Spectrum Approximation
for
Aperiodic Hamiltonians

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Groups, Geometry & Actions

Contributors

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S. Beckus,
Spectral approximation of aperiodic Schrödinger operators,
PhD Thesis, U. Jena, October 6, 2016
arXiv: 1610.05894

Recent Work

S. Beckus, J. Bellissard,
Continuity of the Spectrum of a Field of Self-Adjoint Operators,
Ann. Henri Poincaré, **17**, (2016), 3425-3442.

S. Beckus, J. Bellissard, G. De Nittis,
Spectral Continuity for Aperiodic Quantum Systems I. General Theory,
`arXiv: 1709.00975` *to appear in* J. Funct. Anal.

S. Beckus, J. Bellissard, G. De Nittis,
Spectral Continuity for Aperiodic Quantum Systems II. Periodic approximations in 1D,
`arXiv: 1803.03099`

S. Beckus, J. Bellissard, H. Cornean,
Spectral Lipschitz Continuity for FLC Aperiodic Systems,
in preparation

S. Beckus, J. Bellissard,
Spectral Stability of Schrödinger operators for the Kohmoto model,
in preparation

S. Beckus, J. Bellissard, G. De Nittis,
Computing the Spectrum of 1D-tight binding Models,
in preparation

Content

1. The Kohmoto Model
2. Sturmian Sequences
3. Observable Algebra
4. Extensions ?

I - The Kohmoto Model

M. Kohmoto, L. Kadanoff, Chan Tang, Phys. Rev. Lett., **50**, 1870-72, (1983).

S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, E. D. Siggia,
Phys. Rev. Lett., **50**, 1873-76, (1983).

The Model

- On $\ell^2(\mathbb{Z})$

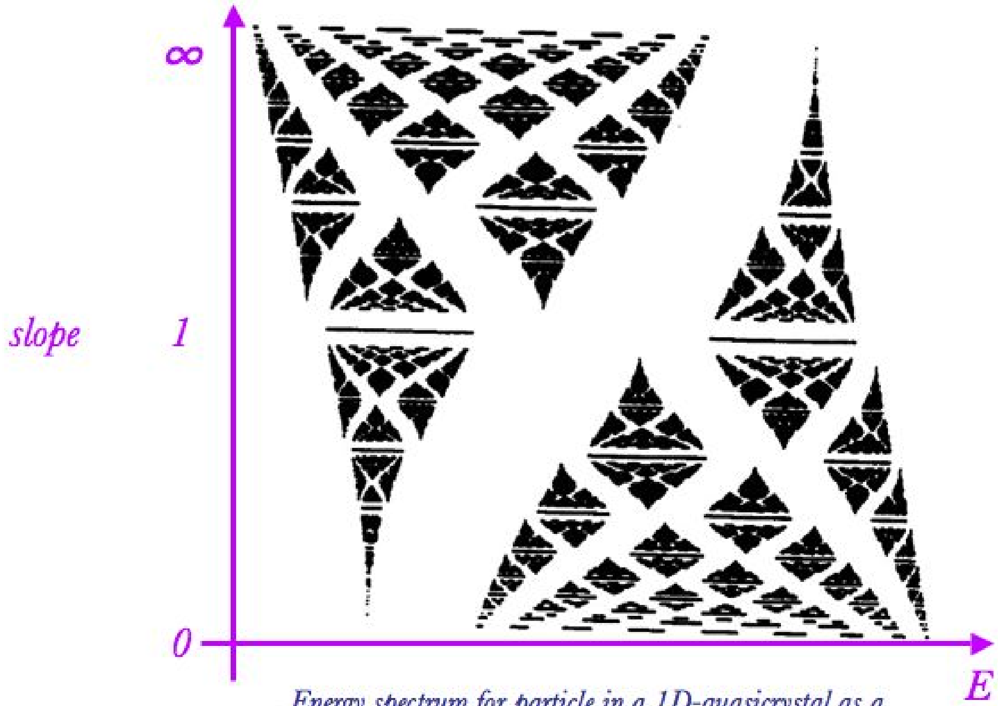
$$H_{\alpha,x}\psi(n) = \psi(n+1) + \psi(n-1) + \lambda\,\chi_{(0,\alpha]}(n\alpha - x)\,\psi(n)$$

- This model describes the electron motion in a *1D quasicrystal*.

- Introduced in 1983 by two groups of Physicists *(Kohmoto et al, '83, Ostlund et al '83)*. A numerical cal culation of the spectrum as a function of α was performed by *(Ostlund & Kim '85)*.

- A *transfer matrix* approach leads to a dynamical system called the *trace map* allowing to study rigorously the spectrum and the spectral measure *(Casdagli '86, Sütó '87, JB-Iochum-Scoppola & Testard '89, Damanik & Gorodetsky, '92-17)*.

The Kohmoto Spectrum



Energy spectrum for particle in a 1D-quasicrystal as a function of the slope in a cut-and-project scheme

Physica Scripta. Vol. T9, 193–198, 1985

Renormalization of Quasiperiodic Mappings

Stellan Ostlund and Seung-hwan Kim

Spectrum of the Kohmoto model
(Fibonacci Hamiltonian)

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\ \chi_{(0,\alpha]}(x - n\alpha)\ \psi(n)$$

as a function of α.

Method:
transfer matrix calculation

The Kohmoto Spectrum

J. Bellissard, B. Iochum, D. Testard, Commun. Math. Phys. **141**, 353-380, (1991).

- The spectrum is independent of x. It is continuous near each *irrational* α.

- The spectrum exhibit a *discontinuity* at each $\alpha = p/q \in \mathbb{Q}$.

- The limits from above or below introduce an *eigenvalue* of multiplicity one *in each gap*.

- The limit $\alpha \to p/q \pm 0$ introduces a *rank one defect* in a q-periodic chain.

The Kohmoto Spectrum

D. Damanik, B. Gorodetsky *et al.*, 1992-2017.

- For the case $\alpha = (\sqrt{5} - 1)/2$ more has been proved

- The spectral measure is singular continuous at any $\lambda > 0$.

- The Hausdorff dimension of the spectral measure has been estimated with high accuracy for every $\lambda > 0$. The asymptotic at $\lambda \sim 0$ and $\lambda \to \infty$ are known exactly.

So why bother further ?

II - Sturmian Sequences

N. Pytheas Fogg, *Substitutions in dynamics, arithmetics, and combinatorics*, see P. Arnoux, Chap. 6, Lect. Notes in Math., Springer, (2002).

Sturmian sequences

N. Pytheas Fogg, *Substitutions in dynamics, arithmetics, and combinatorics*, see P. Arnoux, Chap. 6, Lect. Notes in Math., Springer, (2002).

- Let $\xi = (a_k)_{k\in\mathbb{Z}}$ be a sequence of letters $a_k \in \mathcal{A}$ picked in a *finite alphabet* $\mathcal{A}$.

- For such a ξ
  - it is called *Sturmian* whenever the number of its distinct subwords of length n is exactly $n+1$. Then $\mathcal{A}$ must have only two letters $\mathcal{A} = \{a, b\}$.
  - it will be called called *quasi-Sturmian* whenever the number of its distinct subwords of length n is less than or equal to $n+1$.

- The set Ξ of quasi-Strumian sequences in $\Omega = \mathcal{A}^{\mathbb{Z}}$, is *closed* and *shift invariant*.

Sturmian sequences

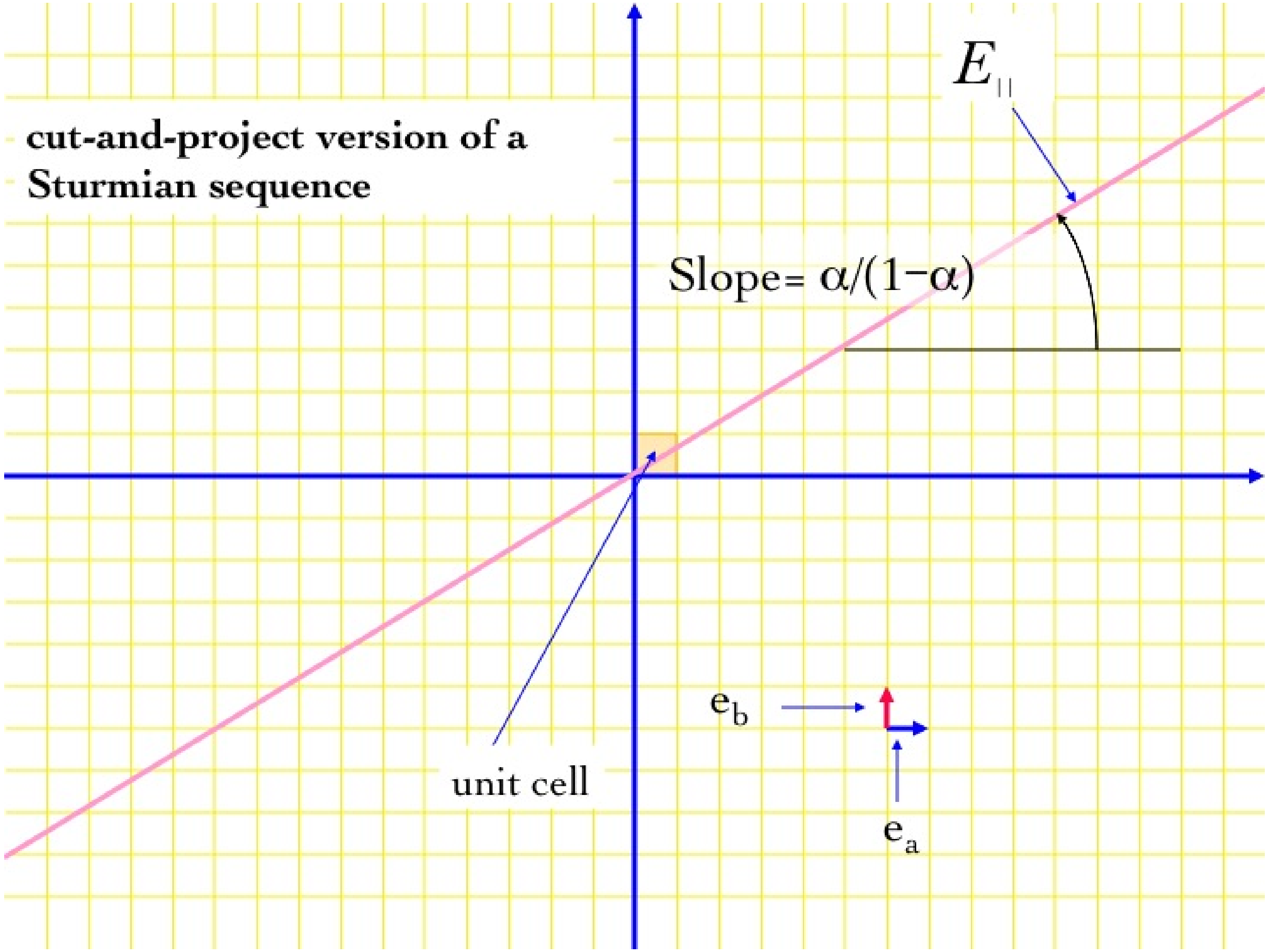
N. Pytheas Fogg, *Substitutions in dynamics, arithmetics, and combinatorics*, see P. Arnoux, Chap. 6, Lect. Notes in Math., Springer, (2002).

