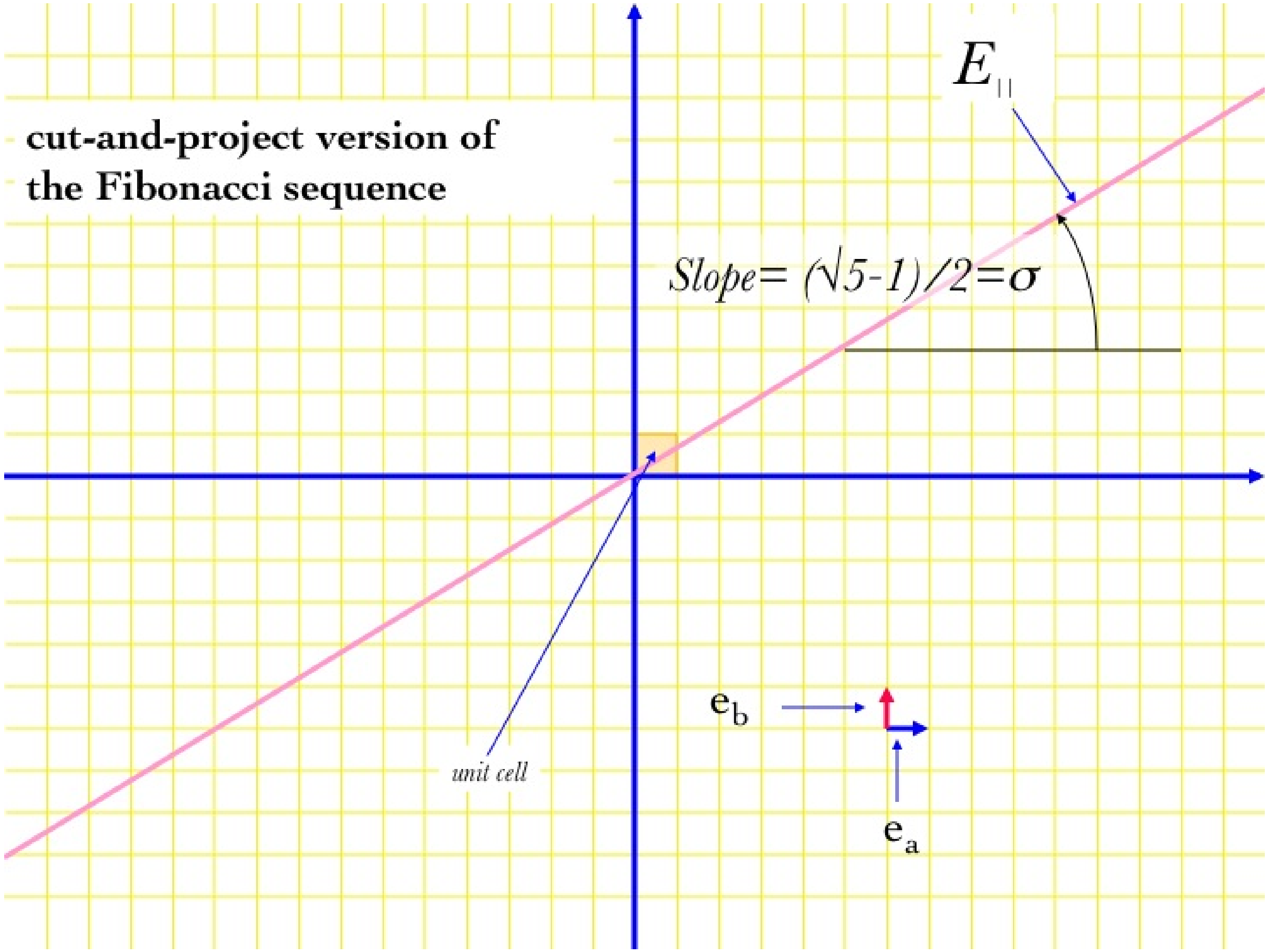


The Fibonacci Substitution

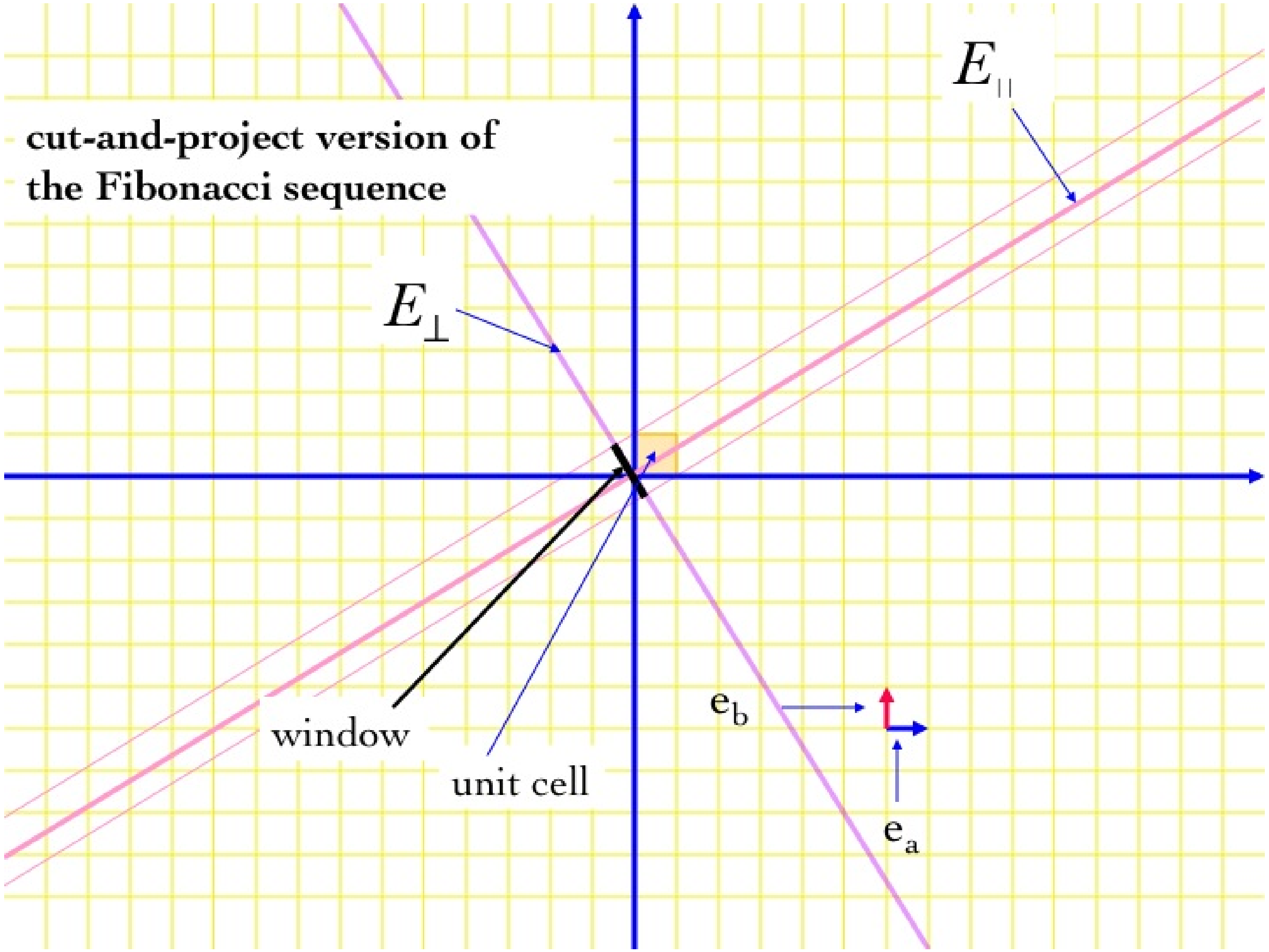
The Fibonacci Tiling



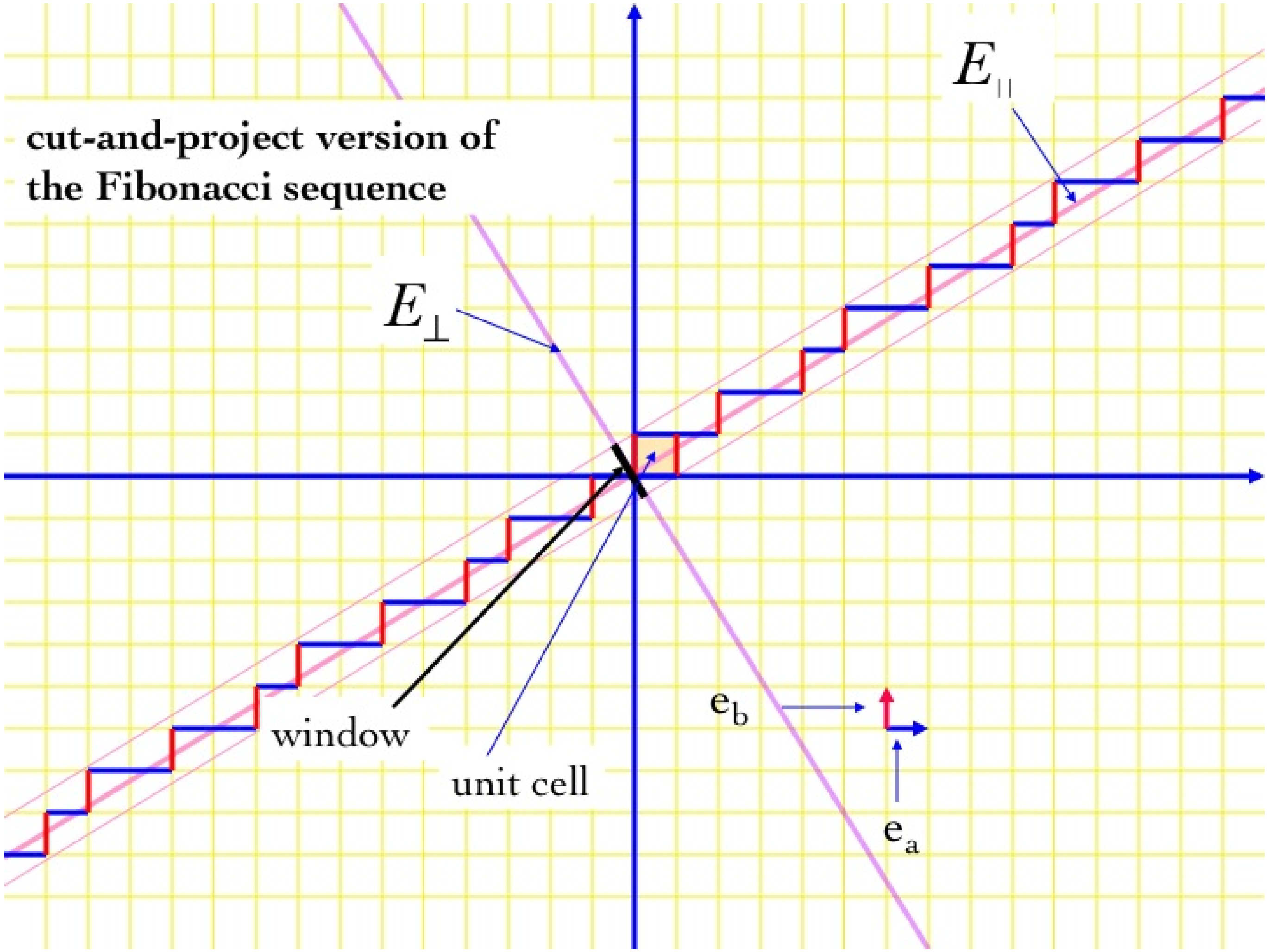
The Fibonacci Tiling



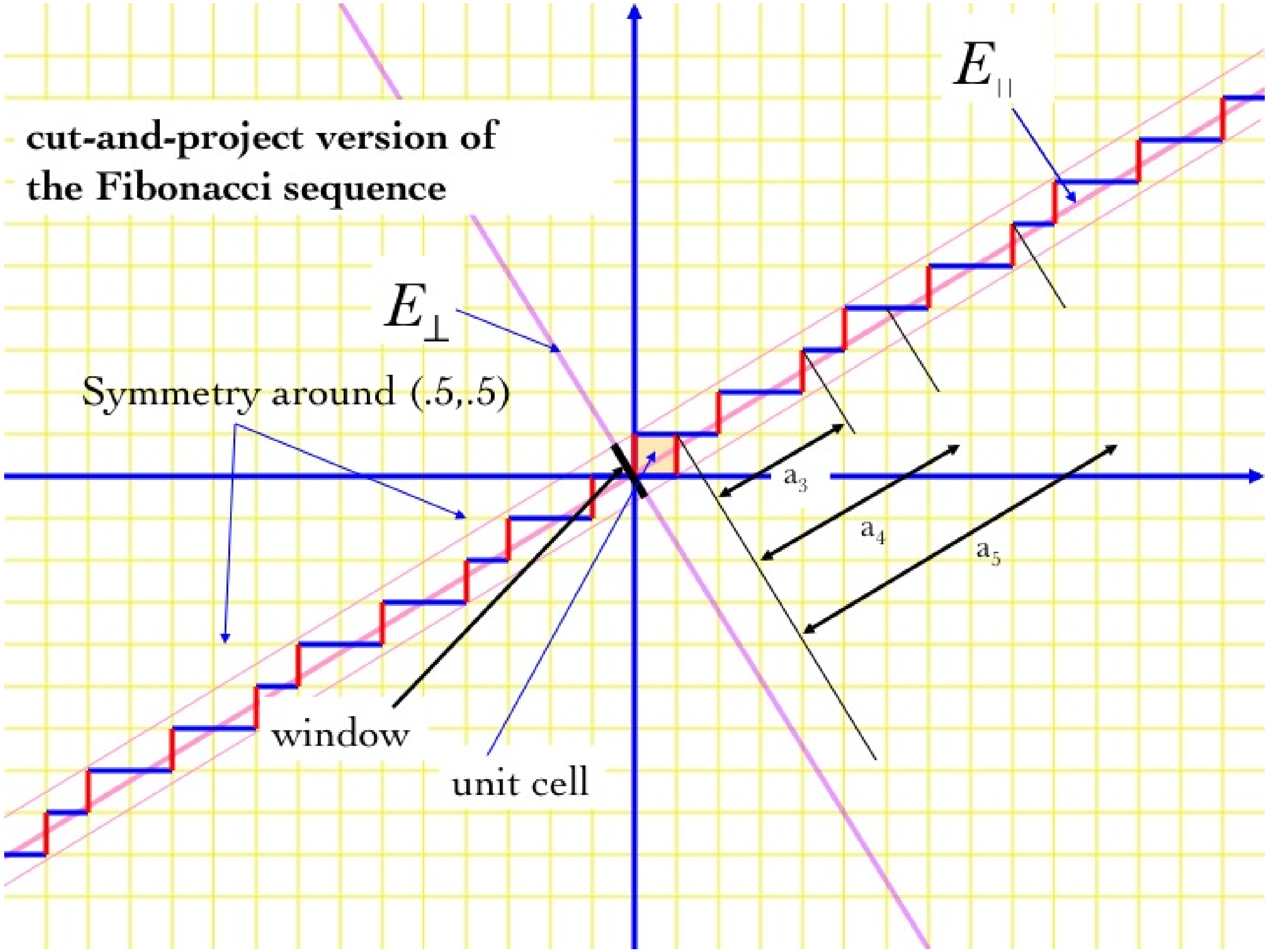
The Fibonacci Tiling



The Fibonacci Tiling



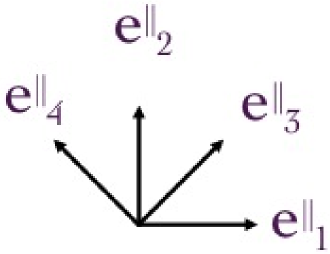
