THE TOPOLOGY of TILING SPACES

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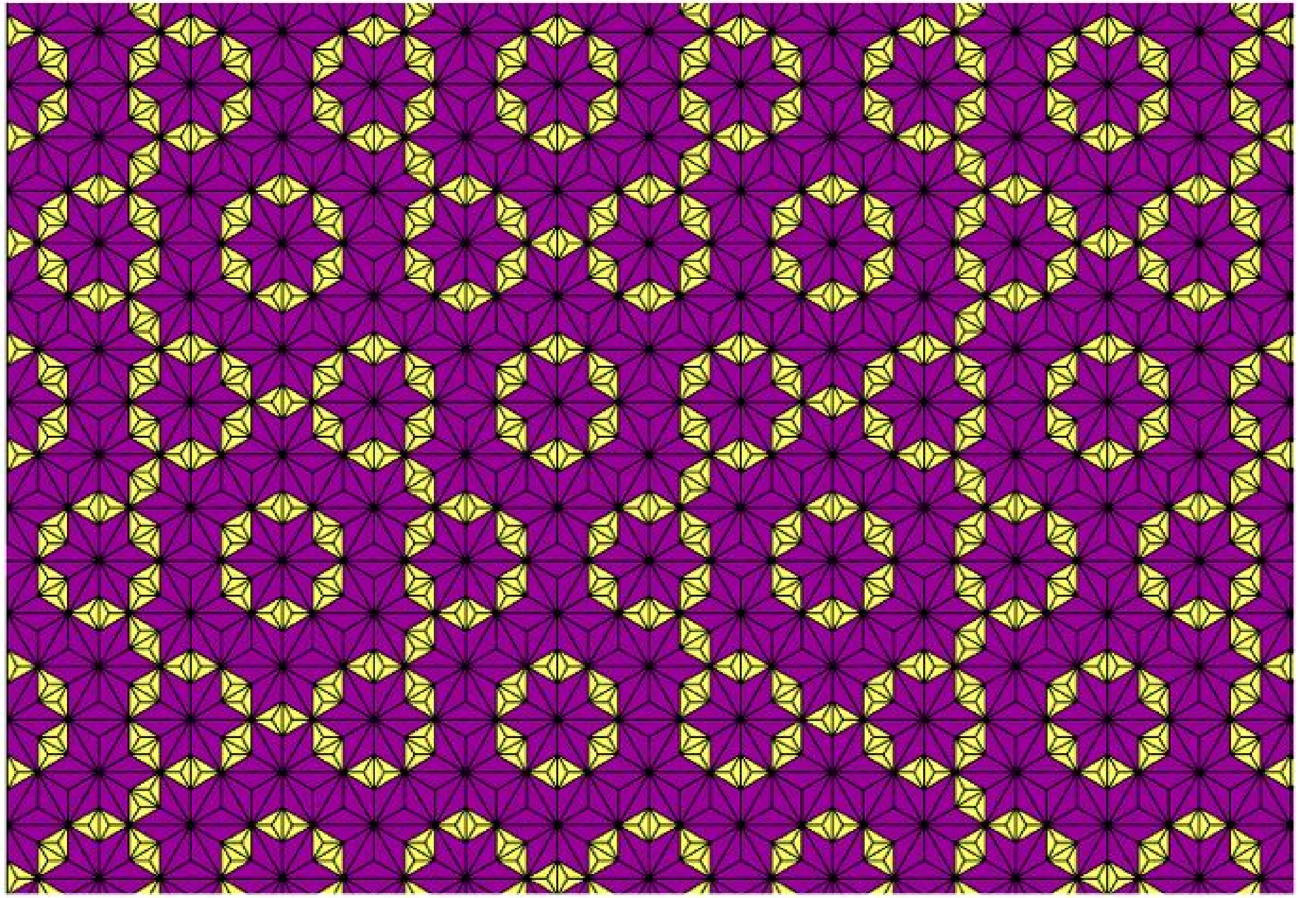
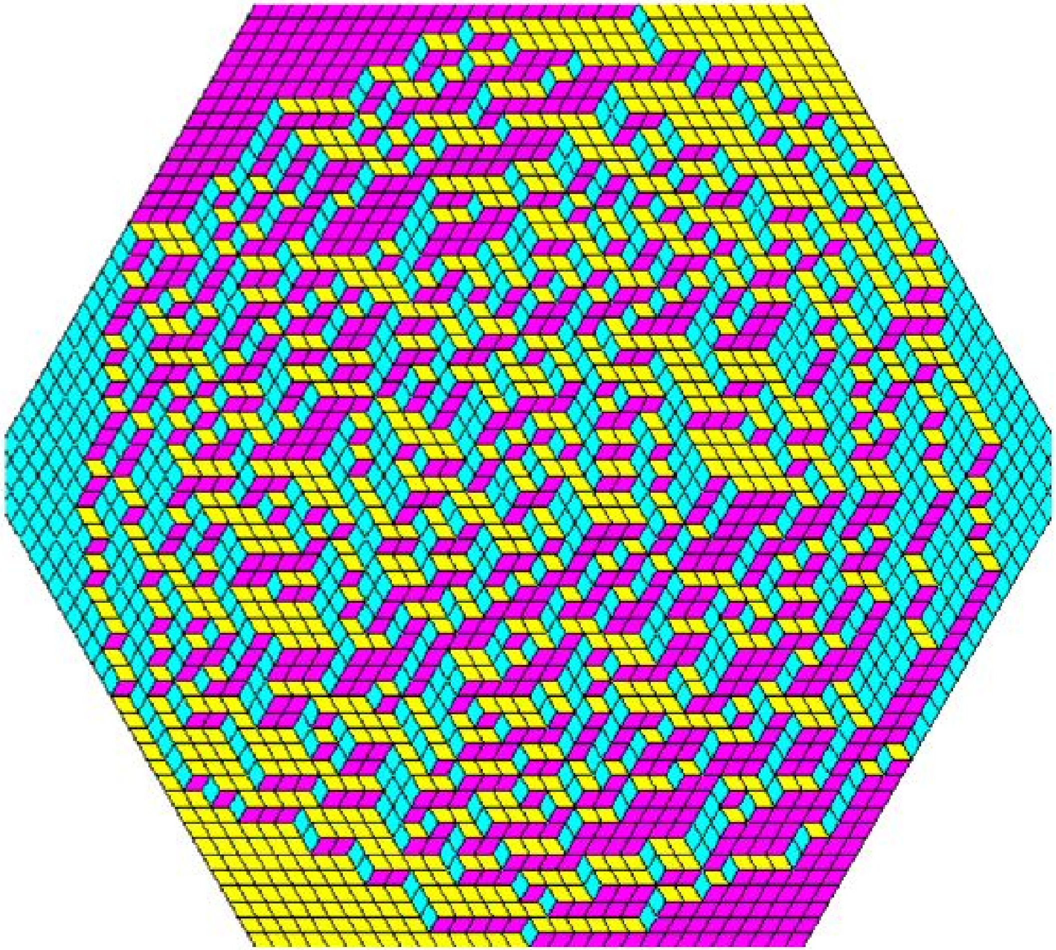
Content

I - Tilings, Tilings,...



- A triangle tiling -



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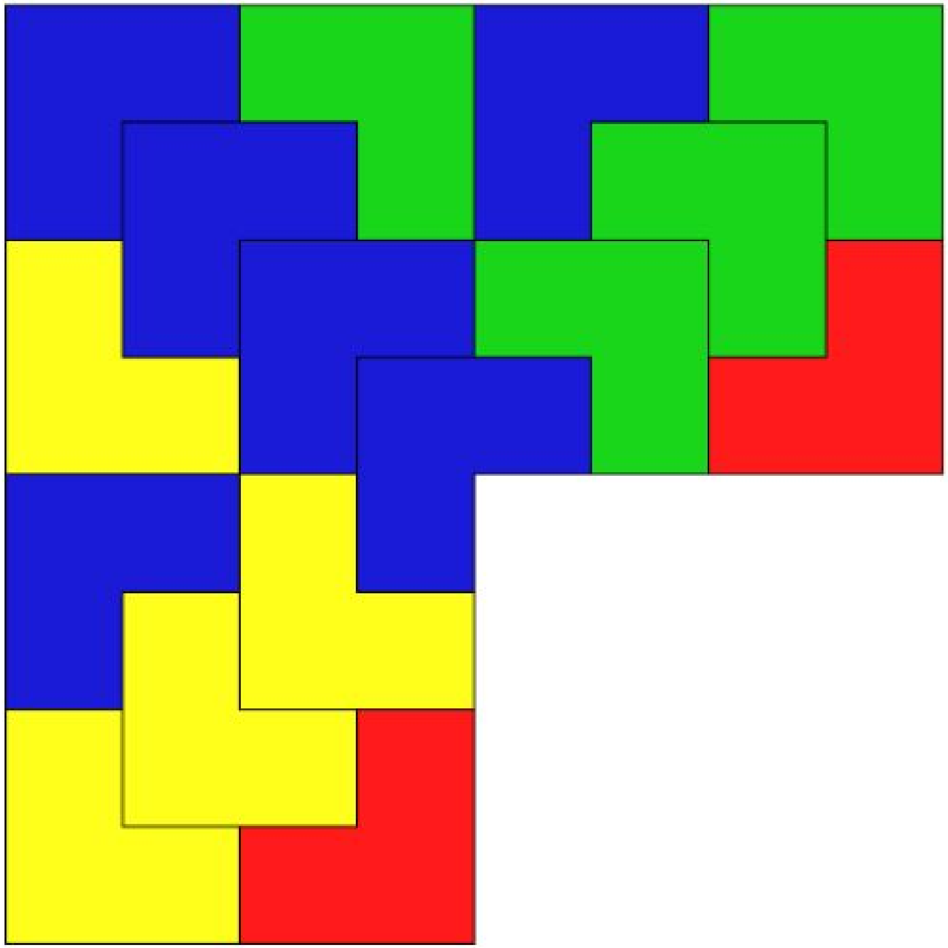
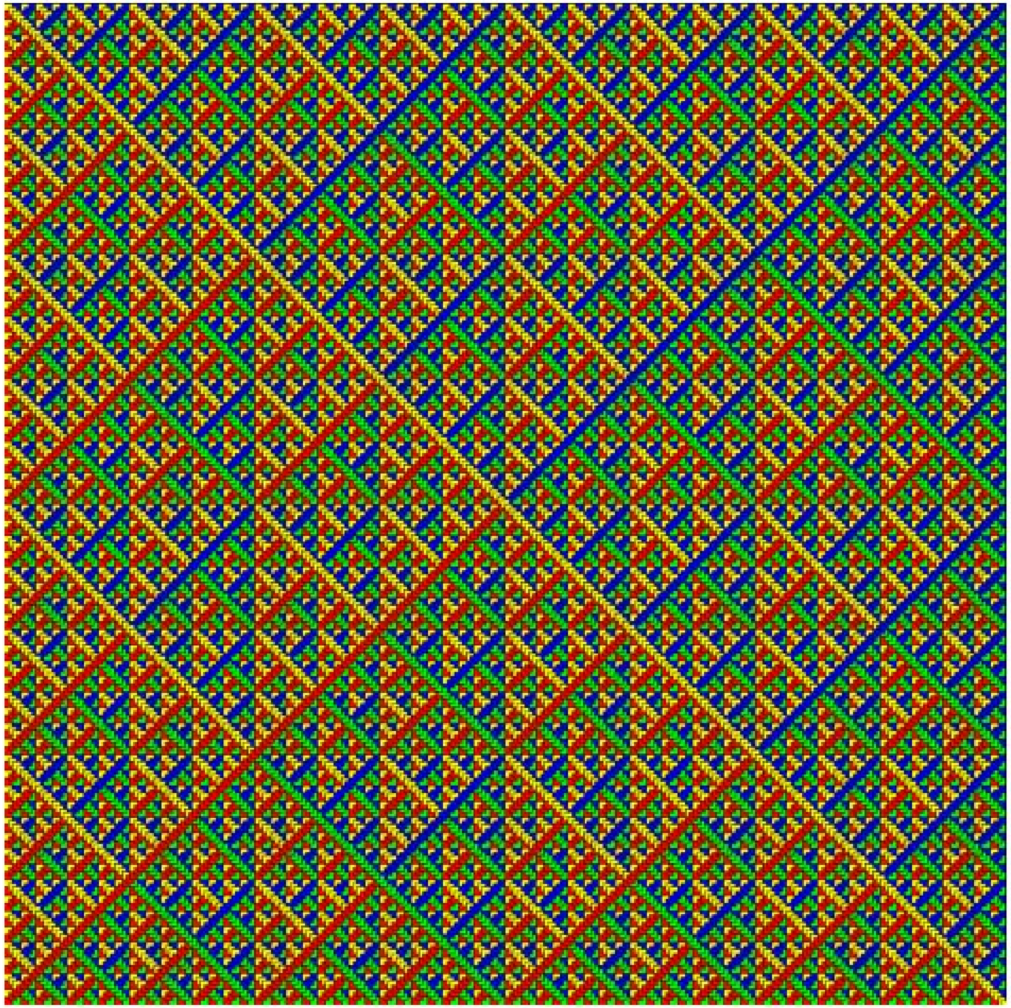