For $\mathcal{A} = \{a, b\}$, set $\epsilon(a) = 0$, $\epsilon(b) = 1$. Then

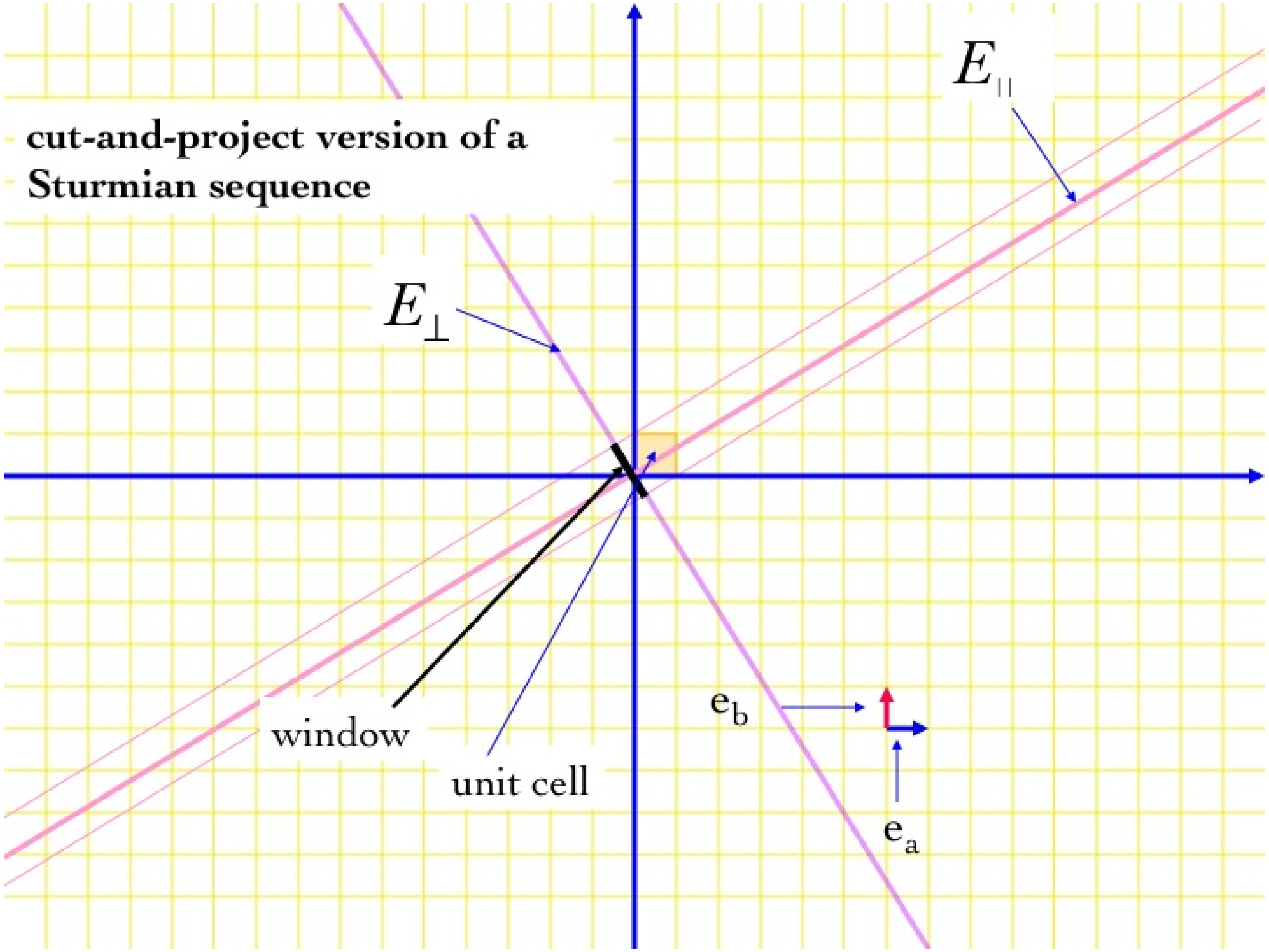
Theorem: *A sequence $\xi = (a_k)_{k\in\mathbb{Z}} \in \Omega$ is Sturmian if and only if there is $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $x \in [0, 1]$ such that $\epsilon(a_k) = \chi_{(0,\alpha]}(k\alpha - x)$*

Remark: *in the cut-and-project representation, the parameter x gives the position of the reference line associated with the sequenceξ*

Cut-and-Project Representation



Cut-and-Project Representation



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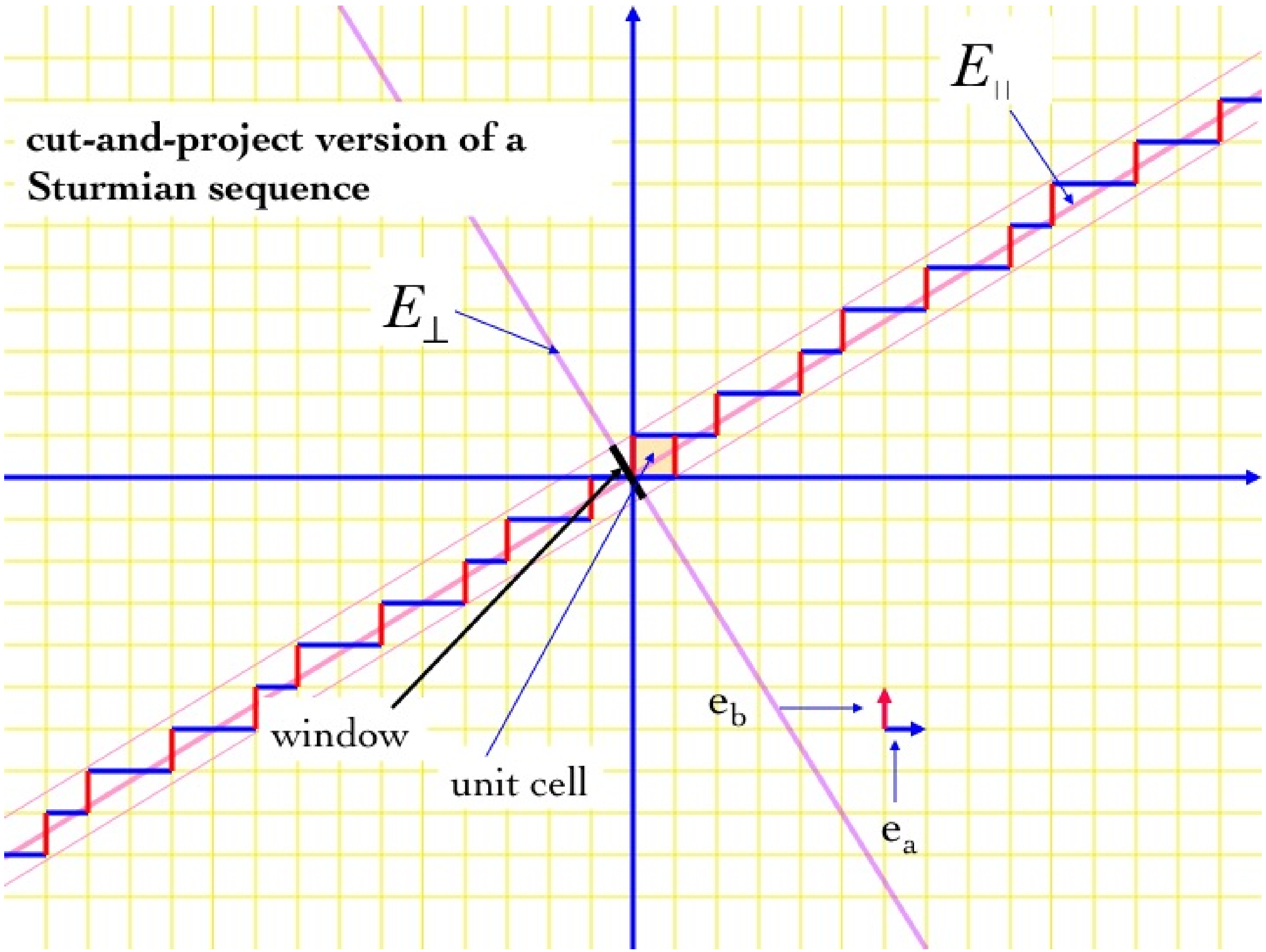