The Fibonacci Tiling

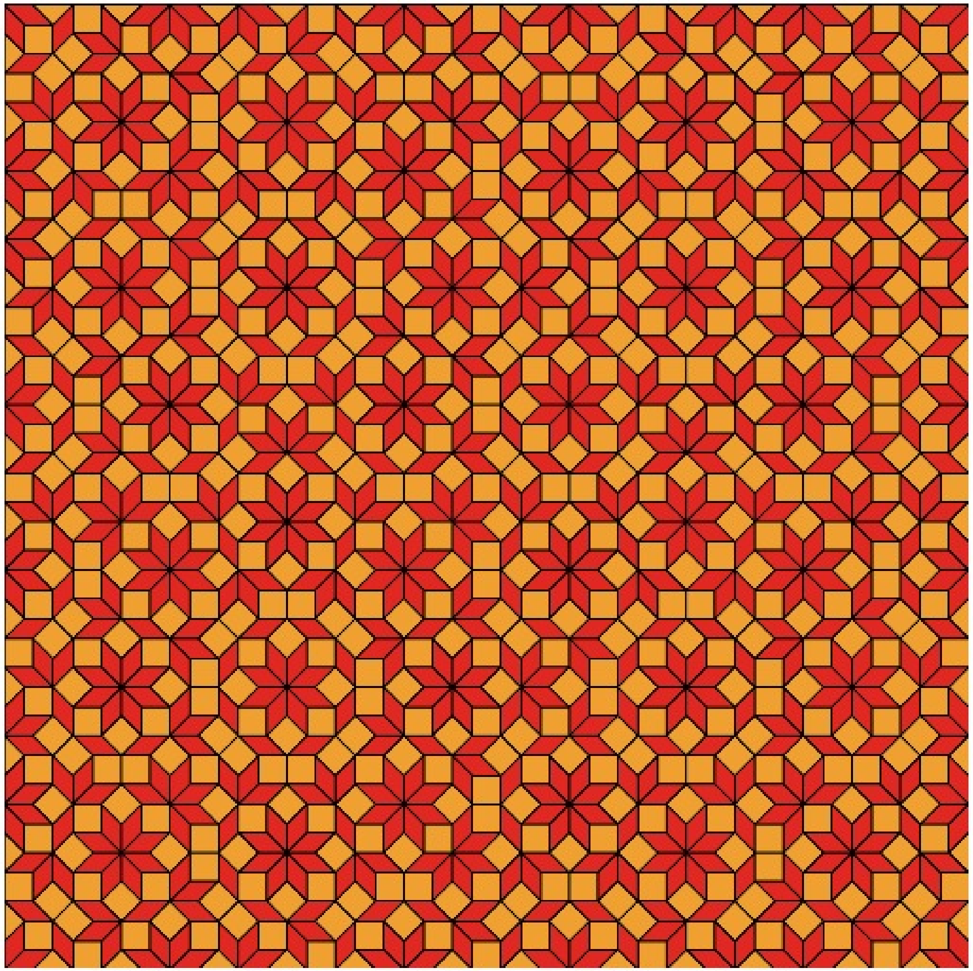


The Octagonal Tiling

Octagonal Lattice

$Z^4 \to R^2$

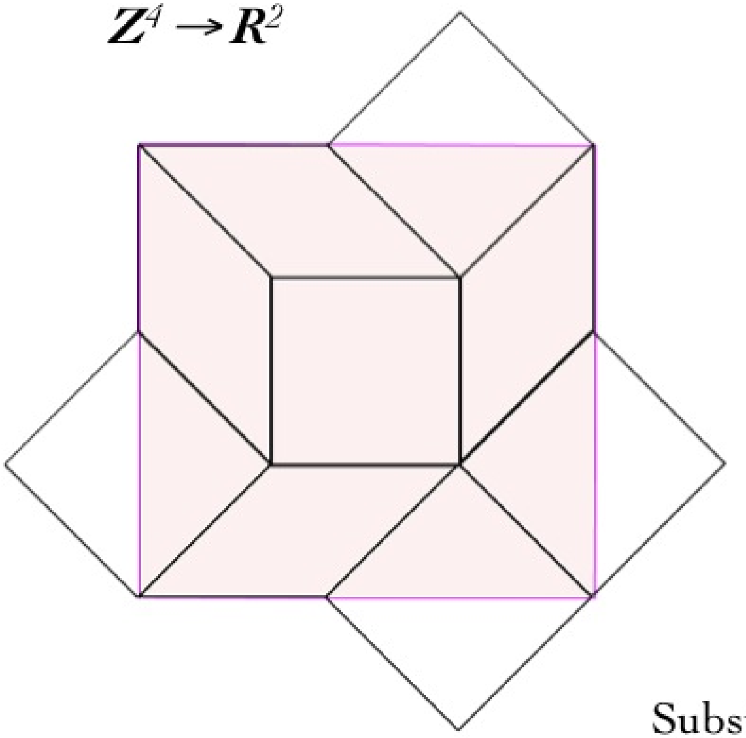


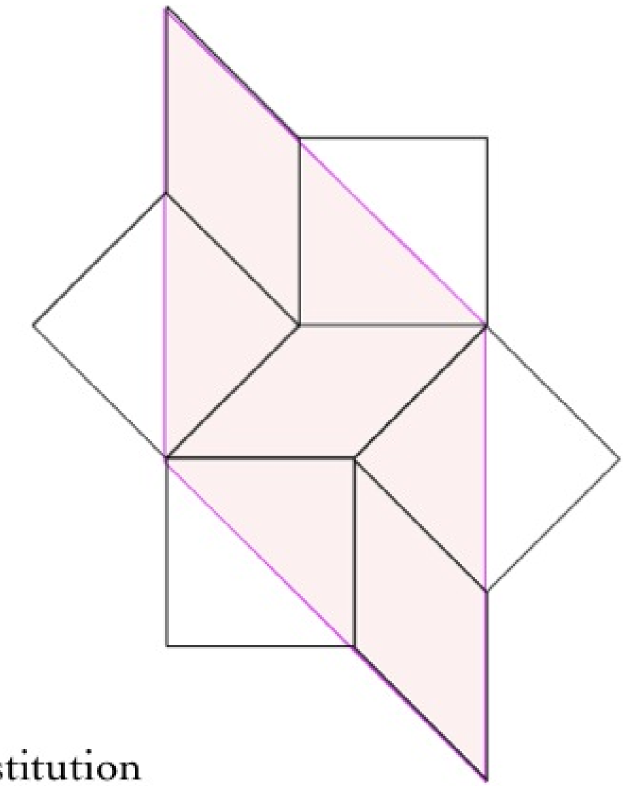


The Octagonal Tiling

Octagonal
Lattice

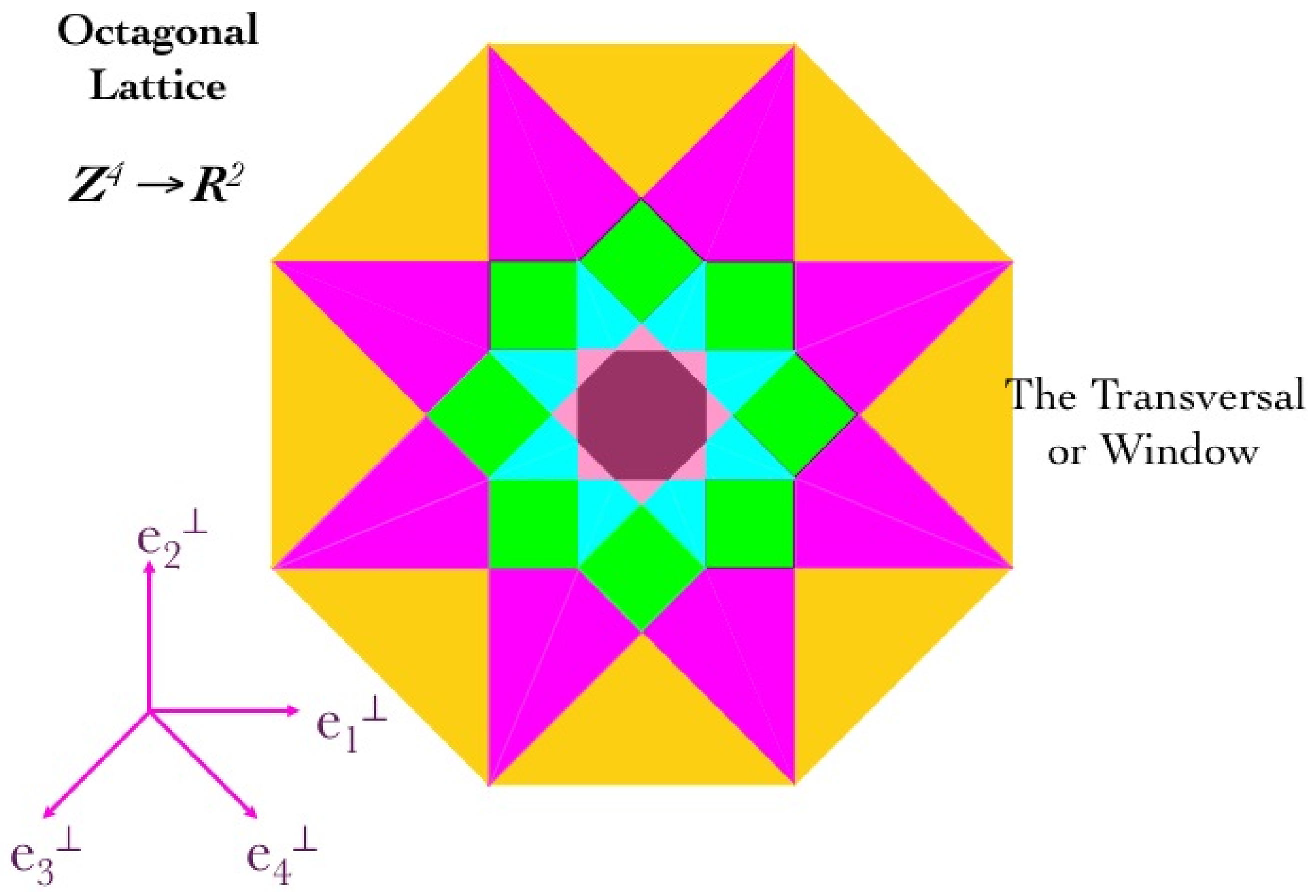
$Z^4 \rightarrow R^2$