- Building the chair tiling -



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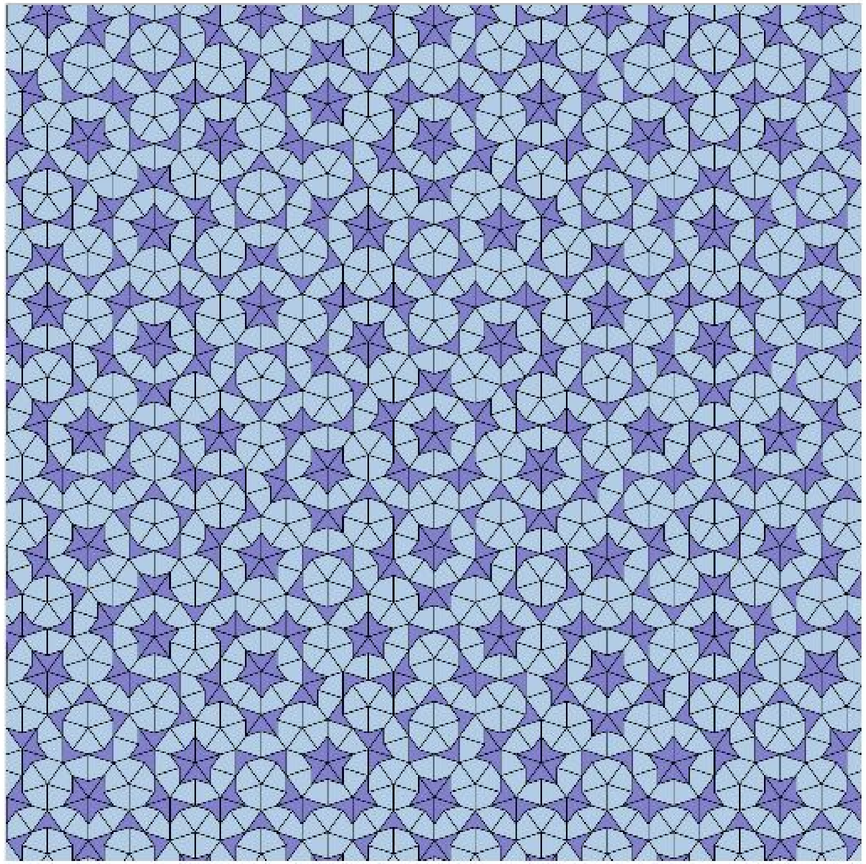

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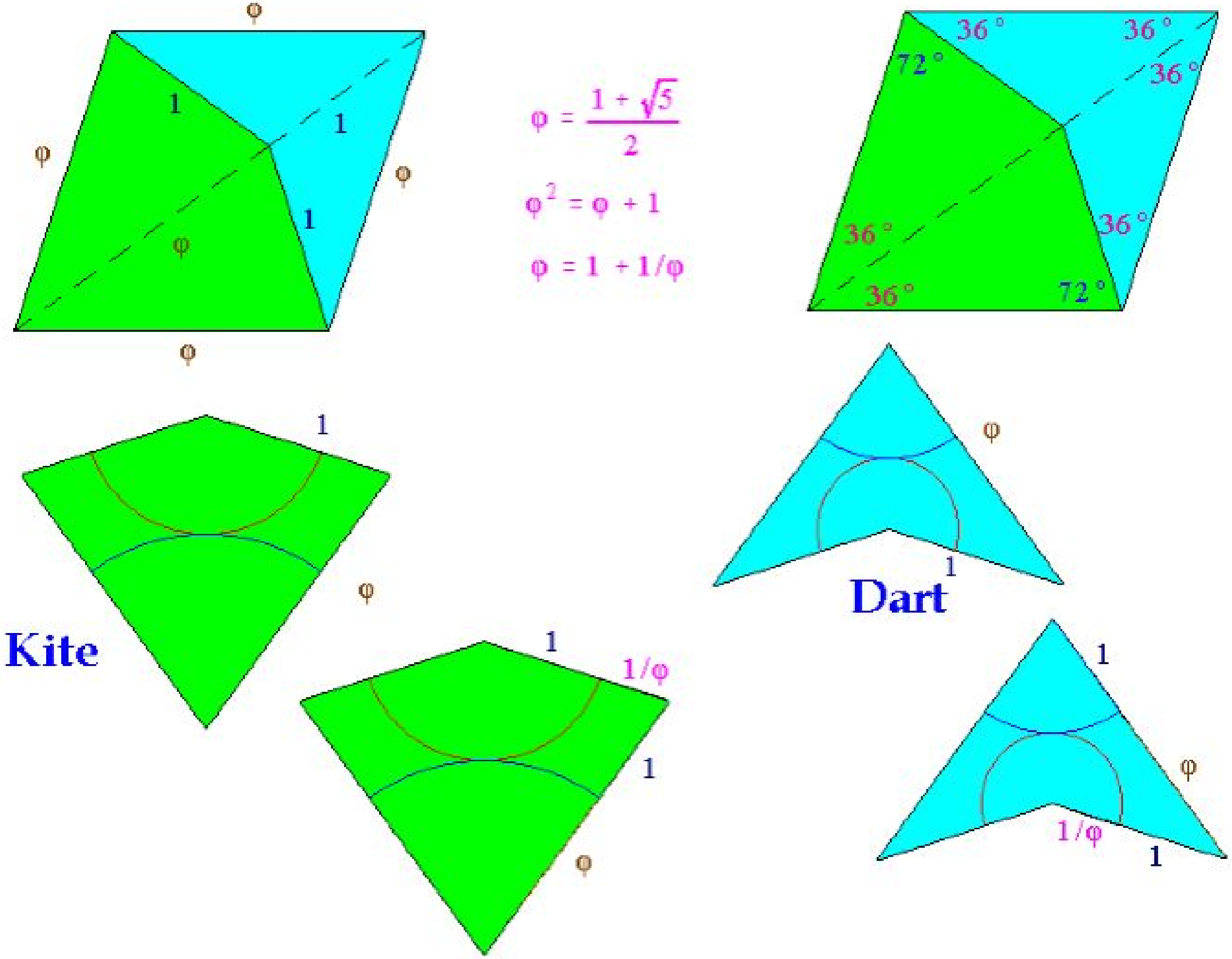
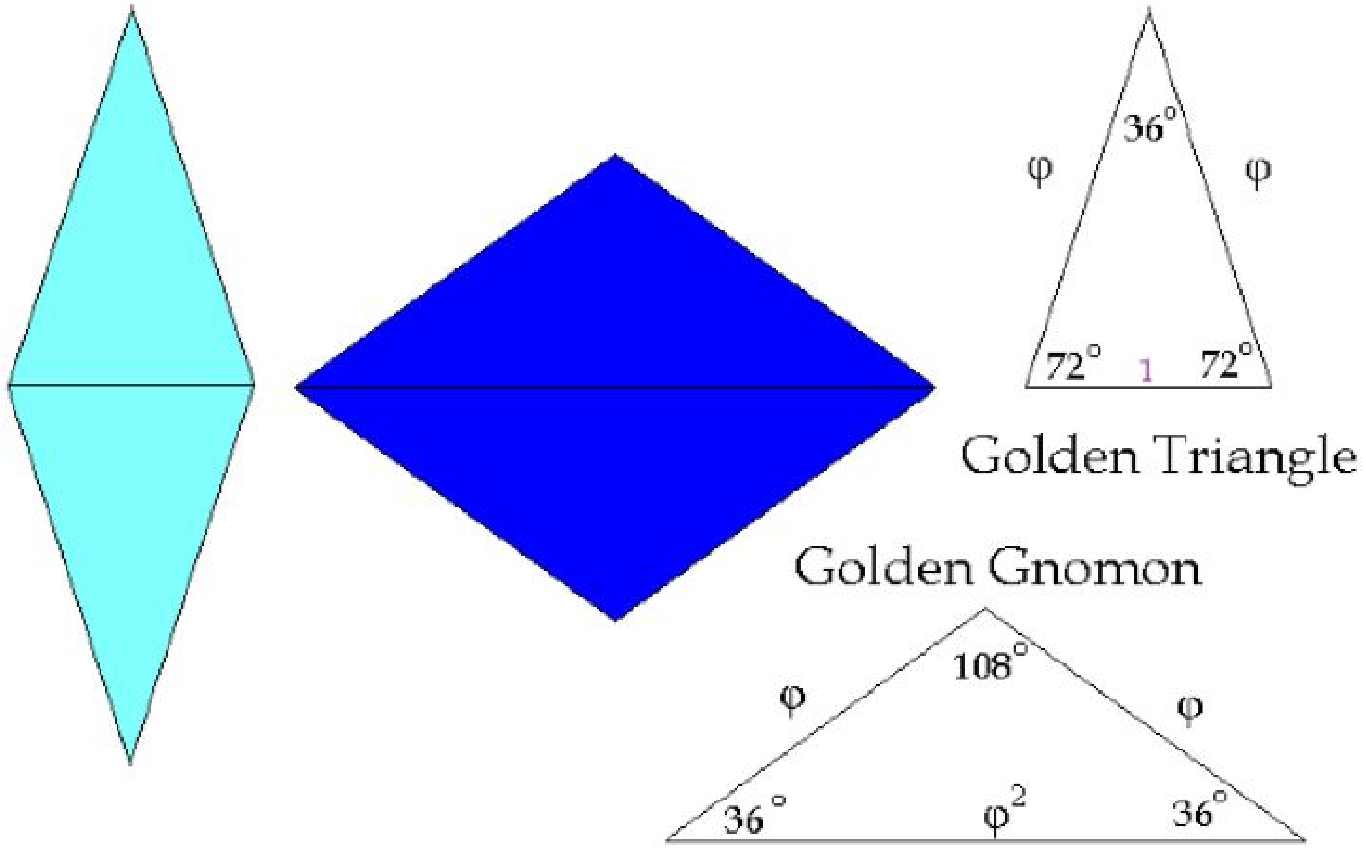
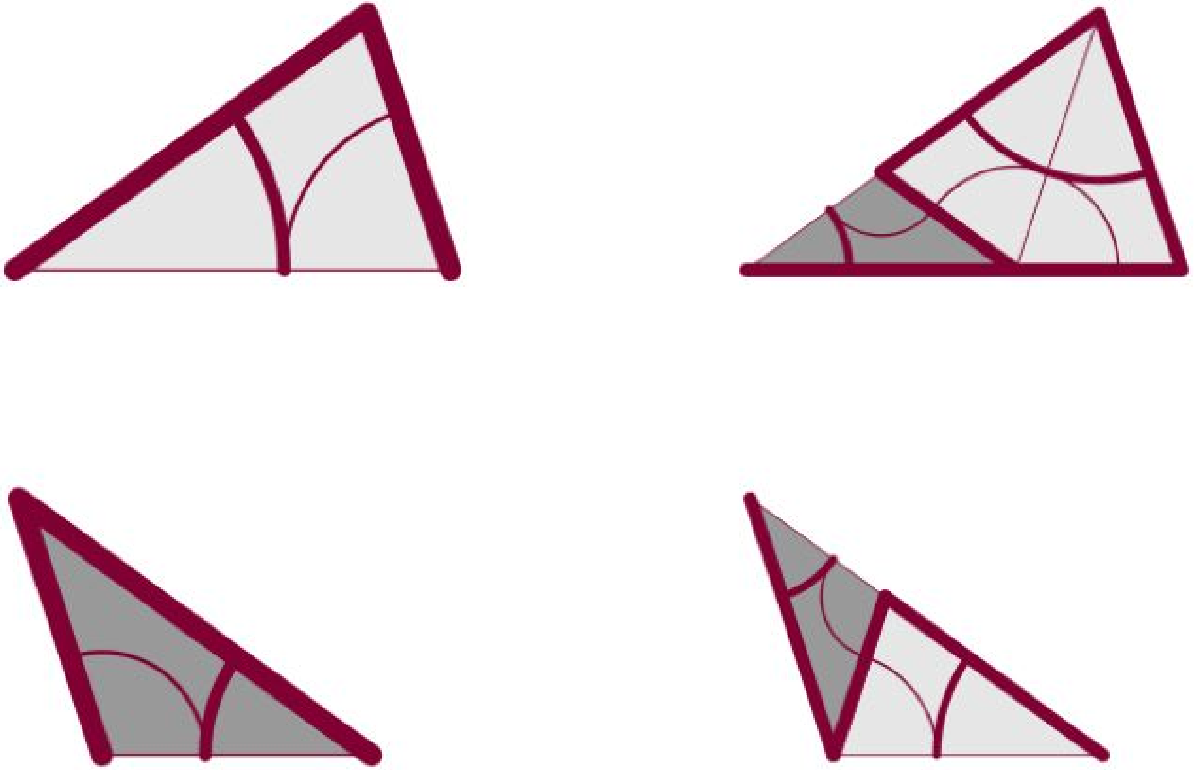