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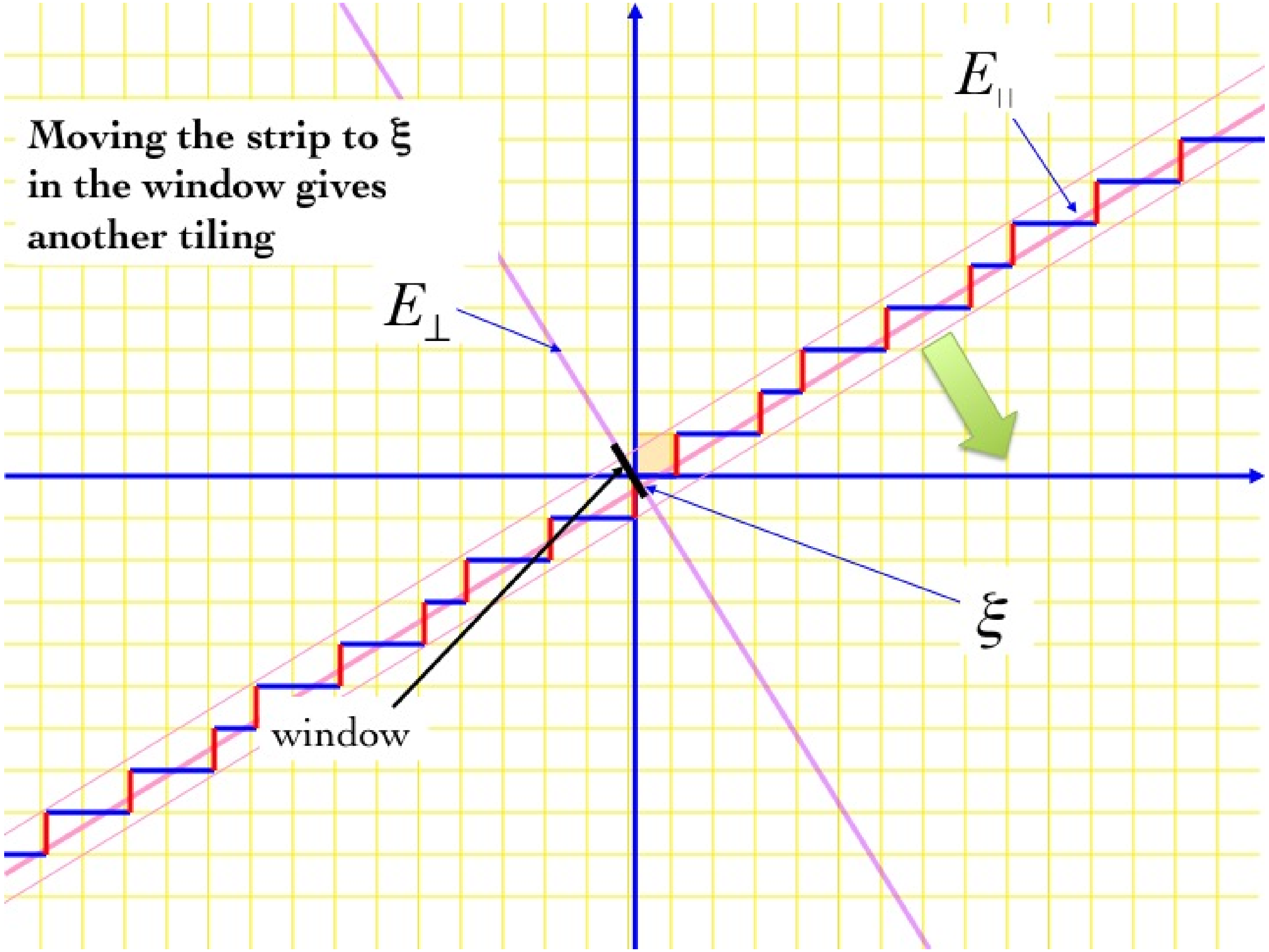

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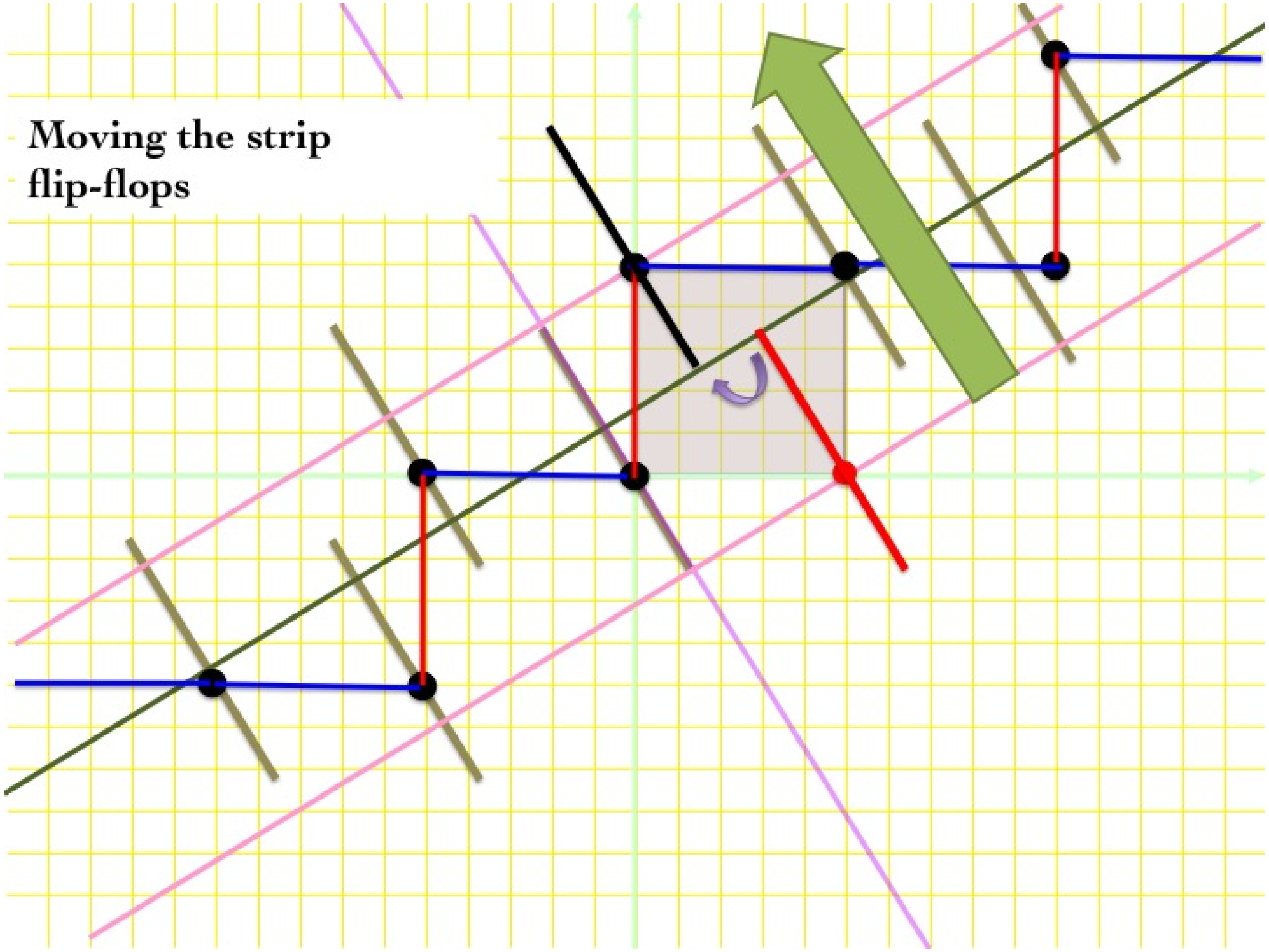