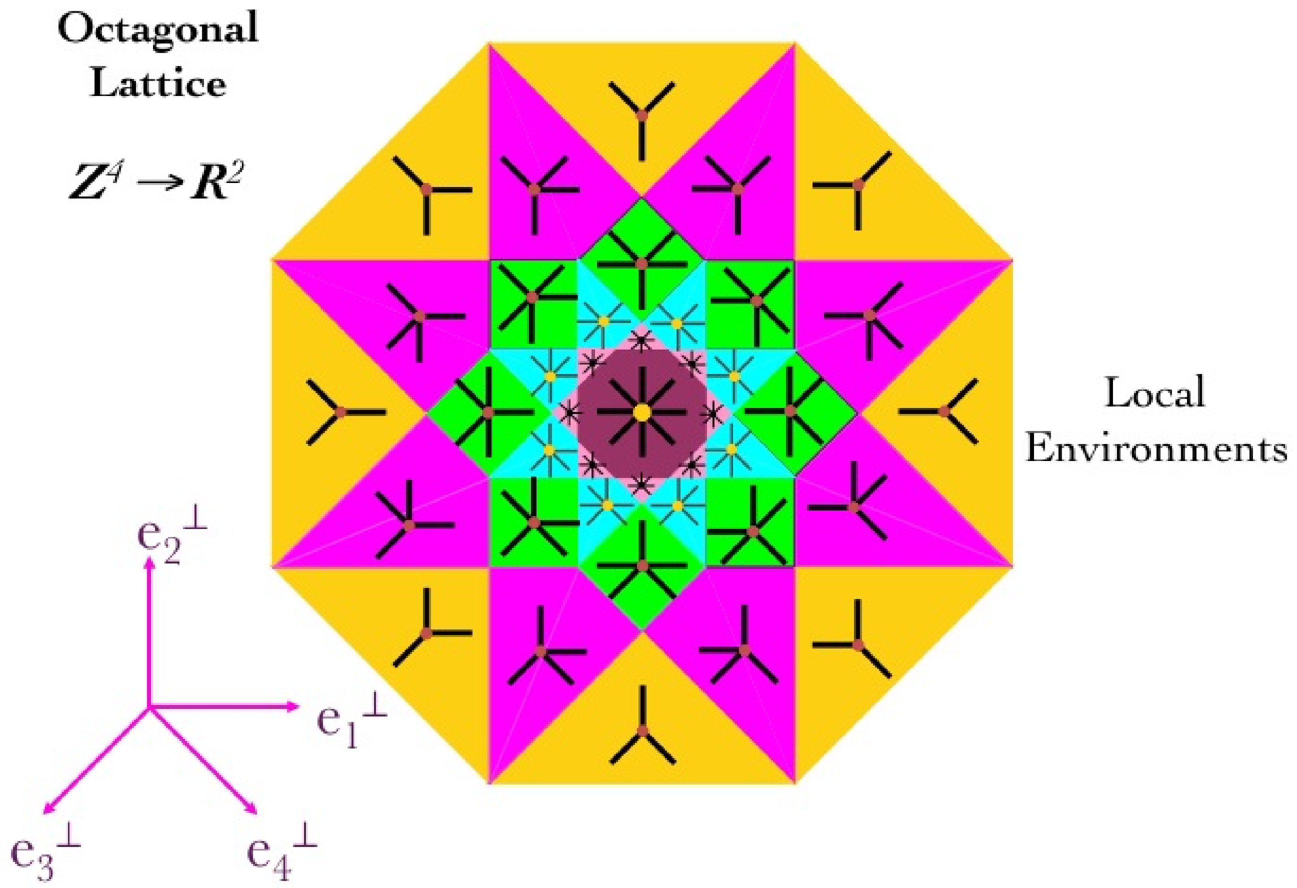


Substitution

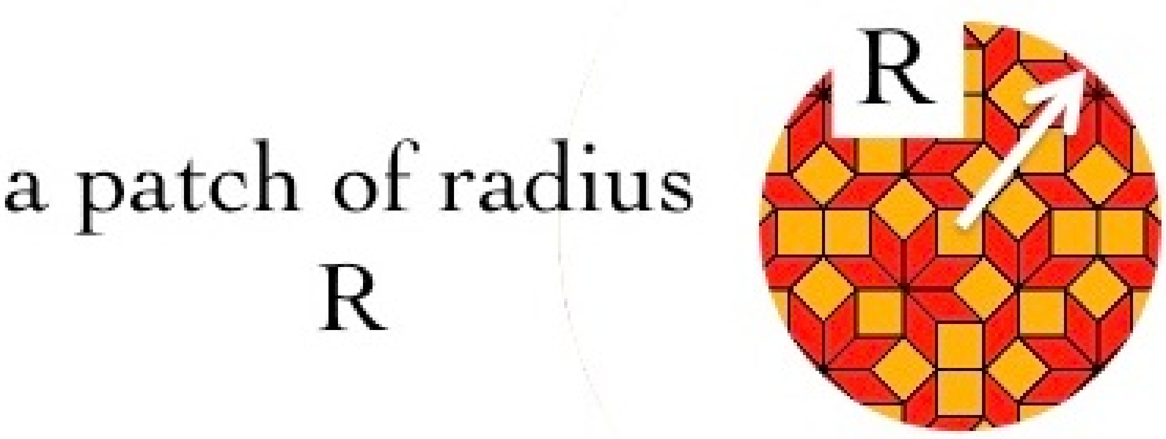
The Octagonal Tiling



The Octagonal Tiling



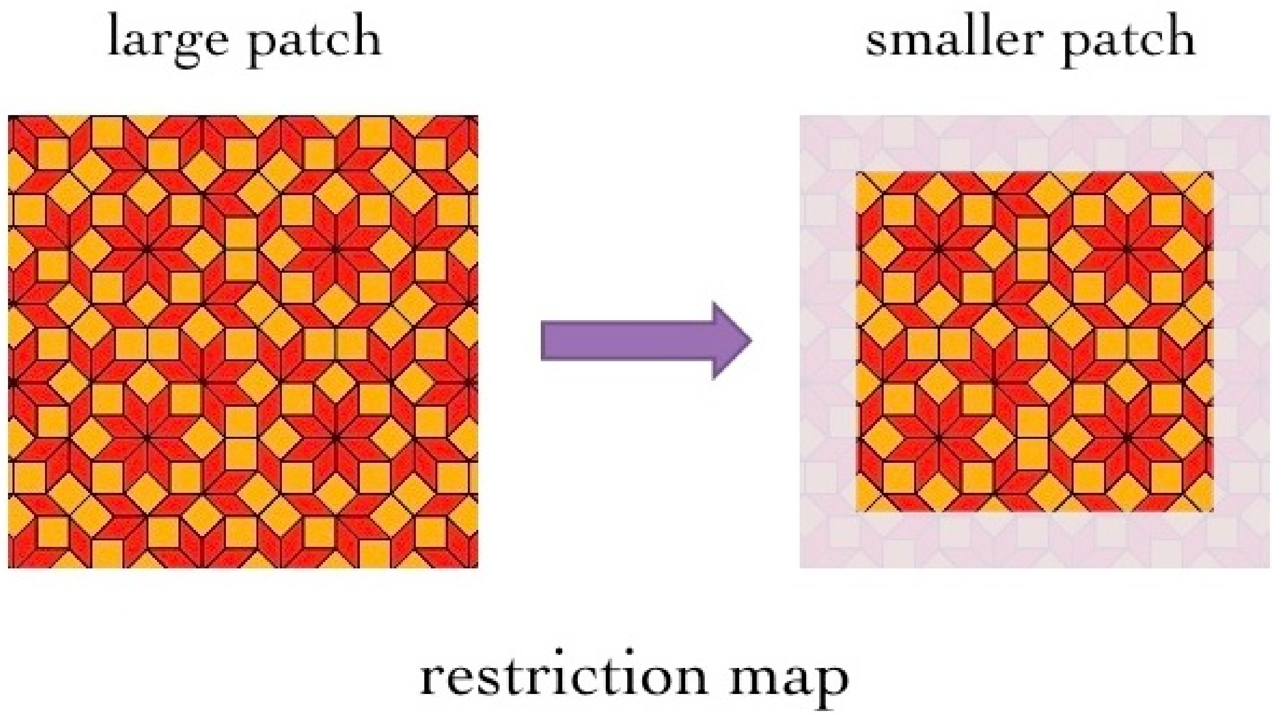
Inverse Limit



Let $\mathcal{P}_R$ be the set of patches of radius R, modulo translation.

The tiling has *finite local complexity* (FLC), if and only if $\mathcal{P}_R$ is a *finite* set for all R. In particular $R \to \mathcal{P}_R$ is *locally constant* and *non-decreasing*. Thus there is a sequence $R_0 = 0 < R_1 < \cdots < R_n < \cdots$ with $R_n \to \infty$ such that $\mathcal{P}_R = \mathcal{P}_n$ for $R_n \leq R < R_{n+1}$.