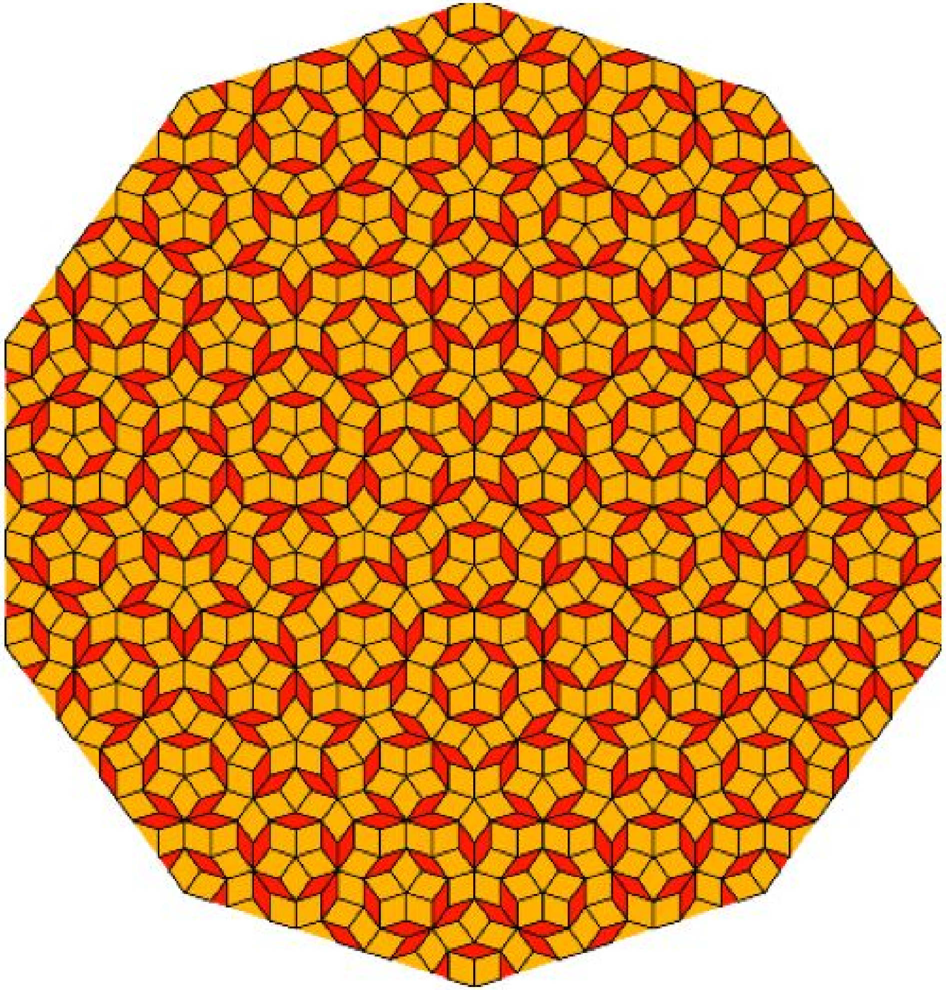
- Kites and Darts -



- Rhombi in Penrose's tiling -



- Inflation rules in Penrose's tiling -



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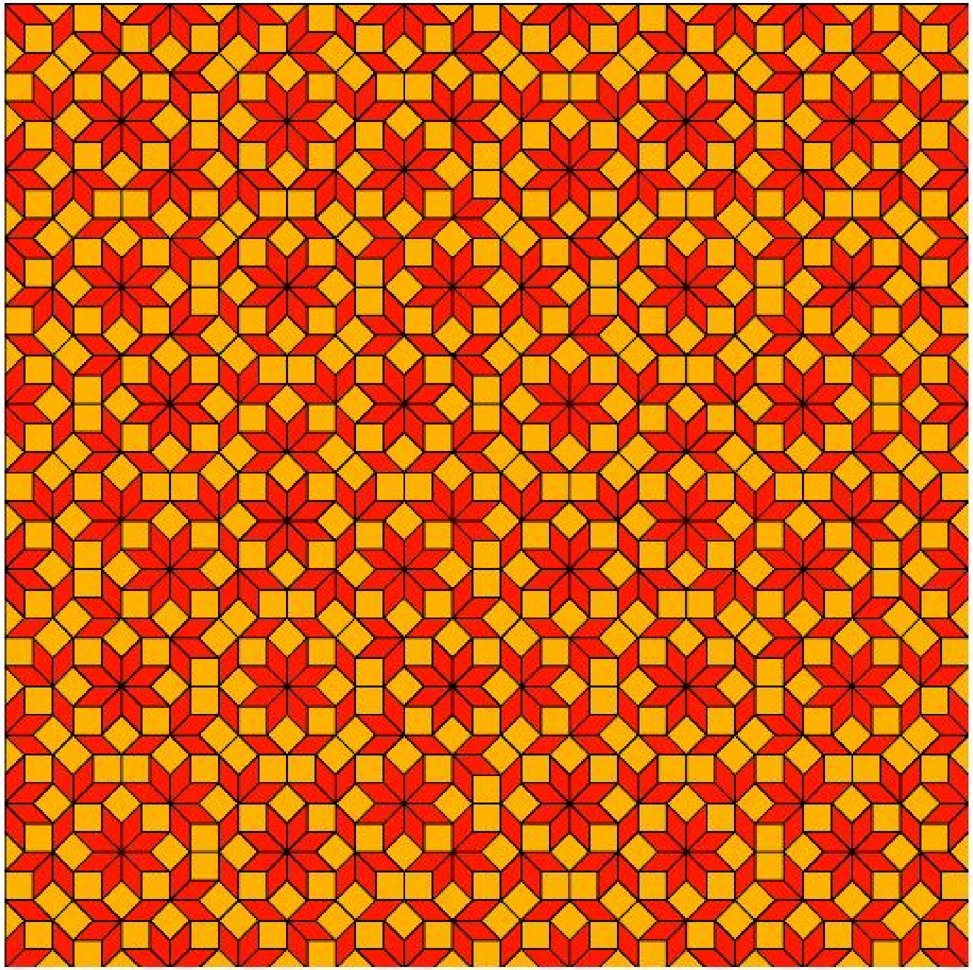
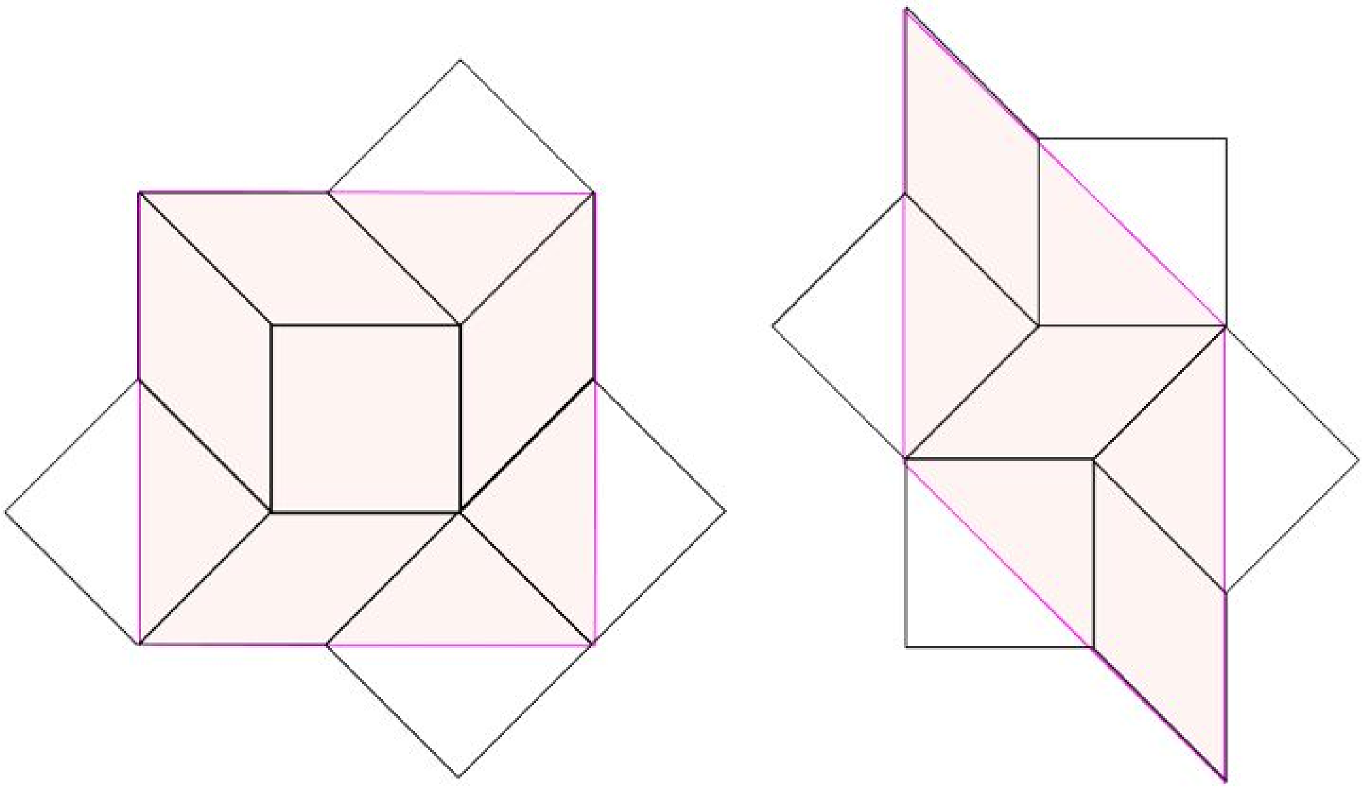
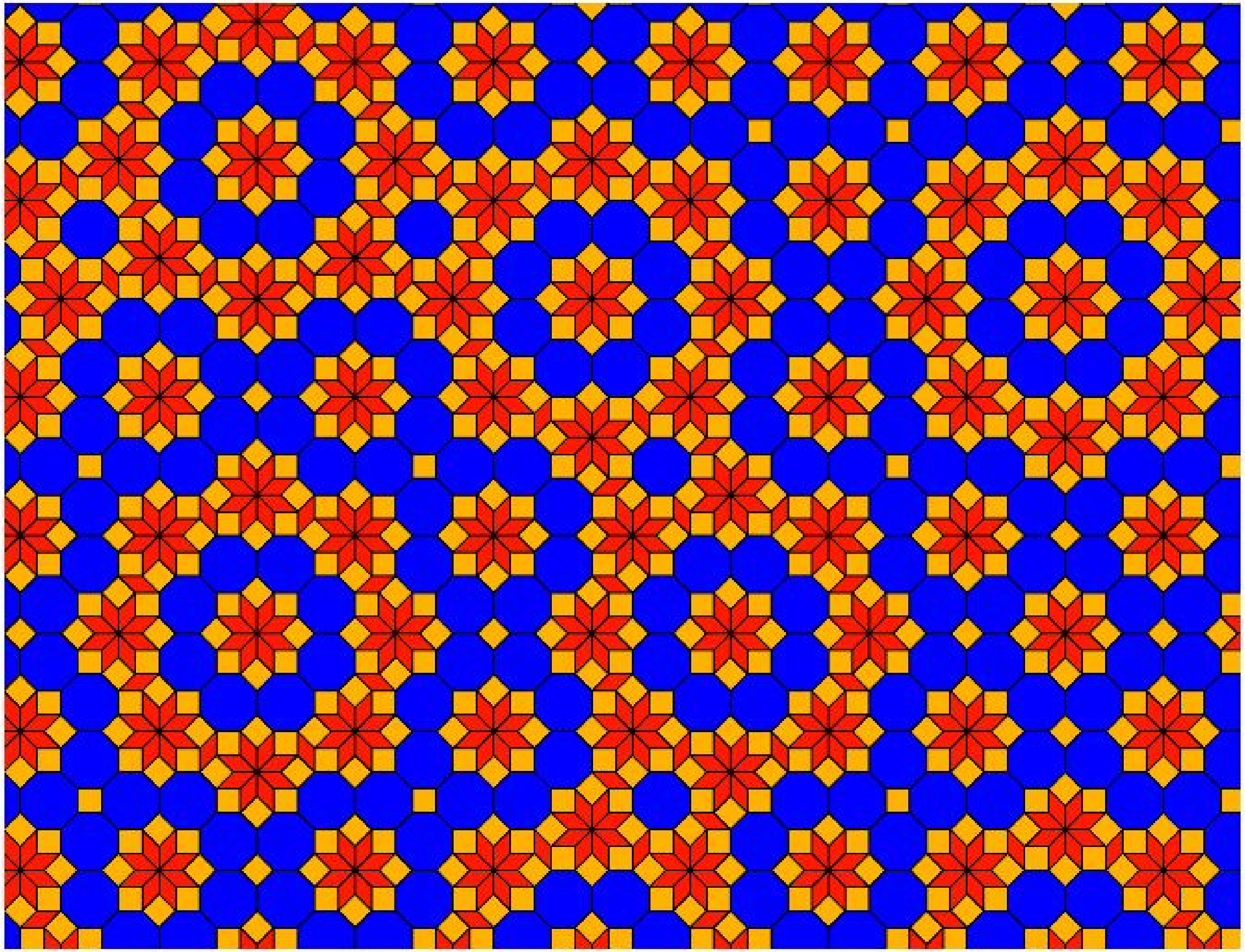