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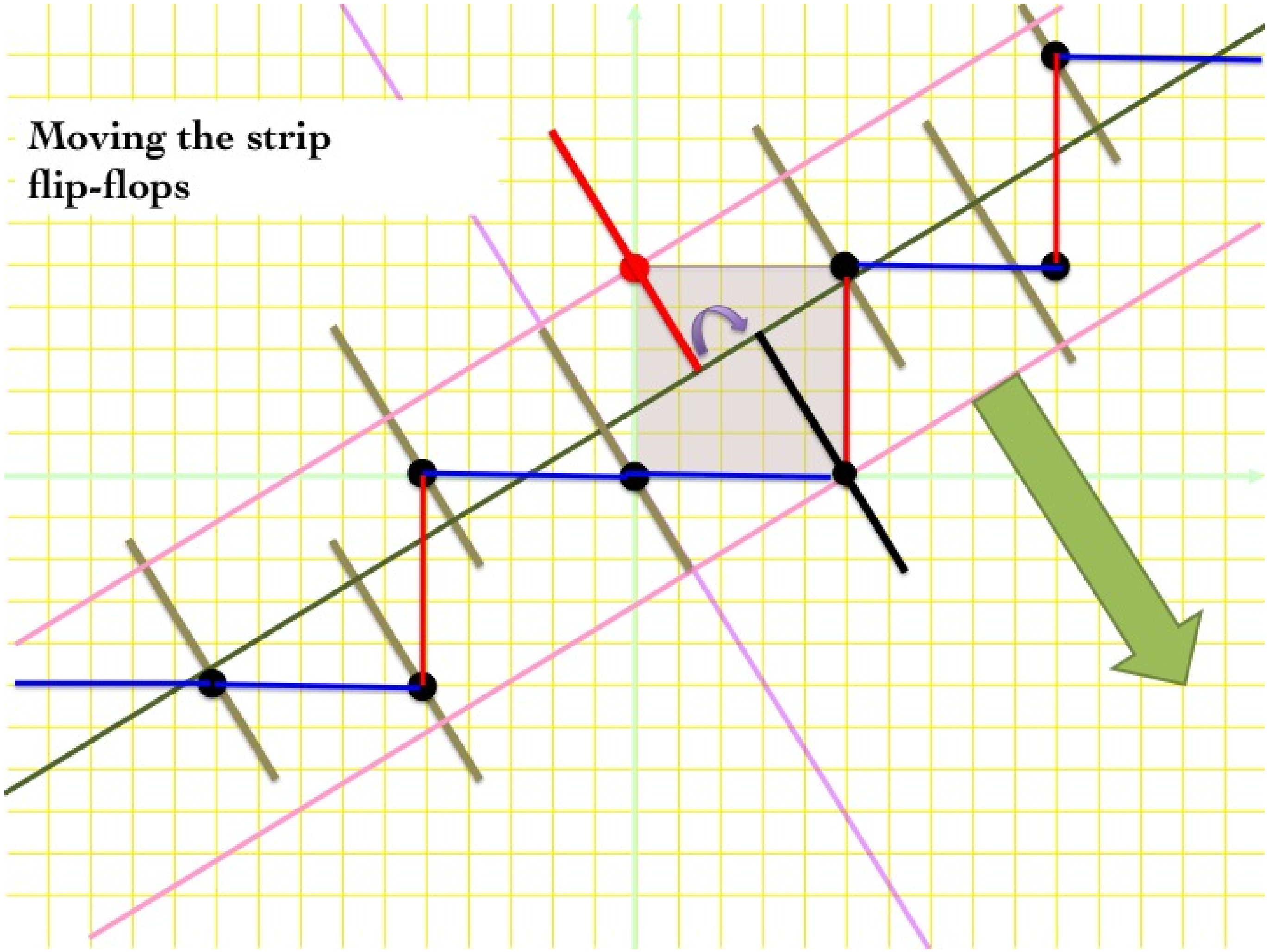

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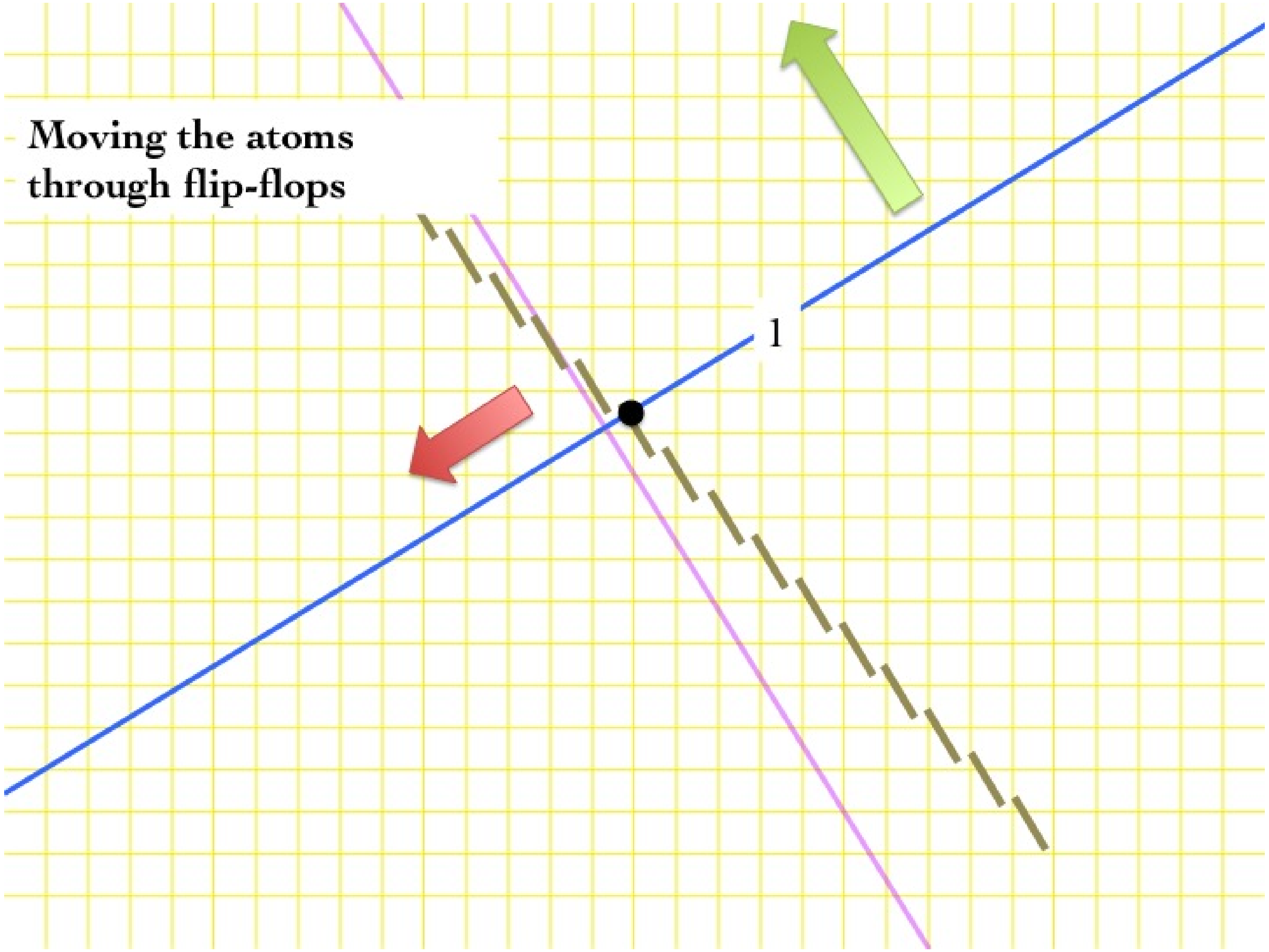

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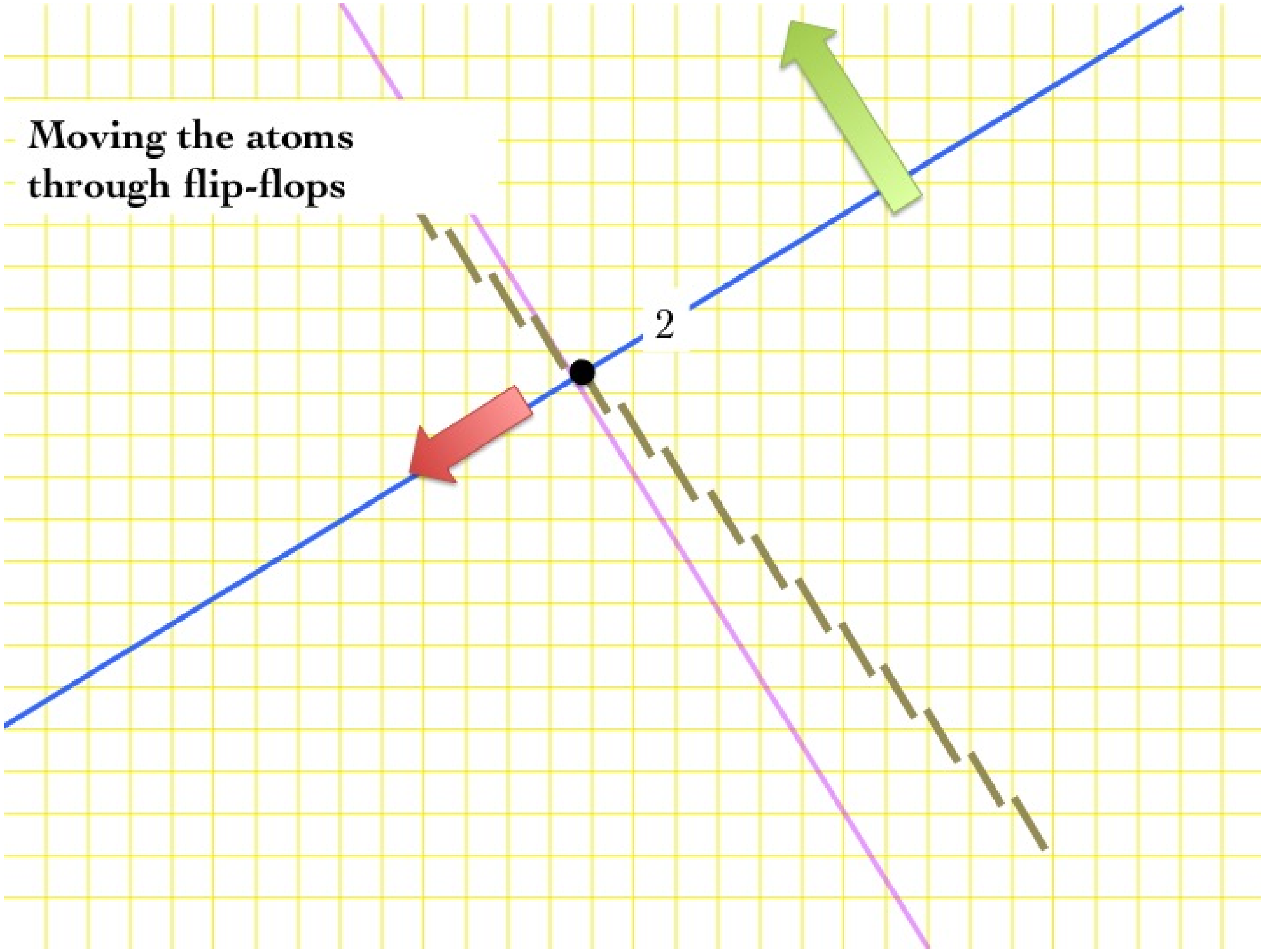