Inverse Limit



There is a restriction map $\pi : \mathcal{P}_{n+1} \to \mathcal{P}_n$. Then the *transversal* is defined by the inverse limit

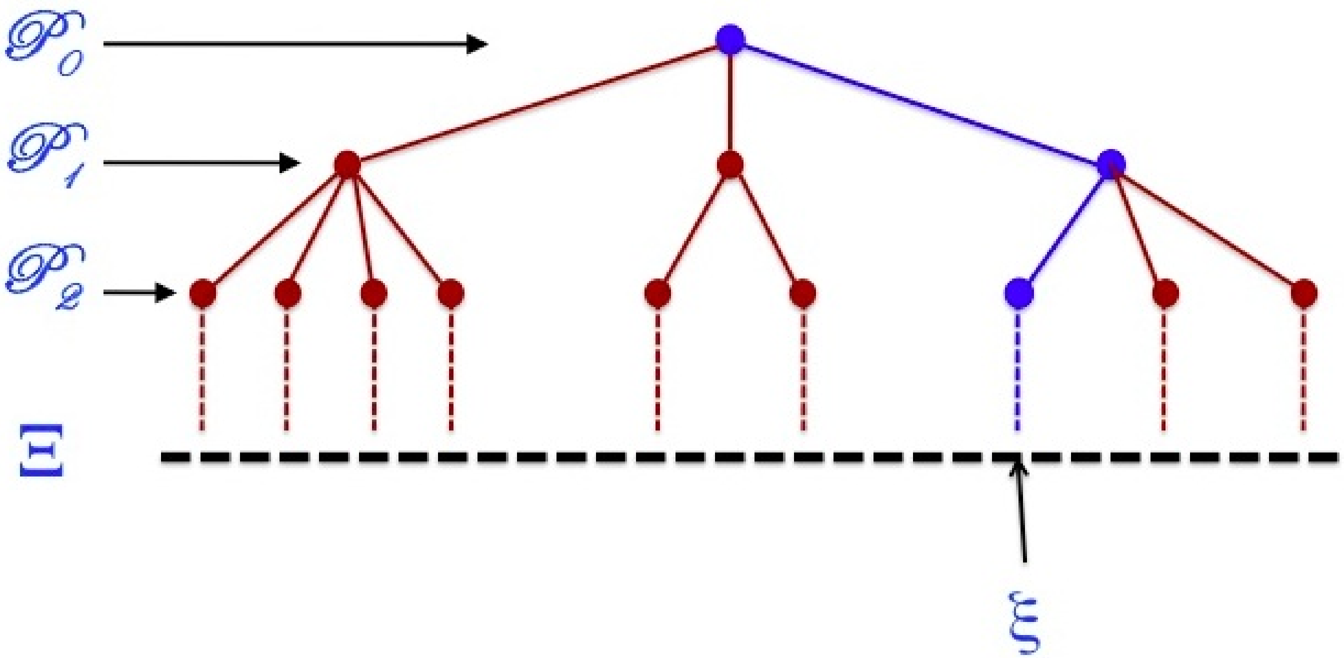
$$\Xi = \lim_{\overleftarrow{\pi}} \mathcal{P}_n$$

Rooted Tree

Since all the $\mathcal{P}_n$'s are finite set, Ξ is a *Cantor set*.

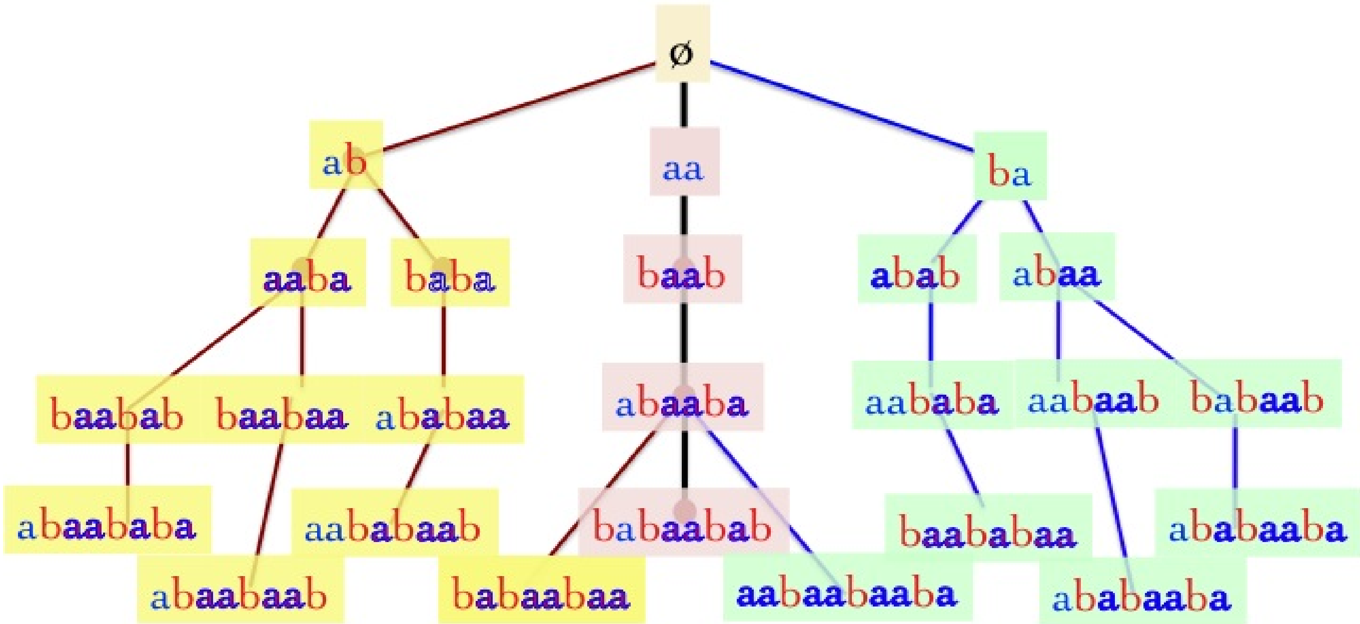
A point of Ξ is an infinite sequence $\xi = (p_n)_{n=0}^{\infty}$ of compatible patches, so it defines a unique *tiling*.

This inverse limit can be represented by a *rooted tree*



Rooted Tree

For the *Fibonacci sequence* this gives



The Fibonacci Tree

II - Spectral Triples

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

- there is a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

- there is a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D is selfadjoint with compact resolvent (*Dirac operator*)

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

- there is a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D is selfadjoint with compact resolvent (*Dirac operator*)
- the set $C^1(X)$ of elements $a \in \mathcal{A}$ leaving the domain of D invariant and such that $\|[D, \pi(a)]\| < \infty$, is dense in $\mathcal{A}$

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

- there is a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D is selfadjoint with compact resolvent (*Dirac operator*)
- the set $C^1(X)$ of elements $a \in \mathcal{A}$ leaving the domain of D invariant and such that $\|[D, \pi(a)]\| < \infty$, is dense in $\mathcal{A}$

Proposition: *Then $C^1(X)$ is a dense $*$-subalgebra of $\mathcal{A}$, invariant under the holomorphic functional calculus.*

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

$$d(x,y) = \sup\{|f(x) - f(y)|\,;\; f \in \mathcal{A},\; \|[D,f]\| \leq 1\}$$

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

$$d(x,y) = \sup\{|f(x) - f(y)|\,;\ f \in \mathcal{A},\ \|[D,f]\| \leq 1\}$$

Actually $\|[D,f]\| = \|\nabla f\|_{L^\infty} = \|f\|_{C_{\mathrm{Lip}}}$ and $C^1(X) = \mathrm{Lip}(M)$.

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

$$d(x,y) = \sup\{|f(x) - f(y)|\,;\ f \in \mathcal{A},\ \|[D,f]\| \leq 1\}$$

Actually $\|[D,f]\| = \|\nabla f\|_{L^\infty} = \|f\|_{C_{\mathrm{Lip}}}$ and $C^1(X) = \mathrm{Lip}(M)$.

Hence the algebra $\mathcal{A}$ encodes the *space*, the Dirac operator D encodes the *metric*. $\mathcal{H}$ is needed to define D.

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

- $\kappa(p)$ is non increasing as p changes from father to son,

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

- $\kappa(p)$ is non increasing as p changes from father to son,
- $\kappa(p)$ converges to zero as p tends to the end of the path.

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

- $\kappa(p)$ is non increasing as p changes from father to son,
- $\kappa(p)$ converges to zero as p tends to the end of the path.

Theorem, (Michon '84) *If $\xi, \eta \in \Xi$ let $\xi \wedge \eta$ be the least common ancestor of the path ξ and η. Then $d_\kappa(\xi, \eta) = \kappa(\xi \wedge \eta)$ defines an ultrametric on Ξ.*

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

- $\kappa(p)$ is non increasing as p changes from father to son,
- $\kappa(p)$ converges to zero as p tends to the end of the path.

Theorem, (Michon '84) *If $\xi, \eta \in \Xi$ let $\xi \wedge \eta$ be the least common ancestor of the path ξ and η. Then $d_\kappa(\xi, \eta) = \kappa(\xi \wedge \eta)$ defines an ultrametric on Ξ.*

Then $\kappa(p)$ is the diameter of the set of tilings compatible with p.