- The octagonal tiling -



- Octagonal tiling: inflation rules -



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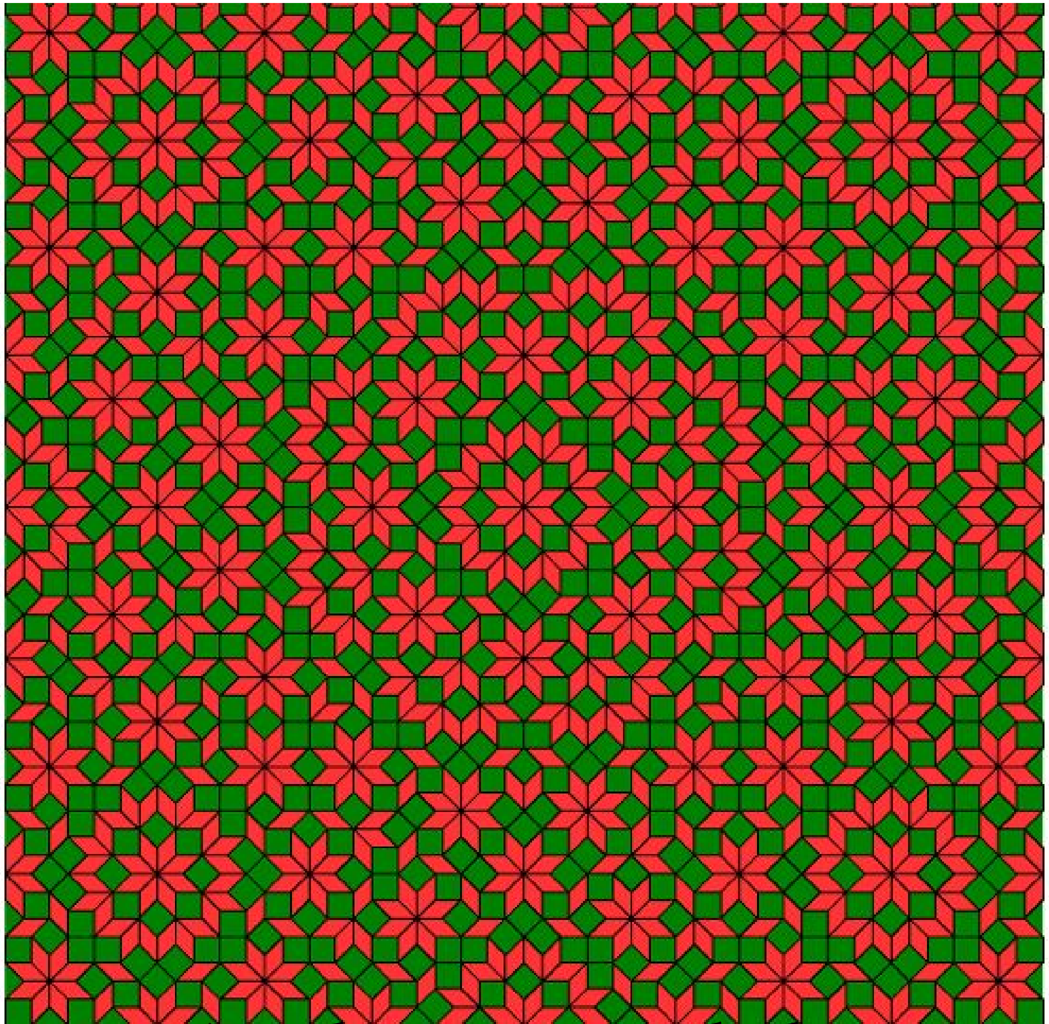

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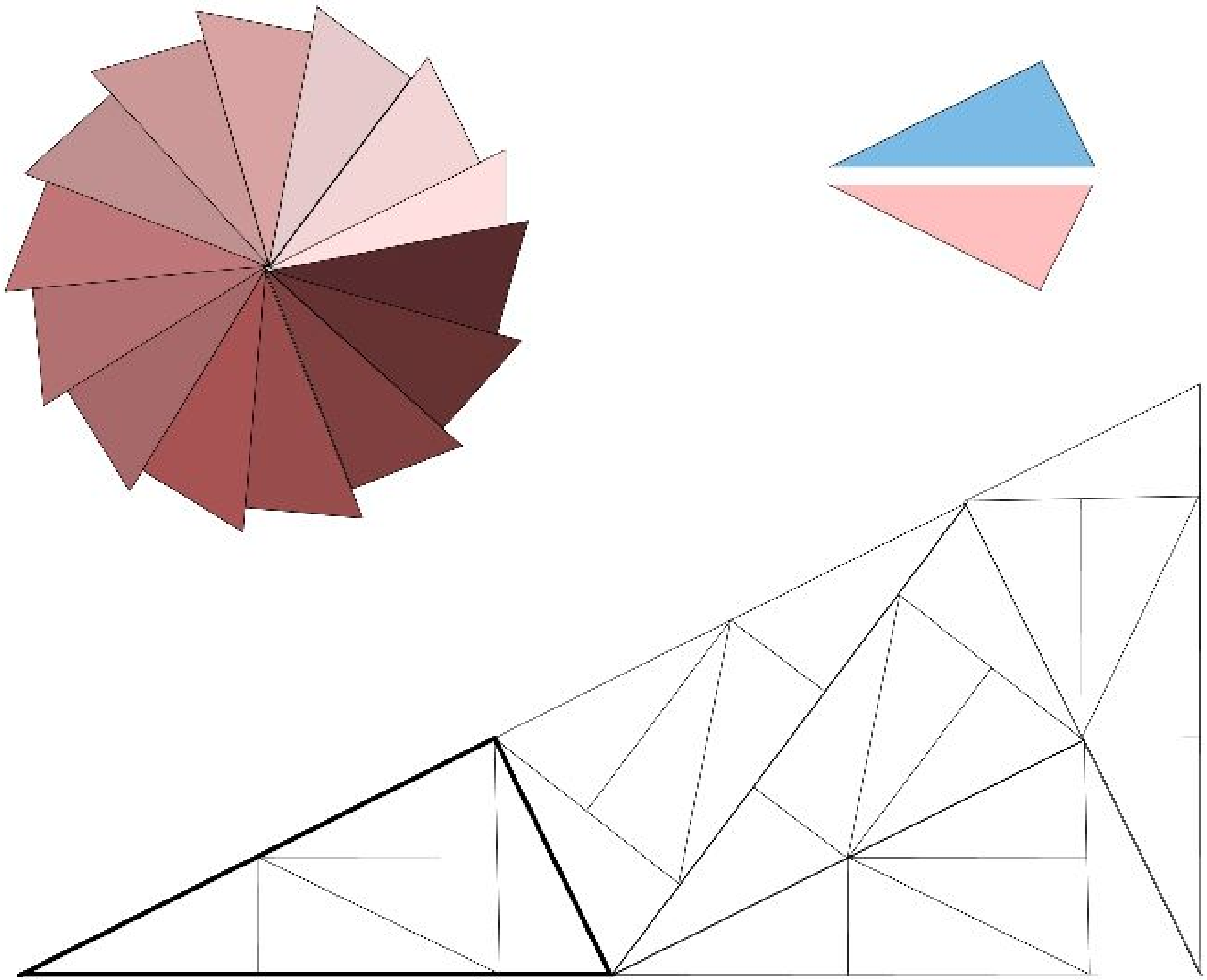
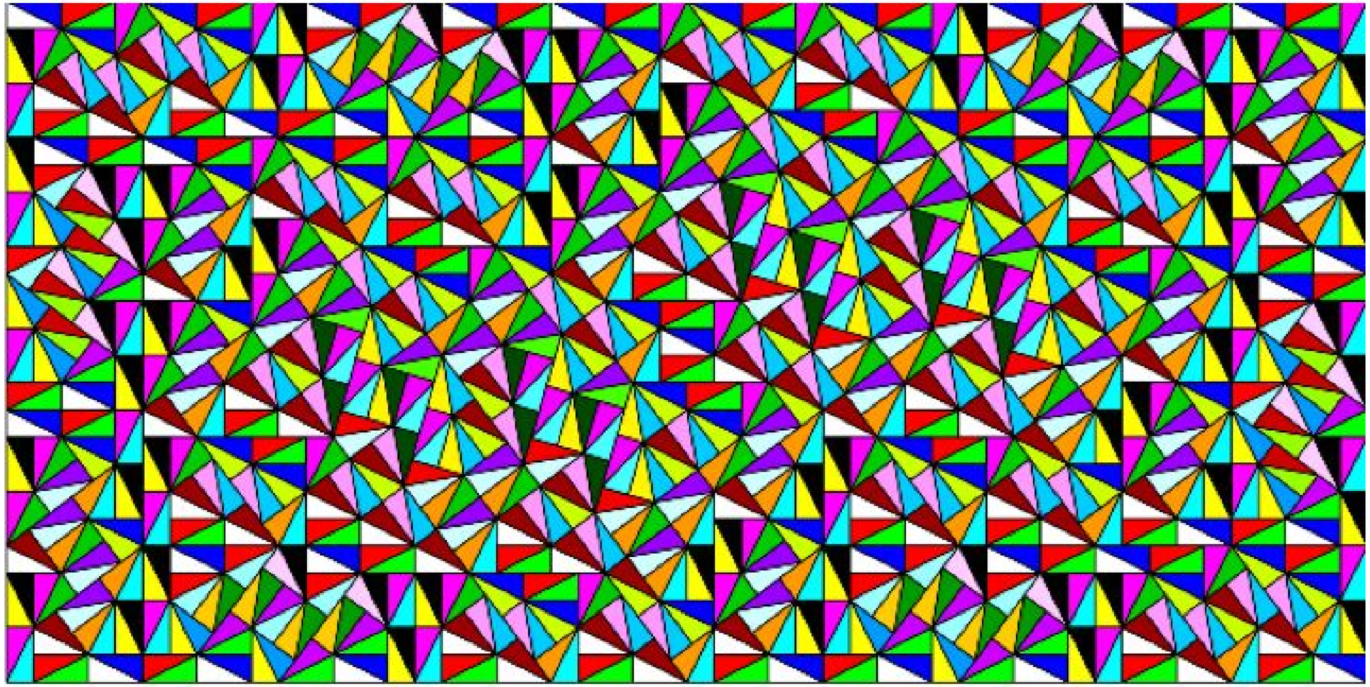


- Building the Pinwheel Tiling -



- The Pinwheel Tiling -

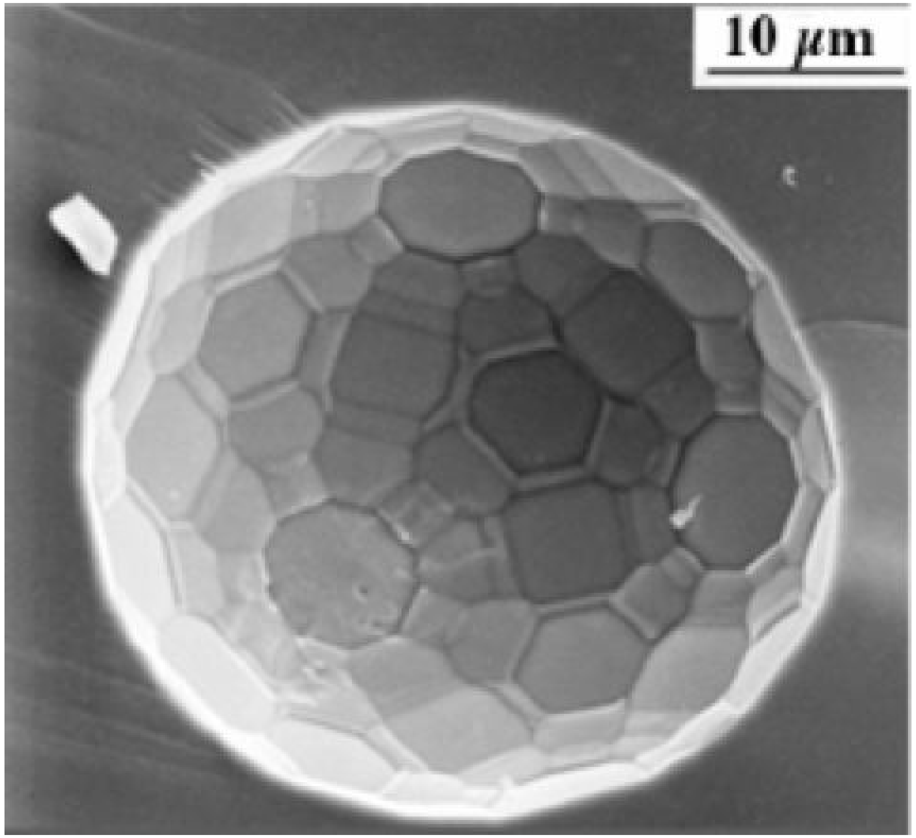
Aperiodic Materials

1. *Periodic Crystals* in d-dimensions:
   translation and crystal symmetries.
   Translation group $\mathcal{T} \simeq \mathbb{Z}^d$.