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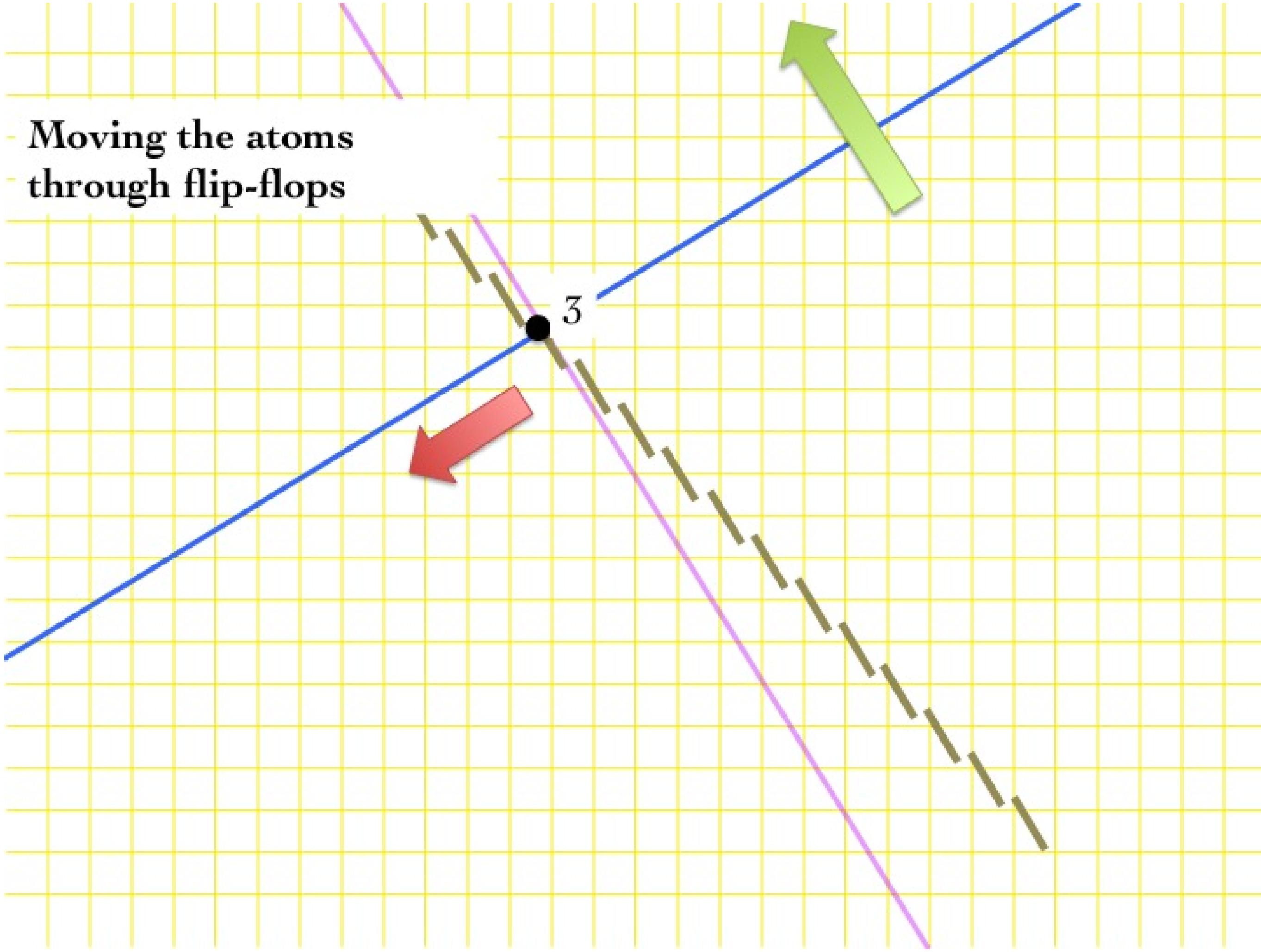
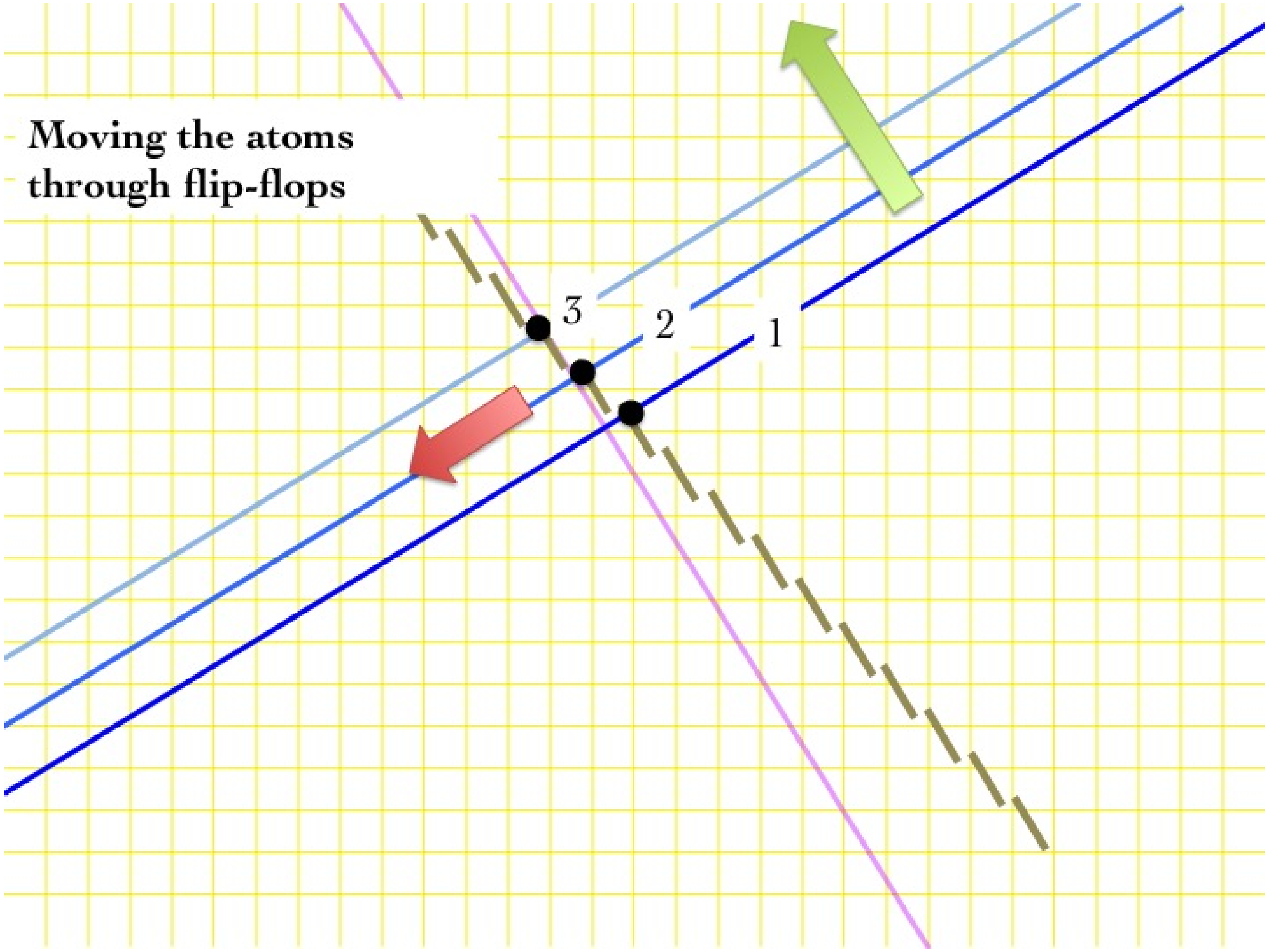