Ultrametric on Ξ

A *weight* on the rooted tree associated with Ξ is an assignement $\kappa(p) \in (0, \infty)$ on each patch p (vertex of the graph), such that

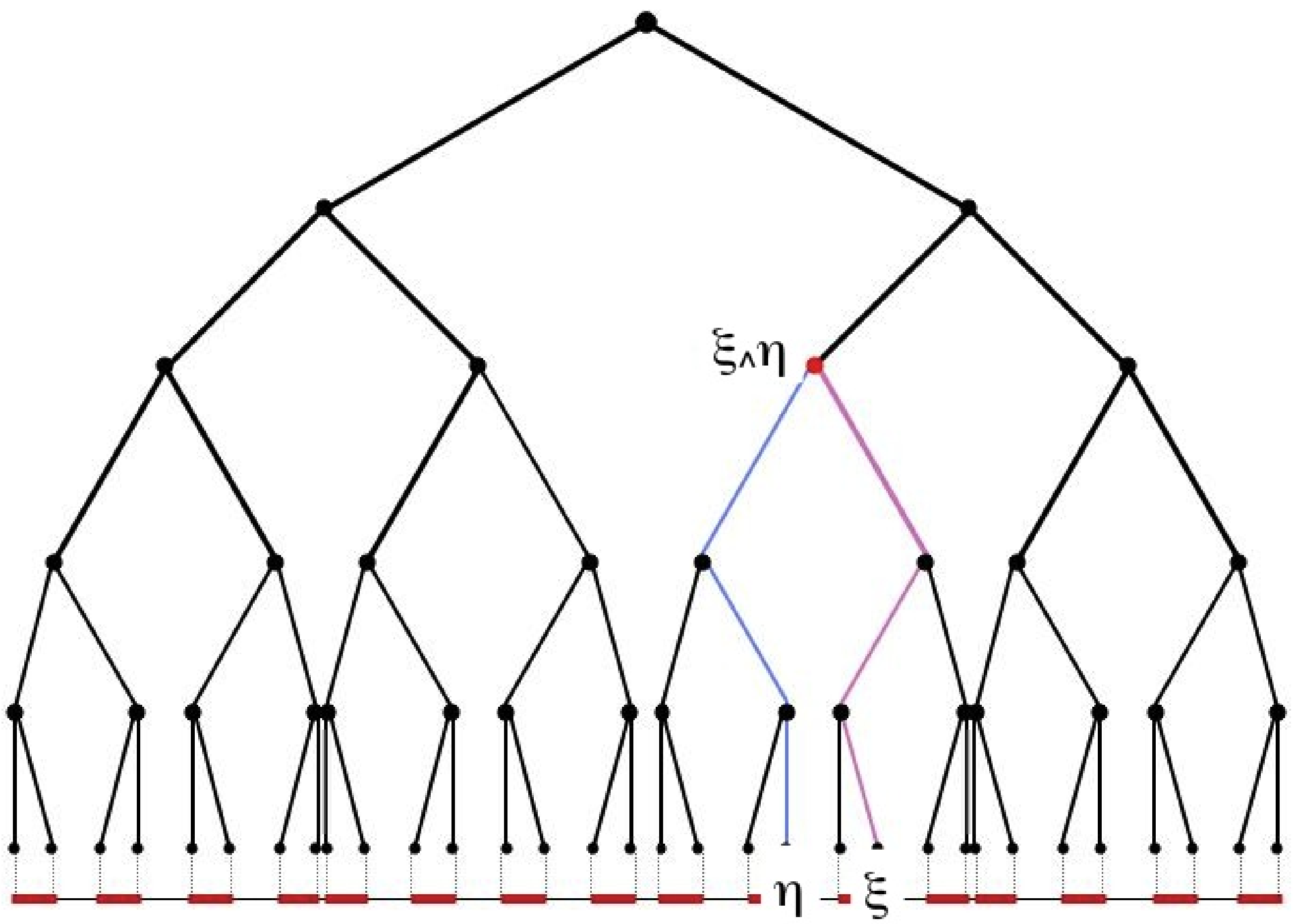
- $\kappa(p)$ is non increasing as p changes from father to son,
- $\kappa(p)$ converges to zero as p tends to the end of the path.

Theorem, (Michon '84) *If $\xi, \eta \in \Xi$ let $\xi \wedge \eta$ be the least common ancestor of the path ξ and η. Then $d_\kappa(\xi, \eta) = \kappa(\xi \wedge \eta)$ defines an ultrametric on Ξ.*

Then $\kappa(p)$ is the diameter of the set of tilings compatible with p.

Each ultrametric on Ξ can be obtained in such a way through a rooted tree defined from the metric.

Ultrametric on Ξ



Ultrametric on Ξ

Examples:

- If p is a patch of radius R, take $\kappa(p) = 1/R$,

Ultrametric on Ξ

Examples:

- If p is a patch of radius R, take $\kappa(p) = 1/R$,

- If p is a patch, take $\kappa(p)$ to be the *maximum potential energy difference* at the origin, produced by atoms outside p on all tilings of Ξ compatible with p.

The Pearson-Palmer Spectral Triple

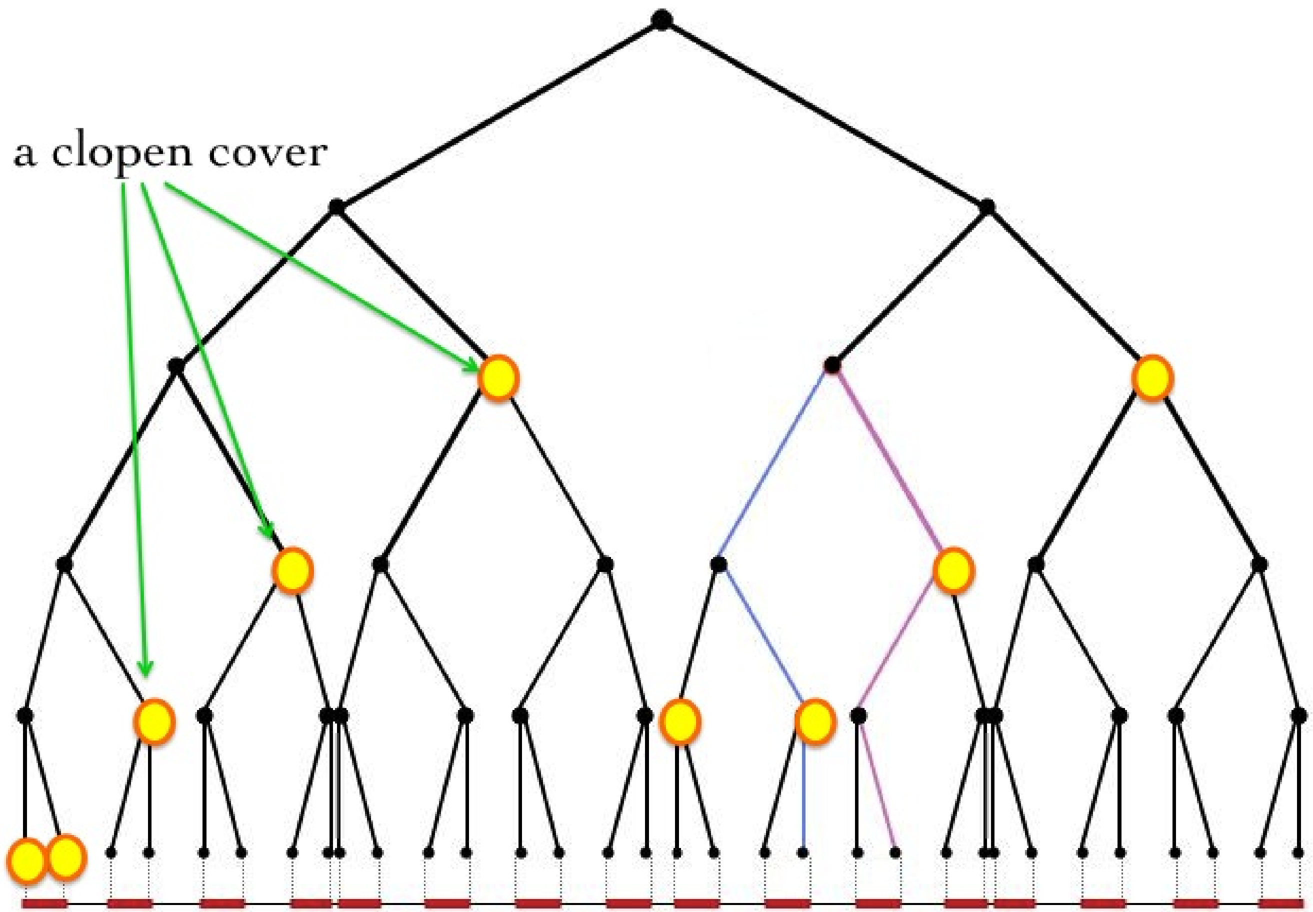
Given p a patch, let $\Xi(p)$ be the set of all tilings in Ξ compatible with p at the origin. The family $(\Xi(p))_{p\in\mathcal{P}}$ is a basis of clopen set for the topology of Ξ.