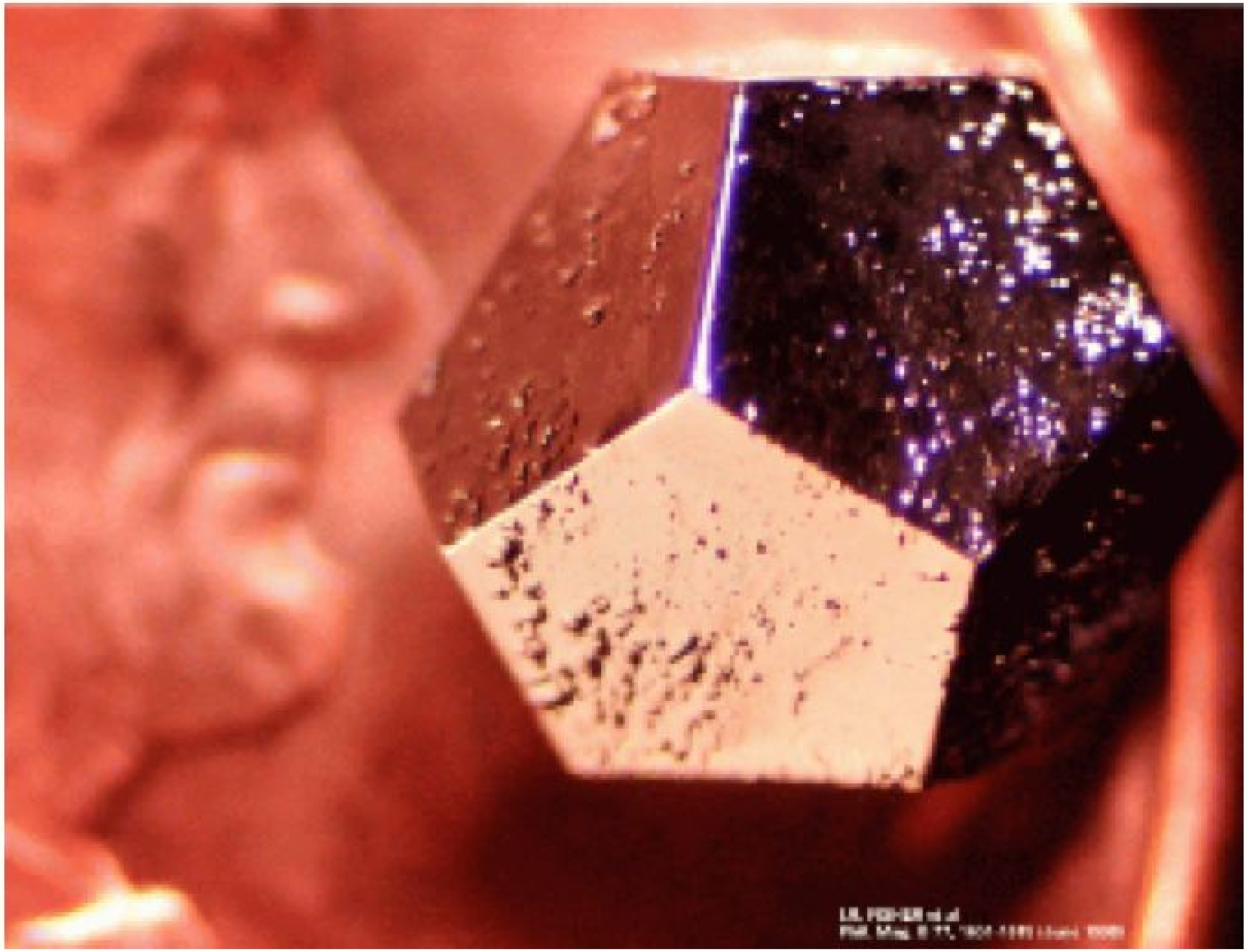
2. *Periodic Crystals in a Uniform Magnetic Field*;
   magnetic oscillations, Shubnikov-de Haas, de Haas-van Alfen.
   The magnetic field breaks the translation invariance to give some quasiperiodicity.

3. *Quasicrystals*: no translation symmetry, but icosahedral symmetry. Ex.:
   (a) $\mathbf{Al_{62.5}Cu_{25}Fe_{12.5}}$;
   (b) $\mathbf{Al_{70}Pd_{22}Mn_{8}}$;
   (c) $\mathbf{Al_{70}Pd_{22}Re_{8}}$;

4. *Disordered Media*: random atomic positions
   (a) Normal metals (with defects or impurities);
   (b) Alloys, bulk metallic glasses (**CuZr**-alloys)
   (c) Doped semiconductors (**Si**, **AsGa**, ...);
   (d) $3D$-topological insulators ($\mathbf{Bi_{0.9}Sb_{0.1}}$)



- The icosahedral quasicrystal *AlPdMn* -



- The icosahedral quasicrystal *HoMgZn*-

II - The Hull as a Dynamical System

Delone Sets

- The set $\mathcal{V}$ of atomic positions is *uniformly discrete* if there is $b > 0$ such that in any ball of radius b there is at *most* one atomic nucleus.

  (Then minimum distance between atoms is $\geq 2b$*)*

- The set $\mathcal{V}$ is *relatively dense* if there is $h > 0$ such that in any ball of radius h there is at *least* one atomic nucleus.

  (Then maximal vacancy diameter is $\leq 2h$*)*

- If $\mathcal{V}$ is both uniformly discrete and relatively dense, it is called a *Delone set*.

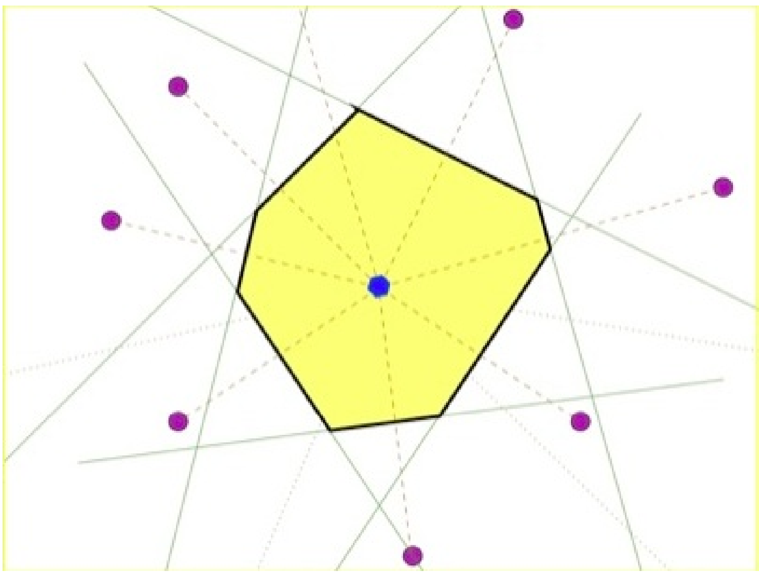
- $\mathrm{Del}_{b,h}$ denotes the set of *Delone sets* with parameters b, h.

Voronoi Cells

- Let $\mathcal{V} \in \mathrm{Del}_{b,h}$. If $x \in \mathcal{V}$ its *Voronoi cell* is defined by

$$V(x) = \{y \in \mathbb{R}^d \, ; \, |y - x| < |y - x'| \, \forall x' \in \mathcal{V} \, , \, x' \neq x\}$$

$V(x)$ is open. Its closure $T(x) = \overline{V(x)}$ is called the *Voronoi tile* of x



Proposition: *If $\mathcal{V} \in \mathrm{Del}_{r_0,r_1}$ the Voronoi tile of any $x \in \mathcal{V}$ is a convex polytope*

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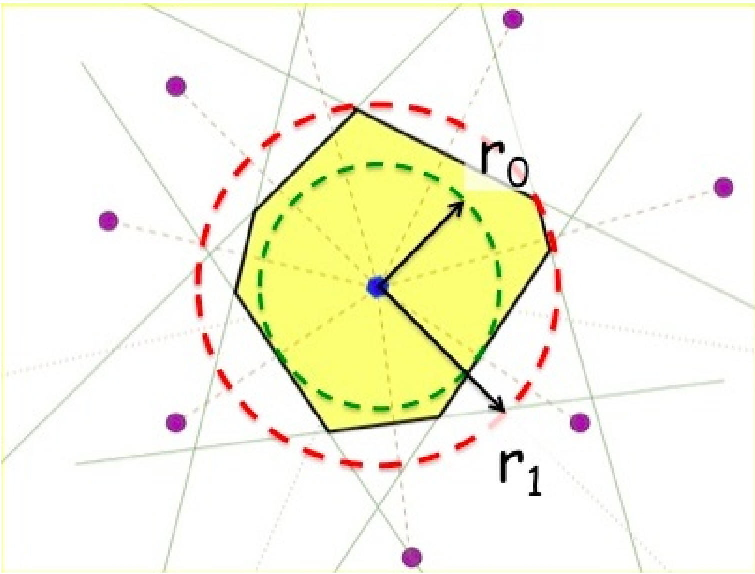
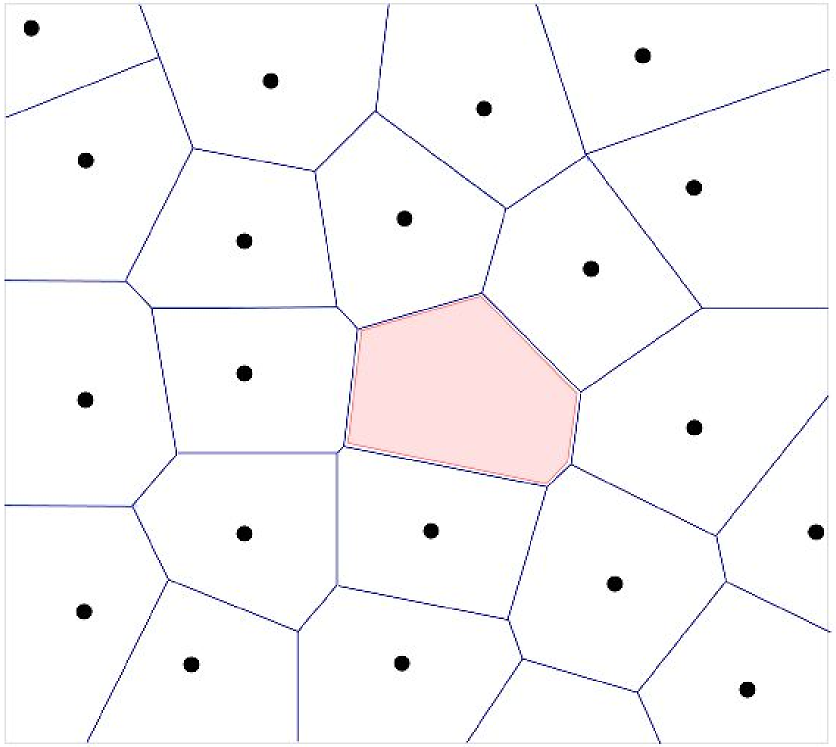