Cut-and-Project Representation



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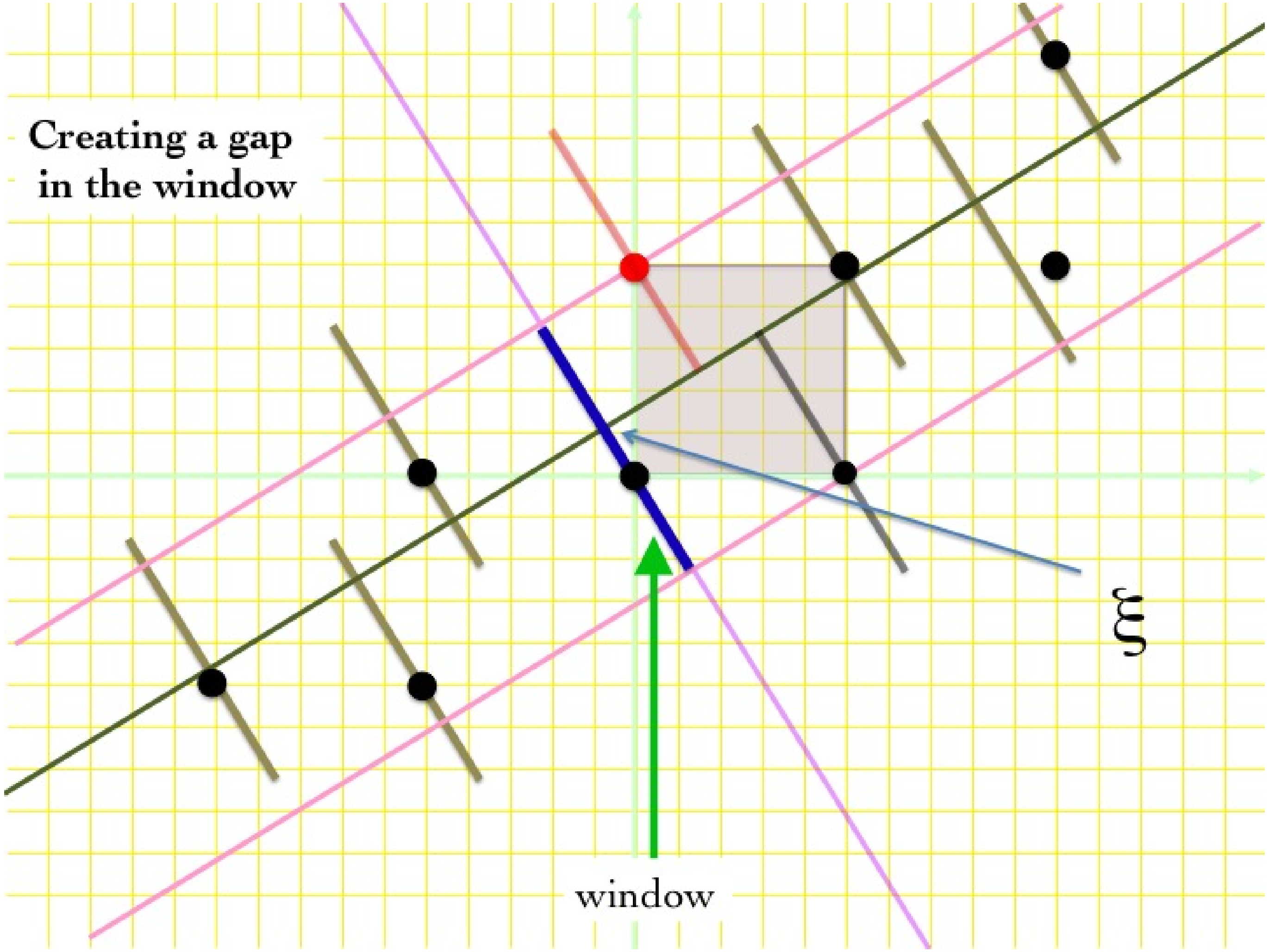

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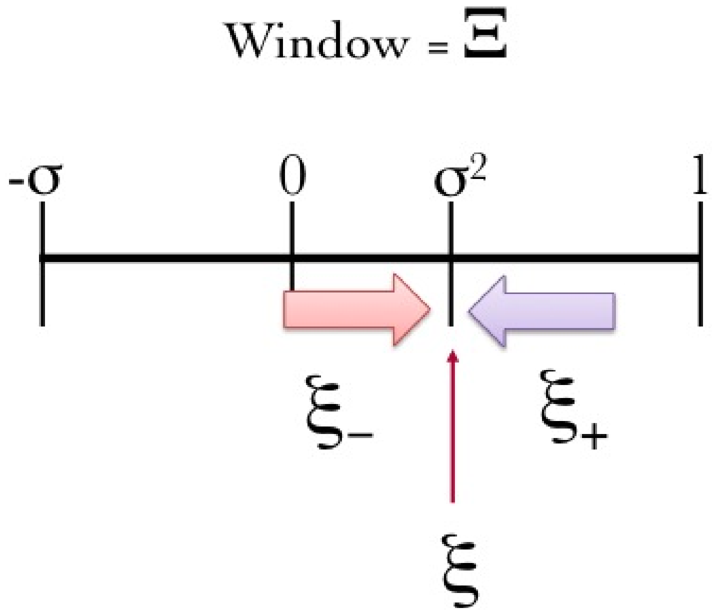

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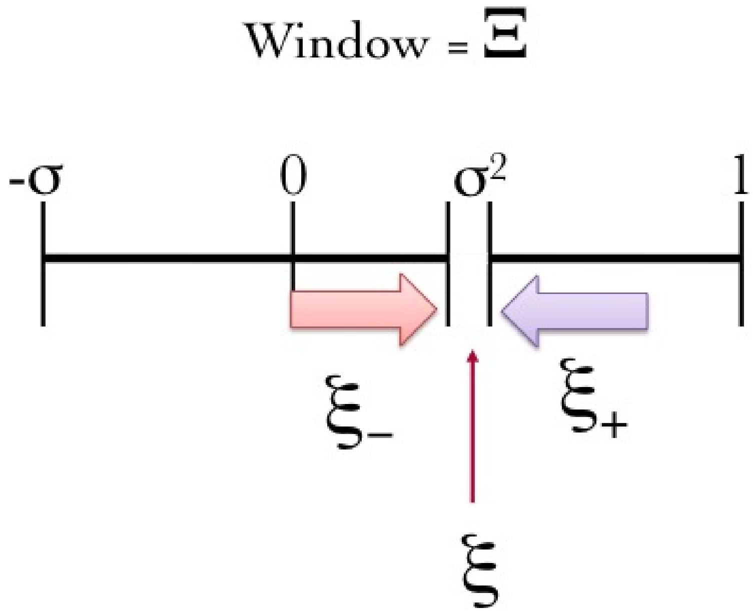
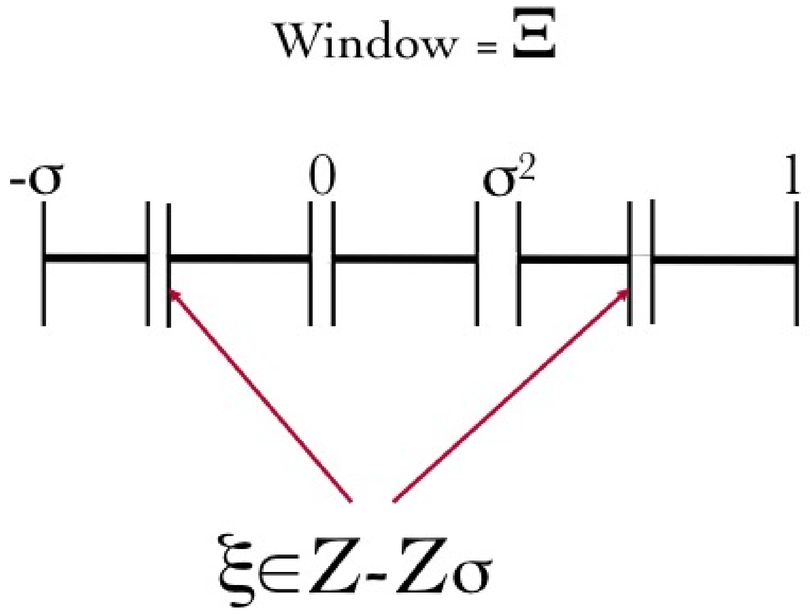


Cut-and-Project Representation



Sturmian sequences

- Endow Ω with the *combinatorial metric* d_{comb} defined, for two sequences $\xi = (a_k)_{k\in\mathbb{Z}}$, $\eta = (b_k)_{k\in\mathbb{Z}}$ in Ω, by

$$d_{\mathrm{comb}}(\xi,\eta) = \inf\left\{\frac{1}{n+1}\,;\, a_k = b_k\,,\ |k| \le n\right\}$$

- d_{comb} is an *ultrametric*.

- Let d_{Hc} denotes the corresponding *Hausdorff metric* defined on the closed subspaces of Ω.

- Let $\mathfrak{J}(\Xi)$ denotes the set of *closed shift invariant* subsets of the quasi-Sturmian set Ξ.

Examples of Closed Shift Invariant Sets

- Any *periodic* sequence η in Ω has a *finite* orbit $\mathrm{Orb}(\eta)$, thus it is closed and invariant, namely $\mathrm{Orb}(\eta) \in \mathfrak{J}(\Xi)$.

- Given any *quasi-Strurmian* sequence ξ, its *orbit closure* $\Xi_\xi = \overline{\{S^n\xi\,;\, n \in \mathbb{Z}\}}$ belongs to $\mathfrak{J}(\Xi)$.

- Let ζ be obtained from a periodic sequence η, by *cutting* it around the origin and inserting a *finite word*, called a *defect*, between the two semi-infinite sequences. Then again $\Xi_\zeta \in \mathfrak{J}(\Xi)$. In addition
  - by *shifting* the defect to *infinity*, the sequence η is recovered showing that $\eta \in \Xi_\zeta$
  - However, since $\Xi_\eta \neq \Xi_\zeta$, it follows that $d_{\mathrm{Hc}}(\Xi_\zeta, \Xi_\eta) > 0$.