The Pearson-Palmer Spectral Triple

Given p a patch, let $\Xi(p)$ be the set of all tilings in Ξ compatible with p at the origin. The family $(\Xi(p))_{p\in\mathcal{P}}$ is a basis of clopen set for the topology of Ξ.

A *clopen cover* $\mathcal{P}$ is a finite family of patches partitionning Ξ.

The Pearson-Palmer Spectral Triple



The Pearson-Palmer Spectral Triple

Given p a patch, let $\Xi(p)$ be the set of all tilings in Ξ compatible with p at the origin. The family $(\Xi(p))_{p\in\mathcal{P}}$ is a basis of clopen set for the topology of Ξ.

A *clopen cover* $\mathcal{P}$ is a finite family of patches partitionning Ξ. Then

$$\operatorname{diam}\mathcal{P} = \max\{\kappa(p)\,;\, p \in \mathcal{P}\}$$

An infinite sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of clopen cover is called *resolving* if $\lim_{n\to\infty} \operatorname{diam}\mathcal{P}_n = 0$.

The Pearson-Palmer Spectral Triple

- **Algebra:** $\mathcal{A} = C(\Xi)$,

The Pearson-Palmer Spectral Triple

- **Algebra:** $\mathcal{A} = C(\Xi)$,
- **Hilbert Space:** $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \ell^2(\mathcal{P}_n) \otimes \mathbb{C}^2$, with $(\mathcal{P}_n)_{n\in\mathbb{N}}$ a resolving sequence of clopen covers.

The Pearson-Palmer Spectral Triple

- **Algebra:** $\mathcal{A} = C(\Xi)$,
- **Hilbert Space:** $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \ell^2(\mathcal{P}_n) \otimes \mathbb{C}^2$, with $(\mathcal{P}_n)_{n\in\mathbb{N}}$ a resolving sequence of clopen covers.
- **Dirac Operator:** for $\psi \in \mathcal{H}$

$$(D\psi)(p) = \frac{1}{\kappa(p)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(p)\,.$$

The Pearson-Palmer Spectral Triple

- **Algebra:** $\mathcal{A} = C(\Xi)$,
- **Hilbert Space:** $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \ell^2(\mathcal{P}_n) \otimes \mathbb{C}^2$, with $(\mathcal{P}_n)_{n\in\mathbb{N}}$ a resolving sequence of clopen covers.
- **Dirac Operator:** for $\psi \in \mathcal{H}$

$$(D\psi)(p) = \frac{1}{\kappa(p)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(p)\,.$$

- **Choice:** it is an assignement, for each $p \in \bigcup_n \mathcal{P}_n$ of two points $\tau(p) = (\xi_p, \eta_p)$, with $\xi_p, \eta_p \in \Xi(p)$ and $\xi_p \wedge \eta_p = p$.

The Pearson-Palmer Spectral Triple

- **Algebra:** $\mathcal{A} = C(\Xi)$,
- **Hilbert Space:** $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \ell^2(\mathcal{P}_n) \otimes \mathbb{C}^2$, with $(\mathcal{P}_n)_{n\in\mathbb{N}}$ a resolving sequence of clopen covers.
- **Dirac Operator:** for $\psi \in \mathcal{H}$

$$(D\psi)(p) = \frac{1}{\kappa(p)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(p)\,.$$

- **Choice:** it is an assignement, for each $p \in \bigcup_n \mathcal{P}_n$ of two points $\tau(p) = (\xi_p, \eta_p)$, with $\xi_p, \eta_p \in \Xi(p)$ and $\xi_p \wedge \eta_p = p$.
- **Representation:** for each choice τ and $f \in C(\Xi)$

$$(\pi_\tau(f)\psi)(p) = \begin{bmatrix} f(\xi_p) & 0 \\ 0 & f(\eta_p) \end{bmatrix} \psi(p)\,.$$

ζ-function

The ζ-function: is defined by

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

ζ-function

The ζ-function: is defined by

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

Theorem: *There is a resolving sequence of clopen covers and an $s > 0$ such that $\zeta(s) < \infty$ if and only if the metric space (Ξ, d_κ) has finite Hausdorff dimension.*

ζ-function

The ζ-function: is defined by

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

Theorem: *There is a resolving sequence of clopen covers and an* $s > 0$ *such that* $\zeta(s) < \infty$ *if and only if the metric space* (Ξ, d_κ) *has finite Hausdorff dimension.*

If so, the abscissa of convergence, defined by $s_0 = \inf\{s > 0\,;\, \zeta(s) < \infty\}$ *satisfies*

$$s_0 \geq \dim_H(\Xi)$$

ζ-function

The ζ-function: is defined by

$$\zeta(s) = \mathrm{Tr}\left(\frac{1}{|D|^s}\right)$$

Theorem: *There is a resolving sequence of clopen covers and an* $s > 0$ *such that* $\zeta(s) < \infty$ *if and only if the metric space* (Ξ, d_κ) *has finite Hausdorff dimension.*

If so, the abscissa of convergence, defined by $s_0 = \inf\{s > 0\,;\, \zeta(s) < \infty\}$ *satisfies*

$$s_0 \geq \dim_H(\Xi)$$

There exists a (non unique) resolving sequence of clopen covers $(\mathcal{P}_n)_{n\in\mathbb{N}}$*, called a Hausdorff sequence, such that* $s_0 = \dim_H(\Xi)$.

The Connes State

The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s} \pi_\tau(f) \right), \qquad f \in C(\Xi)$$

The Connes State

The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s}\, \pi_\tau(f)\right), \qquad f \in C(\Xi)$$

Theorem: *If (Ξ, d_κ) has finite Hausdorff dimension and if $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is a Hausdorff sequence, the Connes state exists if and only if Ξ has a finite nonzero Hausdorff measure.*

The Connes State

The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s}\, \pi_\tau(f) \right), \qquad f \in C(\Xi)$$

Theorem: *If (Ξ, d_κ) has finite Hausdorff dimension and if $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is a Hausdorff sequence, the Connes state exists if and only if Ξ has a finite nonzero Hausdorff measure.*

If so, $\mathcal{T}$ is independent of the choice τ.

The Connes State

The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s} \, \pi_\tau(f) \right), \qquad f \in C(\Xi)$$

Theorem: *If (Ξ, d_κ) has finite Hausdorff dimension and if $(\mathcal{P}_n)_{n \in \mathbb{N}}$ is a Hausdorff sequence, the Connes state exists if and only if Ξ has a finite nonzero Hausdorff measure.*

If so, $\mathcal{T}$ is independent of the choice τ.

If so, $\mathcal{T}$ coincides with the normalized Hausdorff measure on Ξ.

III - The Pearson Laplacian

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\, \psi\, (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\, \psi\, (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*.

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\, \psi\, (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at p.

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\, \psi\, (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at p.
- the set Υ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*.

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\, \psi\, (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at p.
- the set Υ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*.
- $[D, \pi_\tau(f)]$ could be written as $\nabla_\tau f$.

Choice Averaging

- The *choice space* Υ is given by $\prod_p \Upsilon(p)$ where $\Upsilon(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$.

Choice Averaging

- The *choice space* Υ is given by $\prod_p \Upsilon(p)$ where $\Upsilon(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$.
- Let ν_p be the probability measure on $\Upsilon(p)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\Xi(p) \times \Xi(p)$.

Choice Averaging

- The *choice space* Υ is given by $\prod_p \Upsilon(p)$ where $\Upsilon(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$.
- Let ν_p be the probability measure on $\Upsilon(p)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\Xi(p) \times \Xi(p)$.
- This leads to the probability

$$\nu = \bigotimes_p \nu_p$$

Choice Averaging

- The *choice space* Υ is given by $\prod_p \Upsilon(p)$ where $\Upsilon(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$.
- Let ν_p be the probability measure on $\Upsilon(p)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\Xi(p) \times \Xi(p)$.
- This leads to the probability

$$\nu = \bigotimes_p \nu_p$$

Hence ν_p can be interpreted as the average over the tangent unit sphere at p.

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

$$Q_s(f, g) = \int_\Upsilon d\nu(\tau)\ \mathrm{Tr}\left(\frac{1}{|D|^s}[D, \pi_\tau(f)]^*\ [D, \pi_\tau(g)]\right)$$

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

$$Q_s(f,g) = \int_{\Upsilon} d\nu(\tau)\ \mathrm{Tr}\left(\frac{1}{|D|^s}[D, \pi_\tau(f)]^* \ [D, \pi_\tau(g)]\right)$$

Theorem: *If (Ξ, d_κ) has positive finite Hausdorff measure, for each $s \in \mathbb{R}$, the quadratic forms Q_s is densely defined, closable in $L^2(X, \mu_H)$ and is a Dirichlet form.*

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

$$Q_s(f,g) = \int_{\Upsilon} d\nu(\tau) \, \mathrm{Tr} \left(\frac{1}{|D|^s} [D, \pi_\tau(f)]^* \, [D, \pi_\tau(g)] \right)$$

Theorem: *If (Ξ, d_κ) has positive finite Hausdorff measure, for each $s \in \mathbb{R}$, the quadratic forms Q_s is densely defined, closable in $L^2(X, \mu_H)$ and is a Dirichlet form.*

The corresponding positive operator Δ_s has pure point spectrum. It is bounded if and only if $s > \dim_H(\Xi) + 2$ and has compact resolvent otherwise.