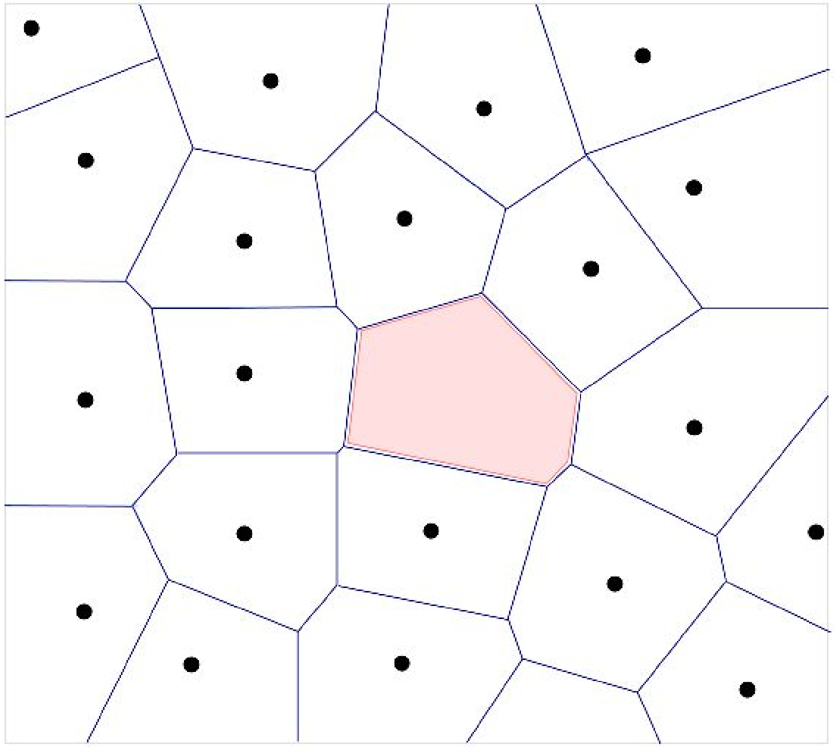
Proposition: *If $\mathcal{V} \in \mathrm{Del}_{r_0,r_1}$ the Voronoi tile of any $x \in \mathcal{V}$ is a convex polytope containing the ball $\overline{\mathrm{B}}(x; r_0)$ and contained in the ball $\overline{\mathrm{B}}(x; r_1)$ with $b = r_0$, $h = r_1$.*

The Delone Graph



Proposition: *the Voronoi tiles of a Delone set touch face-to-face*

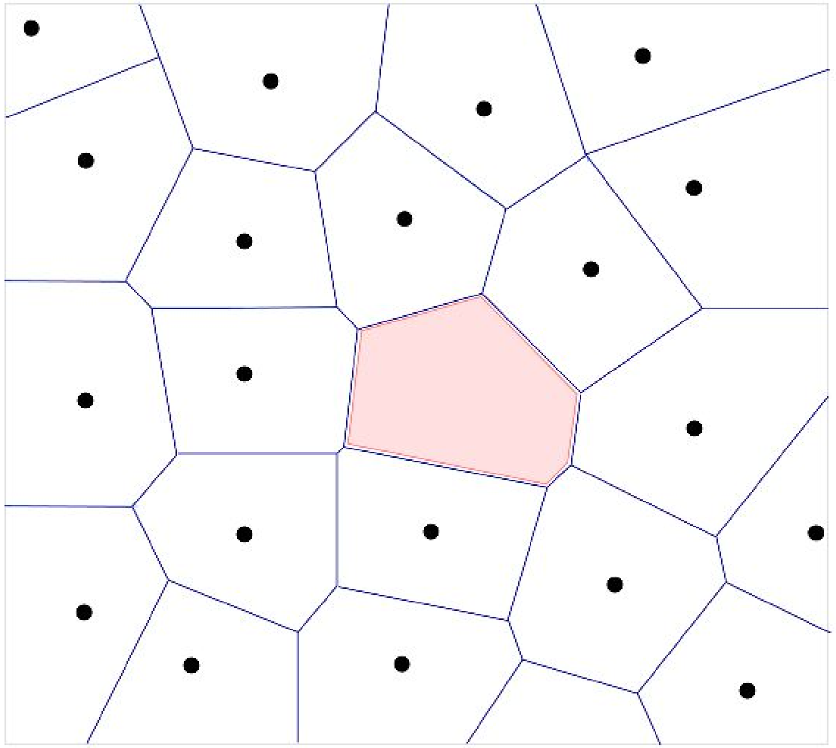
The Delone Graph



Proposition: *the Voronoi tiles of a Delone set touch face-to-face*

Two atoms are *nearest neighbors* if their Voronoi tiles touch along a face of *maximal dimension*.

The Delone Graph



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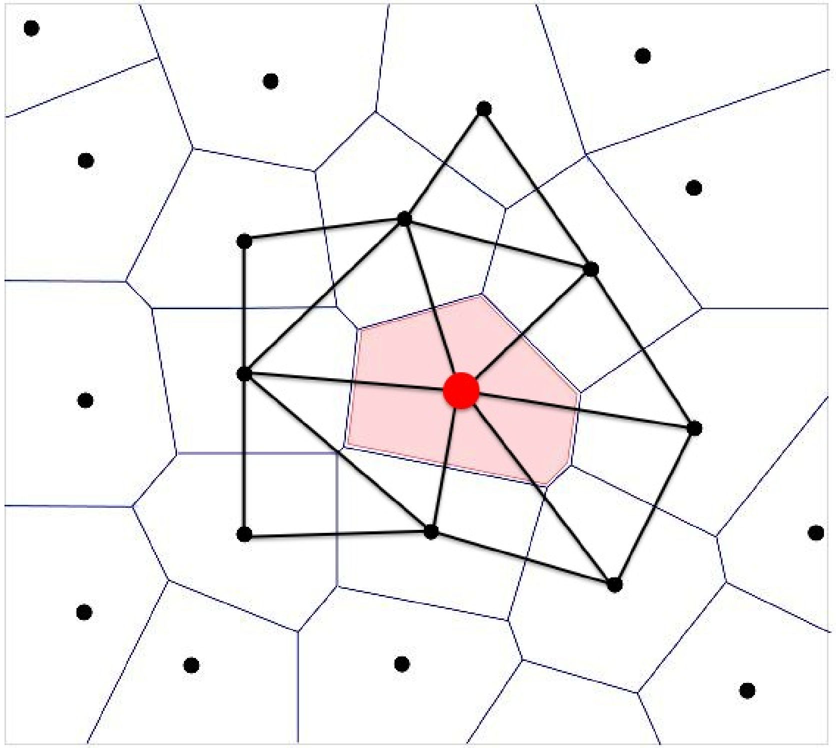

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The family $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the Delone graph.

Finite Local Complexity

A discrete set $\mathcal{V} \in \mathrm{Del}_{b,h}$ has *finite local complexity (FLC)*, whenever its Delone tiling has only finitely many tiles modulo translation

Equivalently $\mathcal{V}$ is FLC iff the set $\mathcal{V} - \mathcal{V}$ of vectors joining two points in $\mathcal{V}$ is *discrete and closed*.

Point Sets and Point Measures

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $C_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{V}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{V}} = \sum_{y \in \mathcal{V}} \delta(x - y) \quad \in \mathfrak{M}(\mathbb{R}^d) .$$

The Hull

A point measure is $\mu \in \mathfrak{M}(\mathbb{R}^d)$ such that $\mu(B) \in \mathbb{N}$ for any ball $B \subset \mathbb{R}^d$. Its support is

1. *Discrete*.
2. *b-Uniformly discrete*: iff $\forall B$ ball of radius b, $\mu(B) \leq 1$.
3. *h-Relatively dense*: iff for each ball B of radius h, $\mu(B) \geq 1$.

$\mathbb{R}^d$ acts on $\mathfrak{M}(\mathbb{R}^d)$ by translation.

Theorem 1 *The set of b-uniformly discrete point measures is compact and $\mathbb{R}^d$-invariant.*
Its subset of h-relatively dense measures is compact and $\mathbb{R}^d$-invariant.

Definition 1 *Given $\mathcal{V}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{V}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $\nu^{\mathcal{V}}$.*

Hence the Hull is a *compact metrizable space* on which $\mathbb{R}^d$ *acts by homeomorphisms*.

Properties of the Hull

If $\mathcal{V} \subset \mathbb{R}^d$ is b-uniformly discrete with Hull Ω, then, using compactness

1. each point $\omega \in \Omega$ *is an b-uniformly discrete* point measure with support $\mathcal{V}_\omega$.
2. if $\mathcal{V} \in \mathrm{Del}_{b,h}$, so are all $\mathcal{V}_\omega$'s.
3. if, in addition, $\mathcal{V}$ is *FLC*, so are all the $\mathcal{V}_\omega$'s.
   Moreover then $\mathcal{V} - \mathcal{V} = \mathcal{V}_\omega - \mathcal{V}_\omega \ \forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{V}_\omega$.*

Theorem 2 *If $\mathcal{V}$ is FLC, then its transversal is completely discontinuous.*

Local Isomorphism Classes and Tiling Space

A *patch* is a finite subset of $\mathcal{V}$ of the form

$$p = (\mathcal{V} - x) \cap \overline{B(0,r)} \qquad x \in \mathcal{V}\,,\ r \geq 0$$

Given $\mathcal{V}$ a repetitive, FLC, Delone set let $\mathcal{W}$ be its set of finite patches: it is called the *the $\mathcal{V}$-dictionary*.

A Delone set (or a Tiling) $\mathcal{V}'$ is *locally isomorphic* to $\mathcal{V}$ if it has the same dictionary. The *Tiling Space* of $\mathcal{V}$ is the set of *Local Isomorphism Classes* of $\mathcal{V}$.

Theorem 3 *The Tiling Space of $\mathcal{V}$ coincides with its Hull.*

Minimality

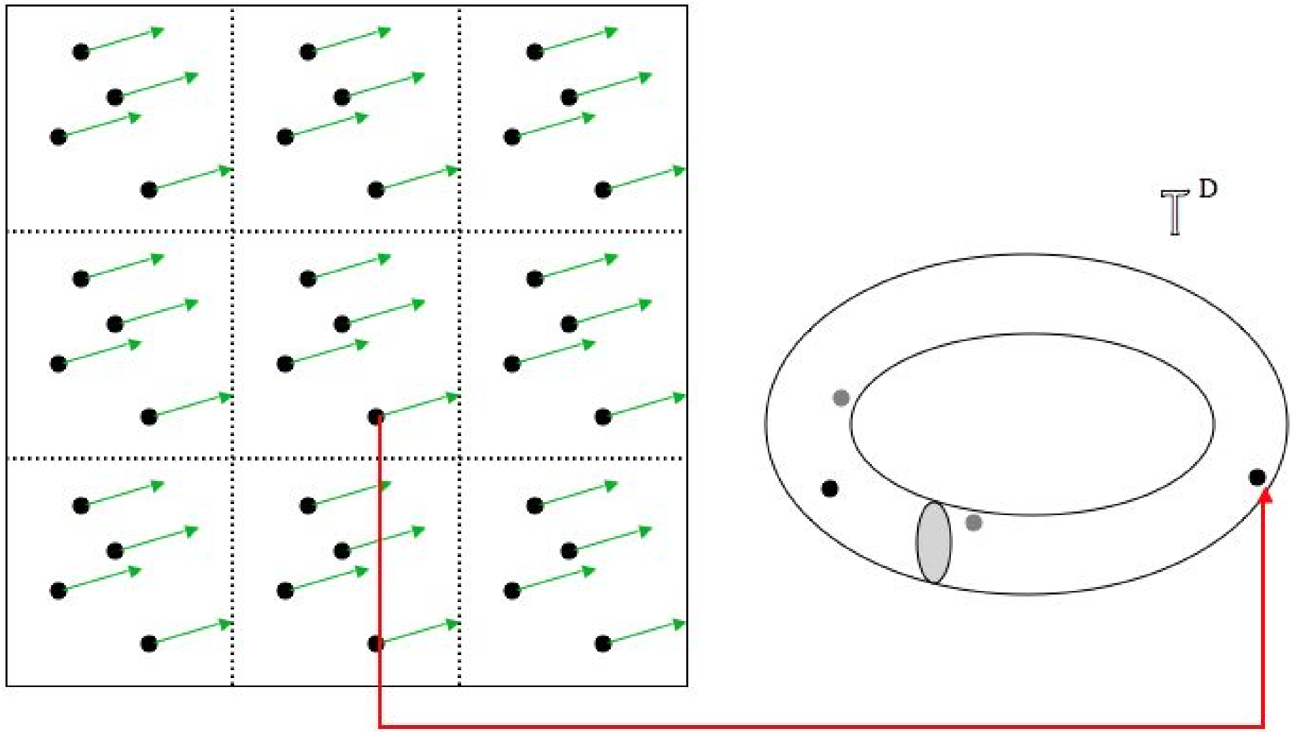
$\mathcal{V}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

Theorem 4 *$\mathbb{R}^d$ acts minimaly on Ω if and only if $\mathcal{V}$ is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Impurities in* **Si** *:* let $\mathcal{V}$ be the lattices sites for Si atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