Examples of Closed Shift Invariant Sets

Consider *two periodic sequences* $\eta_r \neq \eta_\ell$. Produce a new sequence ζ by gluing together the *left part* of η_ℓ on the *left* of the origin and the *right part* of η_r on the *right*. Then $\zeta \in \Xi$ is quasi-Sturmian. However

Lemma: *The orbit closure* Ξ_ζ *contains both* $\eta_\pm$
The Hausdorff combinatorial distance $d_{\mathrm{Hc}}\left(\Xi_\zeta, \Xi_\eta\right) > 0$ *for any periodic sequence* η

In other words there are quasi-Sturmian sequences in Ξ that *cannot be a approximated by periodic ones* in the space $\mathfrak{J}(\Xi)$!!

The Space Ξ

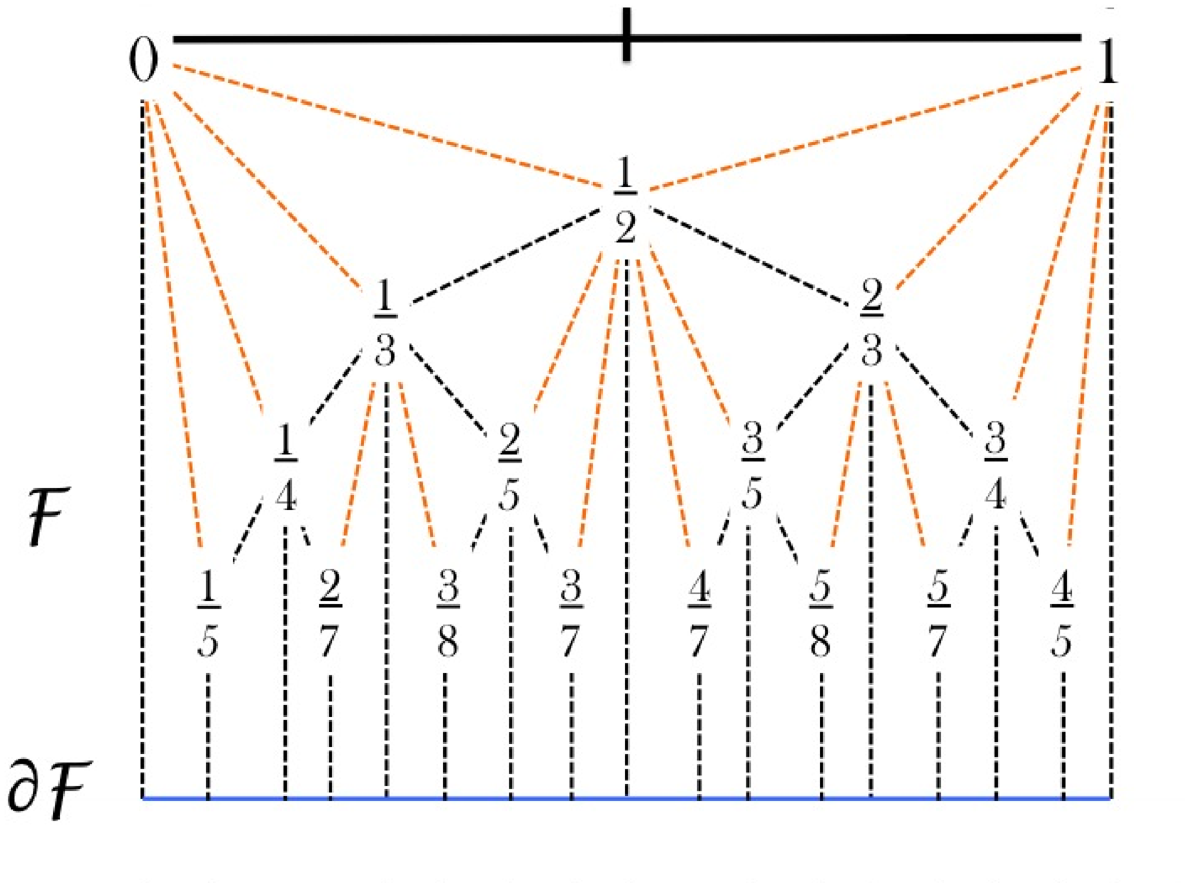
The first part of the following result follows from *(Beckus, JB, De Nittis '18)*, the second part is coming from *(Beckus, JB, in preparation)*,

Theorem: *The space $\mathfrak{J}(\Xi)$, equipped with the metric d_{Hc} is closed and compact.*

It is completely disconnected with isolated points corresponding to periodic Sturmian sequences. It is homeomorphic to the limiting space of the Farey tree.

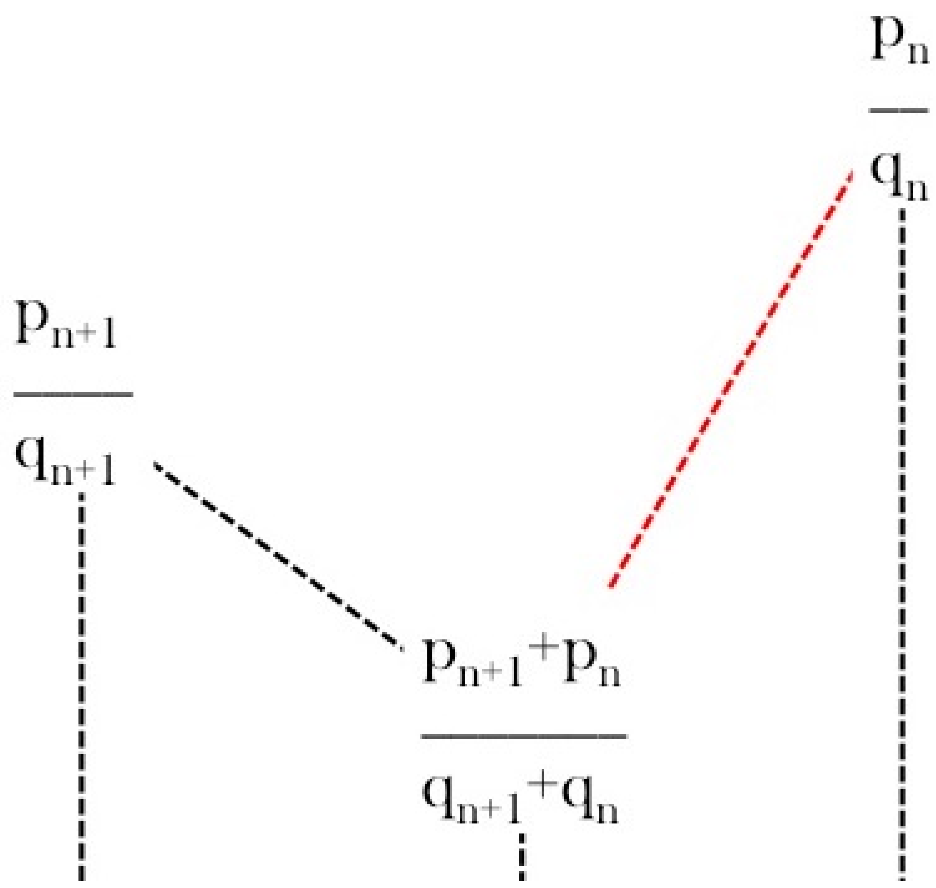
The Farey tree ?

The Farey Tree



The Farey Tree

The algorithm (due independently to *(C. Haros, 1802, A.-L. Cauchy circa 1816)*) is visually provided by



The Farey Tree

- If the red edges are *ignored*, the Farey graph $\mathfrak{F}$ is a *tree* with *root* $1/2$.
- Each vertex is a *fraction* from which *three branches* grow. The central one gives an infinite path *without* further branching.
- The *limit set* $\partial\mathfrak{F}$ is defined as the set of *infinite paths* starting from the root.
- If v is a vertex, $[v]$ denotes the set of path passing *through* v. The family $\{[v]\,;\,v\ -\ a\ vertex-\}$ defines a *basis* for a topology on $\partial\mathfrak{F}$.
- $\partial\mathfrak{F}$ is compact and totally disconnected *(Pearson, JB '08)*. An infinite path with eventually no branching gives an isolated point in $\partial\mathfrak{F}$.
- There is a canonical *continuous surjective* map $\phi : \partial\mathfrak{F} \to [0,1]$ which is *one-to one* on irrational numbers and *3-to-one* on rational ones.

III - Observable Algebra

Quantum Observables

- The Kohmoto Hamiltonian actually does not depends on α but on the *Sturmian sequence* ξ the pair (α, x) defines. Then it can can be abstractly written as

$$H_\xi = U + U^{-1} + \lambda V_\xi \,,$$

where U is the *translation by one* in $\ell^2(\mathbb{Z})$ and

- the *potential* V is defined as a function on Ξ by

$$\widehat{V}(\xi) = (\xi)_0 \,, \qquad V_\xi \, \psi(n) = \epsilon\left(\widehat{V}(S^{-n}\xi)\right) \psi(n) \,.$$

where S denotes the *shift* acting on Ξ.

Quantum Observables

- The set of functions $\{V \circ S^{-n}\,;\, n \in \mathbb{Z}\}$ generates an Abelian $*$-algebra C_0 by pointwise addition, product, complex conjugation, called the set of *pattern equivariant* functions *(Kellendonk '03)*.
- Completing C_0 with the *uniform topology*, gives an Abelian C^*-algebra denoted by C.