The Pearson Quadratic Form

The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

$$Q_s(f,g) = \int_\Upsilon d\nu(\tau)\ \mathrm{Tr}\left(\frac{1}{|D|^s}[D,\pi_\tau(f)]^*\ [D,\pi_\tau(g)]\right)$$

Theorem: *If (Ξ, d_κ) has positive finite Hausdorff measure, for each $s \in \mathbb{R}$, the quadratic forms Q_s is densely defined, closable in $L^2(X, \mu_H)$ and is a Dirichlet form.*

The corresponding positive operator Δ_s has pure point spectrum. It is bounded if and only if $s > \dim_H(\Xi) + 2$ and has compact resolvent otherwise.

The eigenspaces are common to all s's and can be explicitly computed.

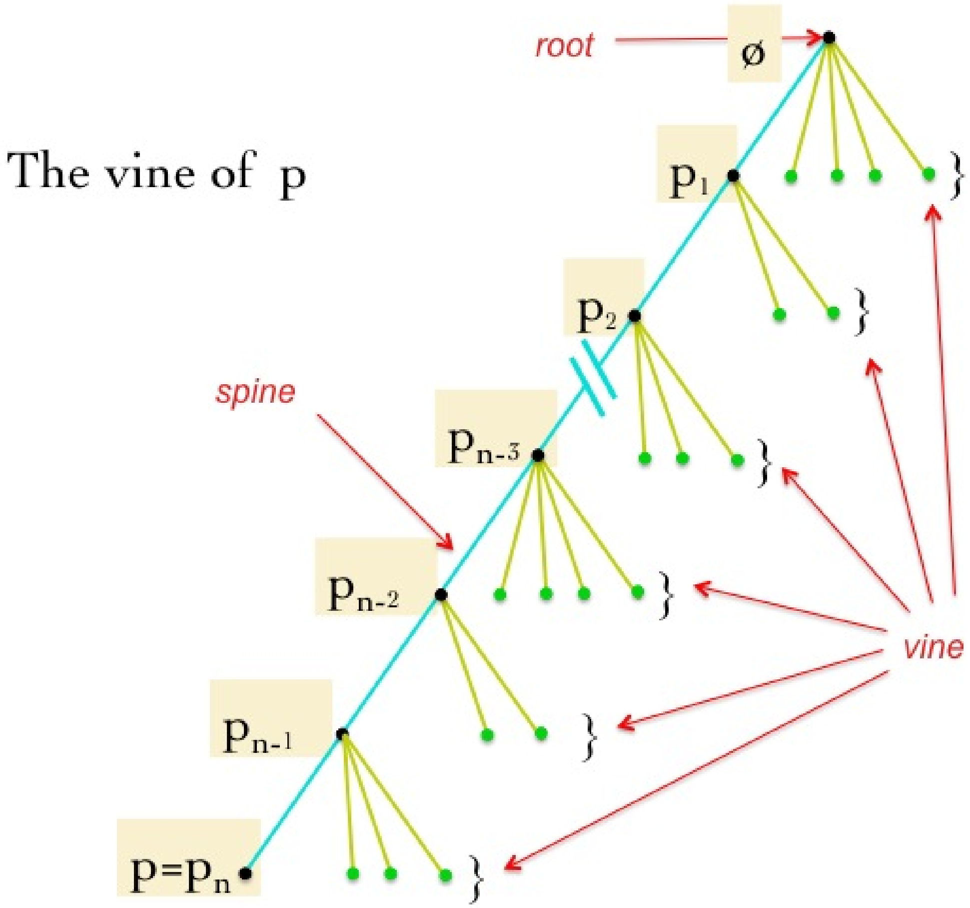
Jump Process

Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in Ξ.

Jump Process

Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in Ξ.

Given a patch p, its *spine* is the set of vertices located along the finite path joining the root to p. The *vine $\mathcal{V}(p)$ of p* is the set of patches, not in the spine, which are children of one vertex of the spine.



The vine of a vertex v

Jump Process

Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in Ξ.

Given a patch p, its *spine* is the set of vertices located along the finite path joining the root to p. The *vine* $\mathcal{V}(p)$ *of* p is the set of patches, not in the spine, which are children of one vertex of the spine.

If χ_p is the characteristic function of $\Xi(p)$, the Pearson operator acts as

$$\Delta_s \chi_p = \sum_{q\in\mathcal{V}(p)} M(p,q)(\chi_q - \chi_p)$$

Jump Process

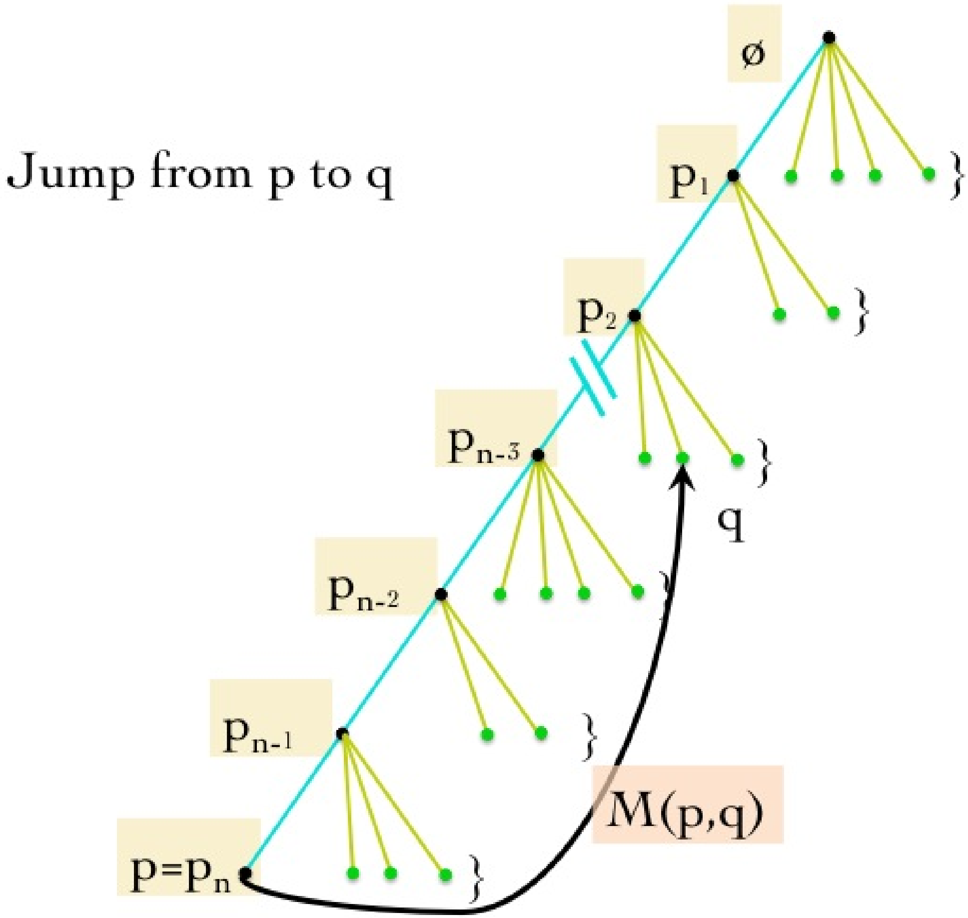
Δ_s generates a Markov semigroup, thus a stochastic process $(X_t)_{t\geq 0}$ where the X_t's takes on values in Ξ.

Given a patch p, its *spine* is the set of vertices located along the finite path joining the root to p. The *vine* $\mathcal{V}(p)$ *of* p is the set of patches, not in the spine, which are children of one vertex of the spine.

If χ_p is the characteristic function of $\Xi(p)$, the Pearson operator acts as

$$\Delta_s \chi_p = \sum_{q \in \mathcal{V}(p)} M(p,q)(\chi_q - \chi_p)$$

where $M(p,q) > 0$ represents the
probability rate (per unit time) for X_t *to jump from* $\Xi(p)$ *to* $\Xi(q)$.



Jump process from v to w

Jump Process

Concretely, if $\hat{q}$ denotes the *father* of q (which belongs to the spine)

$$M(p,q) \;=\; 2\kappa(\hat{q})^{s-2}\;\frac{\mu_p}{Z_{\hat{q}}} \qquad\qquad \mu_p = \mu_H(\Xi(p))$$

where $Z_{\hat{q}}$ is the *normalization constant* for the measure $\nu_{\hat{q}}$ on the set of choices at $\hat{q}$, namely

$$Z_{\hat{q}} = \sum_{q' \neq q'' \in \mathrm{Ch}(\hat{q})} \mu_{q'}\;\mu_{q''}$$

where $\mathrm{Ch}(\hat{q})$ denotes the set of children of $\hat{q}$.

uOttawa

Thanks for Listening !