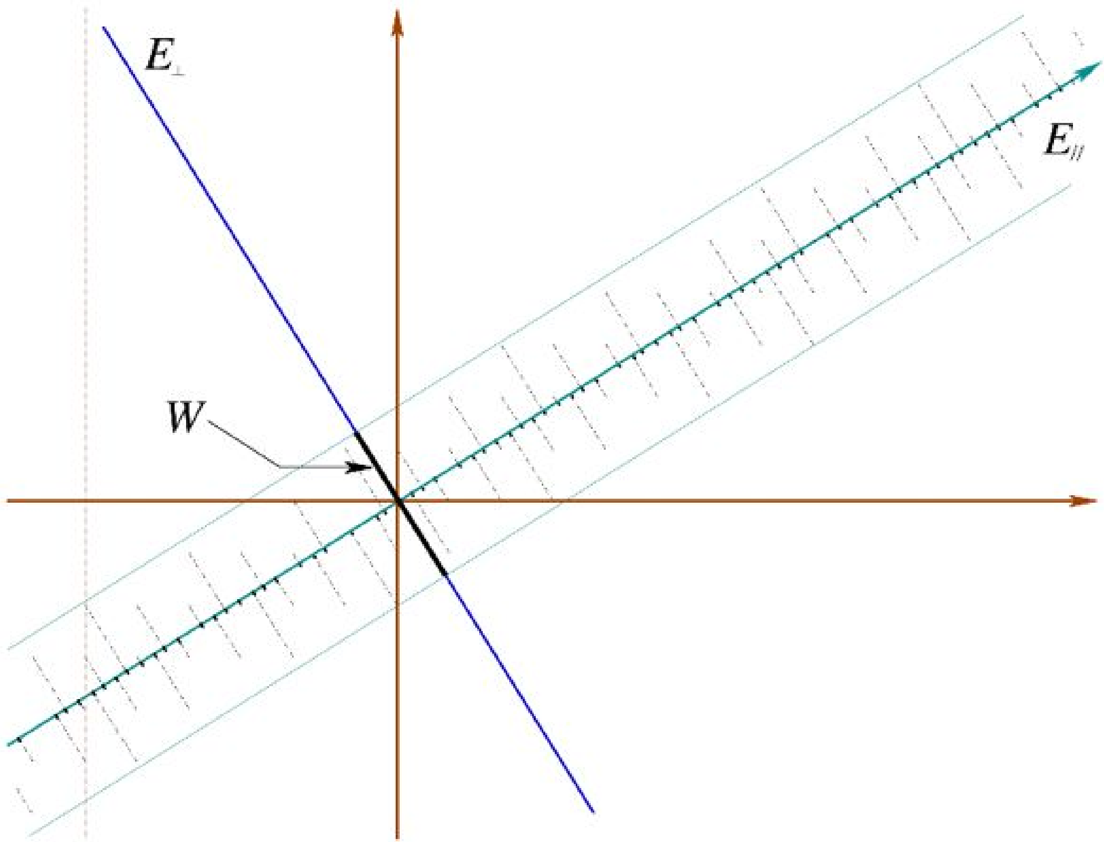
$$\mathbb{R}^d \simeq \mathcal{E}_{||} \xleftarrow{\pi_{||}} \mathbb{R}^n \xrightarrow{\pi_\perp} \mathcal{E}_\perp \simeq \mathbb{R}^{n-d}$$

$$\mathcal{V} \xleftarrow{\pi_{||}} \tilde{\mathcal{V}} \xrightarrow{\pi_\perp} W \quad ,$$

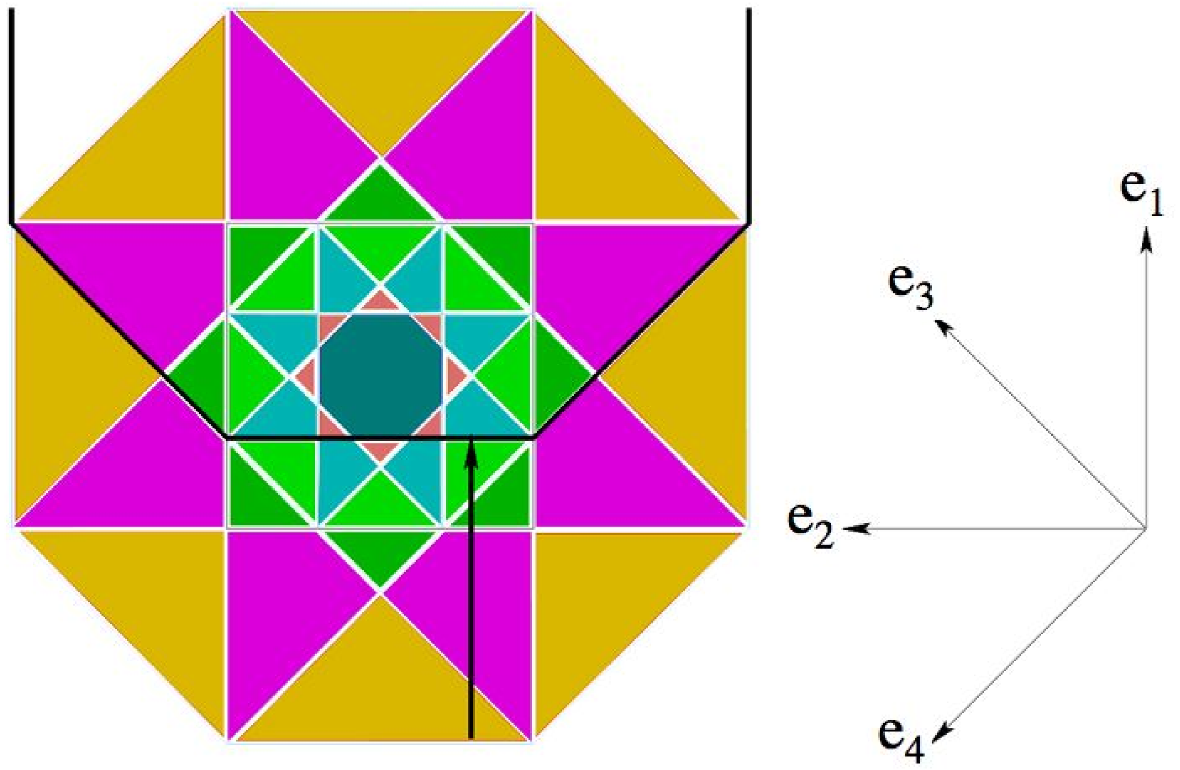
Here

1. $\tilde{\mathcal{V}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{V}$ is the *quasilattice* in $\mathcal{E}_{||}$ defined as

$$\mathcal{V} \;=\; \{\pi_{||}(m) \in \mathcal{E}_{||}\,;\, m \in \tilde{\mathcal{V}}\,,\, \pi_\perp(m) \in W\}$$



– The cut–and–project construction –



- The transversal of the Octagonal Tiling is completely disconnected -

III - The Gap Labeling Theorem

J. Bellissard, R. Benedetti, J.-. Gambaudo, Commun. Math. Phys., **261**, (2006), 1-41.
J. Kaminker, I. Putnam, Michigan Math. J., **51**, (2003), 537-546.
M. Benameur, H. Oyono-Oyono, C. R. Math. Acad. Sci. Paris, **334**, (2002), 667-670.

Schrödinger's Operator

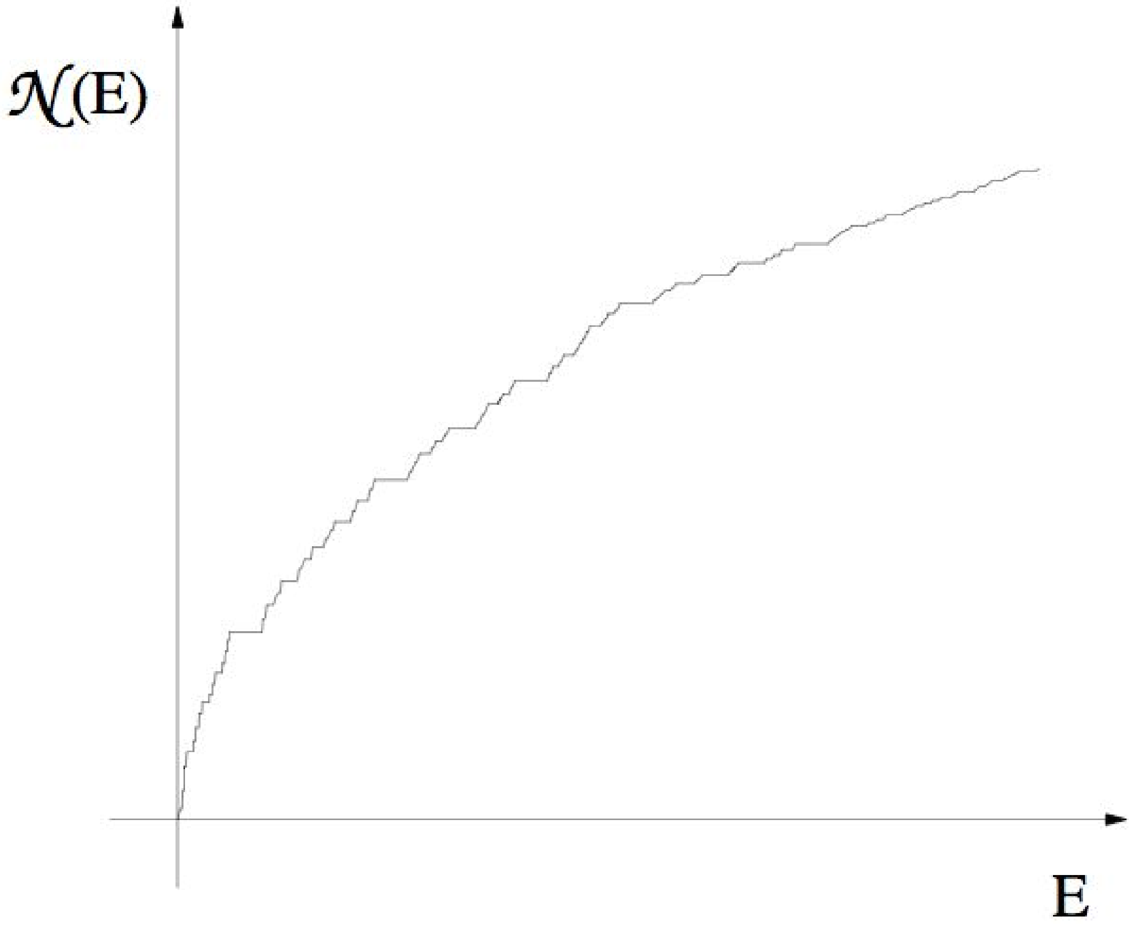
Ignoring electrons-electrons interactions, the one-electron Hamiltonian is given by

$$H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y \in \mathcal{V}_\omega} v(\cdot - y)$$

Its *integrated density of states (IDS)* is defined by

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega \restriction_\Lambda \leq E\}$$

For any $\mathbb{R}^d$*-invariant probability measure* $\mathbb{P}$ on Ω the limit exists a.e. and is independent of ω. It defines a nondecreasing function of E constant on the spectral gaps of H_ω. It is asymptotic at large E's to the IDS of the free Hamiltonian.



- An example of IDS -

Phonons, Vibrational Modes

Atom vibrations in the *harmonic approximation* are solution of

$$M \frac{d^2 \vec{u}_{(\omega,x)}}{dt^2} = \sum_{y \in \mathcal{V}_\omega; y \neq x} K_\omega(x,y) \left(\vec{u}_{(\omega,x)} - \vec{u}_{(\omega,y)} \right)$$

- M= atomic mass,
- $\vec{u}_{(\omega,x)}$ displacement vector of the atom located at $x \in \mathcal{V}_\omega$,
- $K_\omega(x,y)$ is the matrix of *spring constants*.

The *density of vibrational modes (IDVM)* is the IDS of $K_\omega^{1/2}$.

Gap Labels

Theorem 5 *The value of the IDS or of the IDVM on gaps is a linear combination of the occurrence probabilities of finite patches with integer coefficients.*

The proof goes through the group of K-theory of the hull. The result is model independent.

The abstract result goes back to 1982 (J.B). In 1*D, proved in 1993 (JB). Recent proof in any dimension for aperiodic, repetitive, aperiodic tilings by* KAMINKER-PUTNAM, BENAMEUR & OYONO-OYONO, JB-BENDETTI-GAMBAUDO *in 2001.*

IV - Branched Oriented Flat Riemannian Manifolds

Laminations and Boxes

A *lamination* is a foliated manifold with C^∞-structure along the leaves but only finite C^0-structure transversally. The *Hull of a Delone set is a lamination* with $\mathbb{R}^d$-orbits as leaves.

If $\mathcal{V}$ is a *FLC, repetitive, Delone* set, with Hull Ω a *box* is the homeomorphic image of

$$\phi : (\omega, x) \in F \times U \mapsto \mathrm{T}^{-x}\omega \in \Omega$$

if F is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and T denotes the $\mathbb{R}^d$-action on Ω.

A *quasi-partition* is a family $(B_i)_{i=1}^n$ of boxes such that $\bigcup_i \overline{B_i} = \Omega$ and $B_i \cap B_j = \emptyset$.