Theorem *The C^*-algebra C is*

- *$*$-isomorphic to $C(\Xi)$, namely Ξ can be seen as the set of characters of C.*
- *The shift map $\theta_S : f \in C \to f \circ S^{-1}$ extends as a $*$-automorphism*
- *The C^*-algebra generated by the element U, V is the crossed product algebra $C(\Xi) \rtimes_{\theta_S} \mathbb{Z}$*

Continuous Field

- Given a *closed shift invariant subset* $\Sigma \in \mathfrak{J}(\Xi)$, the same construction can be made, namely it gives a C^*-algebra $\mathcal{A}_\Sigma = C(\Sigma) \rtimes_{\theta_S} \mathbb{Z}$
- If $\xi \in \Xi$, then the Hamiltonian H_ξ is *shift covariant* namely

$$U H_\xi U^{-1} = H_{S\xi} \,,$$

- In particular the spectrum of H_ξ *depends only upon the orbit* of ξ. By *strong continuity* it contains the spectrum of H_η for any η in the orbit closure Ξ_ξ
- Thus it gives an element of the C^*-algebra $\mathcal{A}_{\Xi_\xi}$

Continuous Field

From the general theory *(Beckus, JB, De Nittis, '18 to appear in JFA)* it follows that

- The field of C^*-algebra $(\mathcal{A}_\Sigma)_{\Sigma \in \mathfrak{J}(\Xi)}$ is *continuous* for the *Hausdorff topology* of $\mathfrak{J}(\Xi)$
- The family $H_\Sigma = (H_\xi)_{\xi \in \Sigma}$ defines a continuous section of this field.
- The *spectrum* of H_Σ, seen as an element of the C^*-algebra $\mathcal{A}_\Sigma$ satisfies

$$\sigma(H_\Sigma) = \bigcup_{\xi \in \Sigma} \sigma(H_\xi)$$

- The map $\Sigma \in \mathfrak{J}(\Xi) \to \sigma(H_\Sigma) \subset \mathbb{R}$ is *continuous* in the *Hausdorff metric*.

Continuous Field

From *(Beckus, JB, Cornean, '18 in preparation)* it follows that

- If H is a *polynomial* in U, then H is called *finite range*.

- Functions in the $*$-algebra C_0 are called *pattern equivariant* *(Kellendonk '03)*

- If then H is called *pattern equivariant* whenever all its coefficients are *pattern equivariant*

Continuous Field

Main Theorem:
For H a continuous field of self-adjoint elements of the field $(\mathcal{A}_\Sigma)_{\Sigma \in \mathfrak{J}(\Xi)}$, which are both finite range and pattern equivariant, the map $\Sigma \in \mathfrak{J}(\Xi) \to \sigma(H_\Sigma) \subset \mathbb{R}$ is Lipshitz continuous in the Hausdorff metrics

IV - Extensions ?

Decorations of $\mathbb{Z}^d$

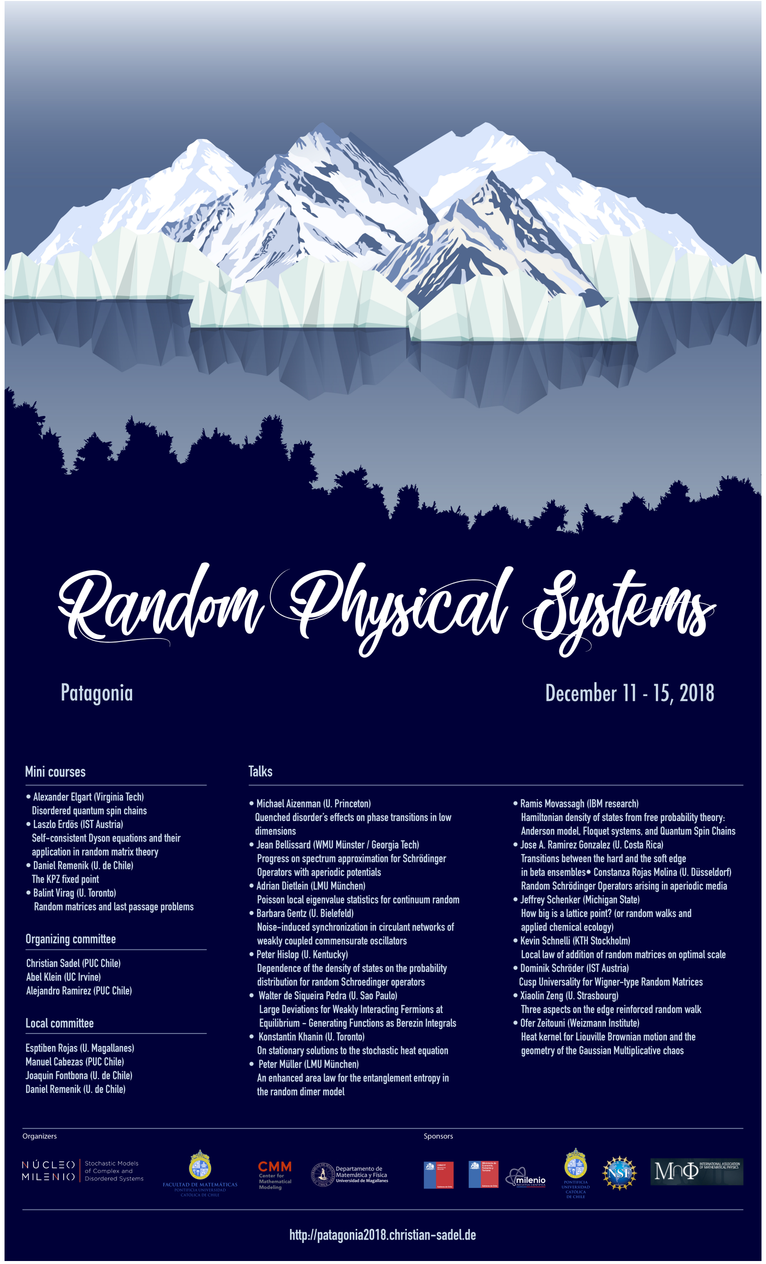
- Let $\mathcal{A}$ be a *finite alphabet*. On the *lattice* $\mathbb{Z}^d$, the full shift is provided by the space $\Omega = \mathcal{A}^{\mathbb{Z}^d}$, endowed with the obvious *shift-action* of $\mathbb{Z}^d$.
- A *subshift* is a *closed* $\mathbb{Z}^d$*-invariant* subset Ξ.
- The same formalism applies: the space of closed invariant subsets $\mathcal{J}(\Omega)$, the C^**-algebras* , the *continuous field*, the concept of *finite range* and of *pattern equivariance*.
- *Both continuity and Lipshitz continuity Theorems apply.* *(Beckus, JB, Cornean, in preparation)*

Quasicrystals

- The *cut-and-project* method can be generalized to $E_{||} \simeq \mathbb{R}^d$ while $\mathbb{Z}^2$ becomes $\mathbb{Z}^N$ with $N > d$.
- The concept of *window* exists as well, together with the concept of *flip-flop*, and the *Cantorian* character of the window.
- The *space of slopes* is replaced by the *Grassmannian* $\mathcal{G}_d^N$ of linear subspaces of $\mathbb{R}^N$ with dimension$d$.
- The opening of the *gaps* occurs at *latticial* d-spaces, namely spaces defined by a subset of vectors of $\mathbb{Z}^N$. But it occurs also at any d-spaces having a *non-generic intersection* with some latticial spaces.
- The usual quasi-lattices like, the *Penrose*, the *octagonal*, or their $3D$-versions, those with *symmetries*, are *non-generic* elements of $\mathcal{G}_d^N$.

Quasicrystals

- The *Farey tree* can be replaced by an analog the end of it representing the Cantor set Ξ obtained by *creating suitable gaps at each non-generic point* of $\mathcal{G}_d^N$.
- Like for the Kohmoto model, the set $\mathcal{J}(\Xi)$ can be defined.
- All the same formalism applies for the *C^*-algebras* , the *continuous fields of self adjoint operators* and the *continuity Theorem*.
- The *Lipshitz continuity* for finite range, pattern equivariant continuous fields of self adjoint element *should be easy* to generalize.


Random Physical Systems
Patagonia
December 11 - 15, 2018
Mini courses
• Alexander Elgart (Virginia Tech)
Disordered quantum spin chains
• Laszlo Erdös (IST Austria)
Self-consistent Dyson equations and their application in random matrix theory
• Daniel Remenik (U. de Chile)
The KPZ fixed point
• Balint Virag (U. Toronto)
Random matrices and last passage problems
Organizing committee
Christian Sadel (PUC Chile)
Abel Klein (UC Irvine)
Alejandro Ramirez (PUC Chile)
Local committee
Esptiben Rojas (U. Magallanes)
Manuel Cabezas (PUC Chile)
Joaquin Fontbona (U. de Chile)
Daniel Remenik (U. de Chile)
Talks
• Michael Aizenman (U. Princeton)
Quenched disorder's effects on phase transitions in low dimensions
• Jean Bellissard (WMU Münster / Georgia Tech)
Progress on spectrum approximation for Schrödinger Operators with aperiodic potentials
• Adrian Dietlein (LMU München)
Poisson local eigenvalue statistics for continuum random
• Barbara Gentz (U. Bielefeld)
Noise-induced synchronization in circulant networks of weakly coupled commensurate oscillators
• Peter Hislop (U. Kentucky)
Dependence of the density of states on the probability distribution for random Schroedinger operators
• Walter de Siqueira Pedra (U. Sao Paulo)
Large Deviations for Weakly Interacting Fermions at Equilibrium - Generating Functions as Berezin Integrals
• Konstantin Khanin (U. Toronto)
On stationary solutions to the stochastic heat equation
• Peter Müller (LMU München)
An enhanced area law for the entanglement entropy in the random dimer model
• Ramis Movassagh (IBM research)
Hamiltonian density of states from free probability theory: Anderson model, Floquet systems, and Quantum Spin Chains
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Thanks !