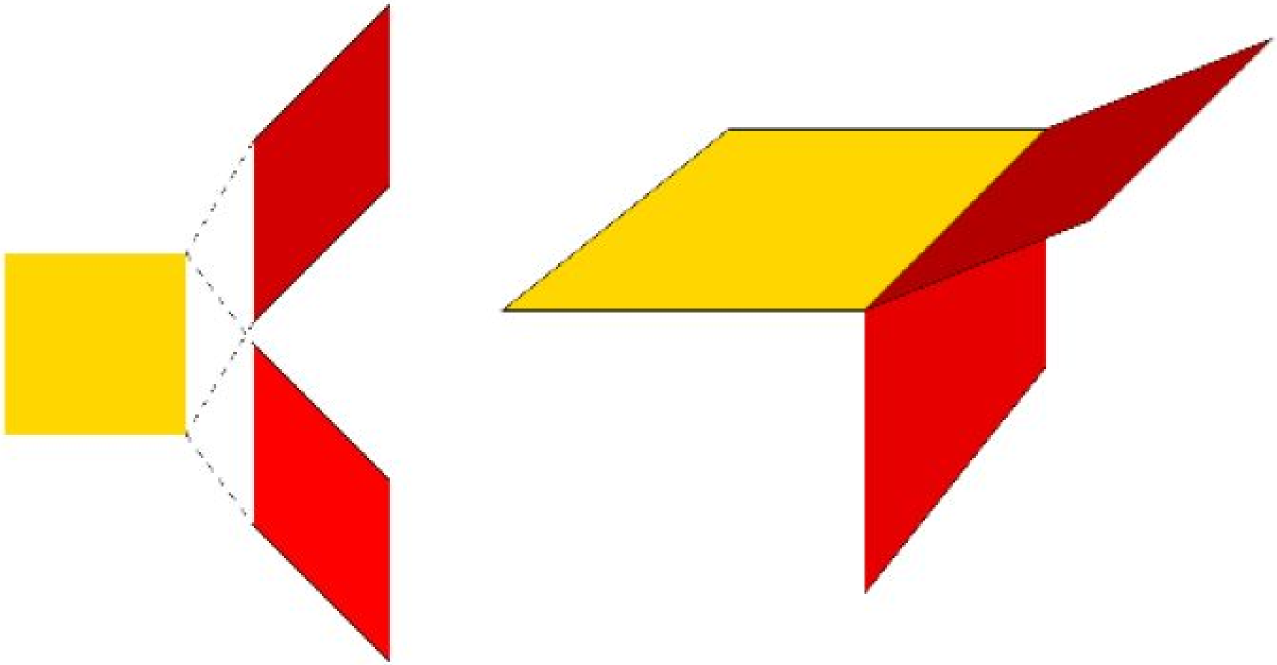
Theorem 6 *The Hull of a FLC, repetitive, Delone set admits a finite quasi-partition. It is always possible to choose these boxes as $\phi(F \times U)$ with U a d-rectangle.*

Branched Oriented Flat Manifolds

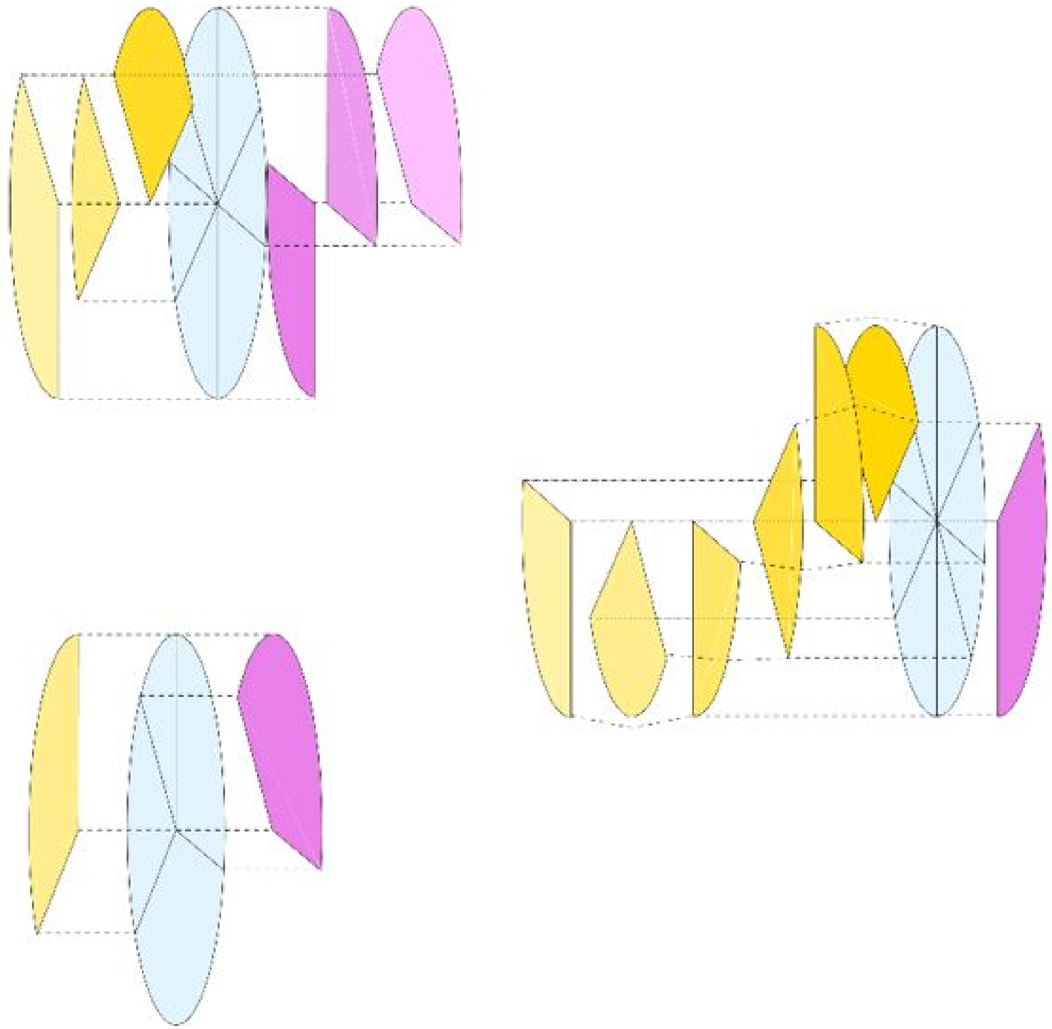
Flattening a box decomposition along the transverse direction leads to a *Branched Oriented Flat manifold*. Such manifolds can be built from the tiling itself as follows

Step 1:

1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.



- Branching -



- Vertex branching for the octagonal tiling -

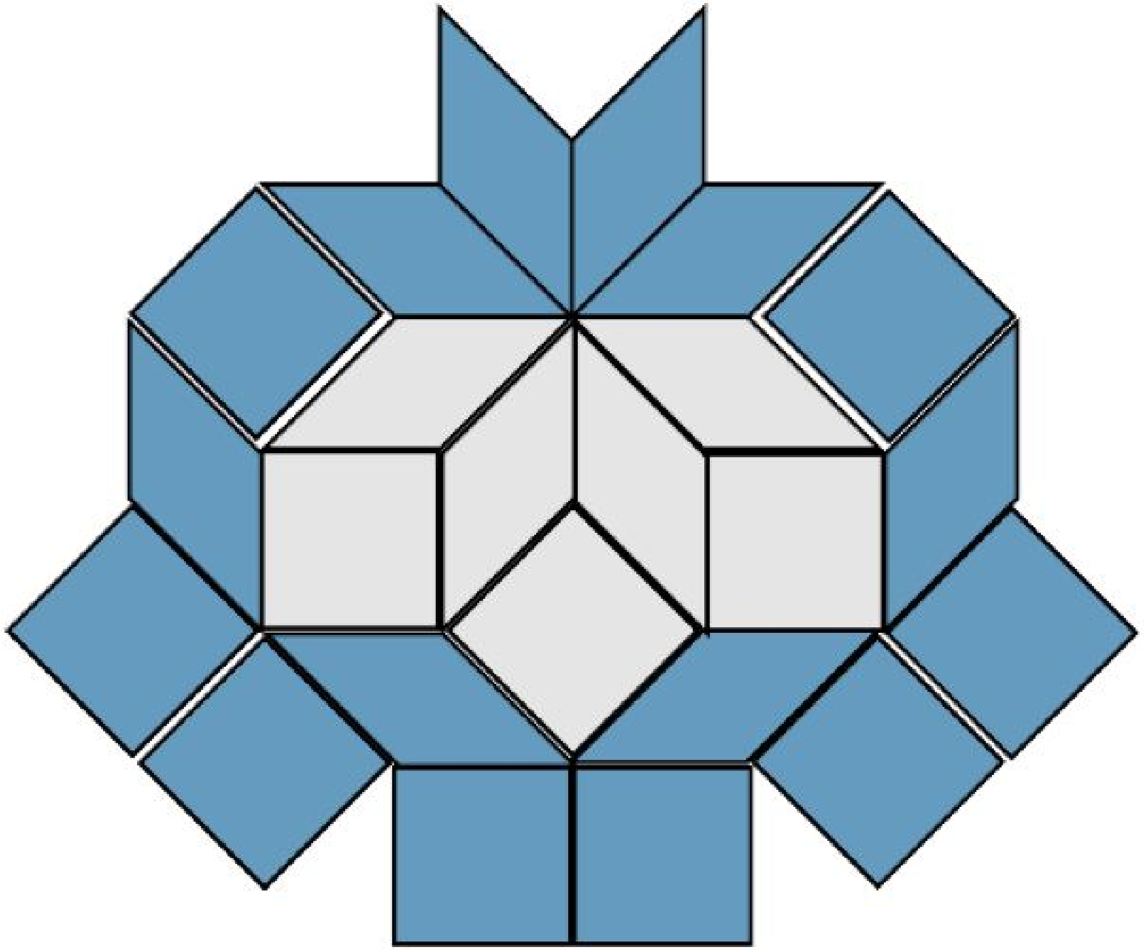
Step 2:

1. Having defined the patch p_n for $n \geq 0$, let $\mathcal{V}_n$ be the subset of $\mathcal{V}$ of points centered at a translated of p_n. By repetitivity this is a FLC repetitive Delone set too. Its prototiles are tiled by tiles of $\mathcal{V}$ and define a finite family $\mathfrak{P}_n$ of patches.

2. Each patch in $\mathcal{T} \in \mathfrak{P}_n$ will be collared by the patches of $\mathfrak{P}_{n-1}$ touching it from outside along its frontier. Call such a patch *modulo translation* a *a collared patch* and $\mathfrak{P}_n^c$ their set.

3. Proceed then as in Step 1 by replacing prototiles by collared patches to get the BOF-manifold B_n.

4. Then choose p_{n+1} to be the collared patch in $\mathfrak{P}_n^c$ containing p_n.



- A collared patch -

Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n. Note that $Df_n = \mathbf{1}$.
2. Call Ω the *projective limit* of the sequence

$$\cdots \stackrel{f_{n+1}}{\to} B_{n+1} \stackrel{f_n}{\to} B_n \stackrel{f_{n-1}}{\to} \cdots$$

3. $X_1, \cdots X_d$ are the commuting constant vector fields on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action τ on Ω.

Theorem 7 *The dynamical system*

$$(\Omega, \mathbb{R}^d, \tau) = \varprojlim (B_n, f_n)$$

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

V - Cohomology and *K*-Theory

Čech Cohomology of the Hull

Let $\mathcal{U}$ be an *open covering* of the Hull. If $U \in \mathcal{U}$, $\mathcal{F}(U)$ is the space of integer valued locally constant function on U.

For $n \in \mathbb{N}$, the n-chains are the element of $C^n(\mathcal{U})$, namely the *free abelian group* generated by the elements of $\mathcal{F}(U_0 \cap \cdots \cap U_n)$ when the U_i varies in $\mathcal{U}$. A differential is defined by

$$d : C^n(\mathcal{U}) \mapsto C^{n+1}(\mathcal{U})$$

$$df(\bigcap_{i=0}^{n+1} U_i) = \sum_{j=0}^{n} (-1)^j f(\bigcap_{i:i \neq j} U_i)$$

This defines a *complex* with cohomology $\check{H}^n(\mathcal{U},\mathbb{Z})$. The Čech cohomology group of the Hull Ω is defined as

$$\check{H}^n(\Omega,\mathbb{Z}) = \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U},\mathbb{Z})$$

with ordering given by *refinement* on the set of open covers. Thanks to properties of the cohomology, if f_n^* is the map induced by f_n on the cohomology

$$\check{H}^n(\Omega,\mathbb{Z}) = \varinjlim_{n} \left(\check{H}^n(B_n,\mathbb{Z}), f_n^*\right)$$

Examples

J. E. ANDERSON, I. PUTNAM, Ergodic Theory Dynam. Systems, **18**, (1998), 509-537.
L. SADUN, Topology of Tiling Spaces. AMS (2008)

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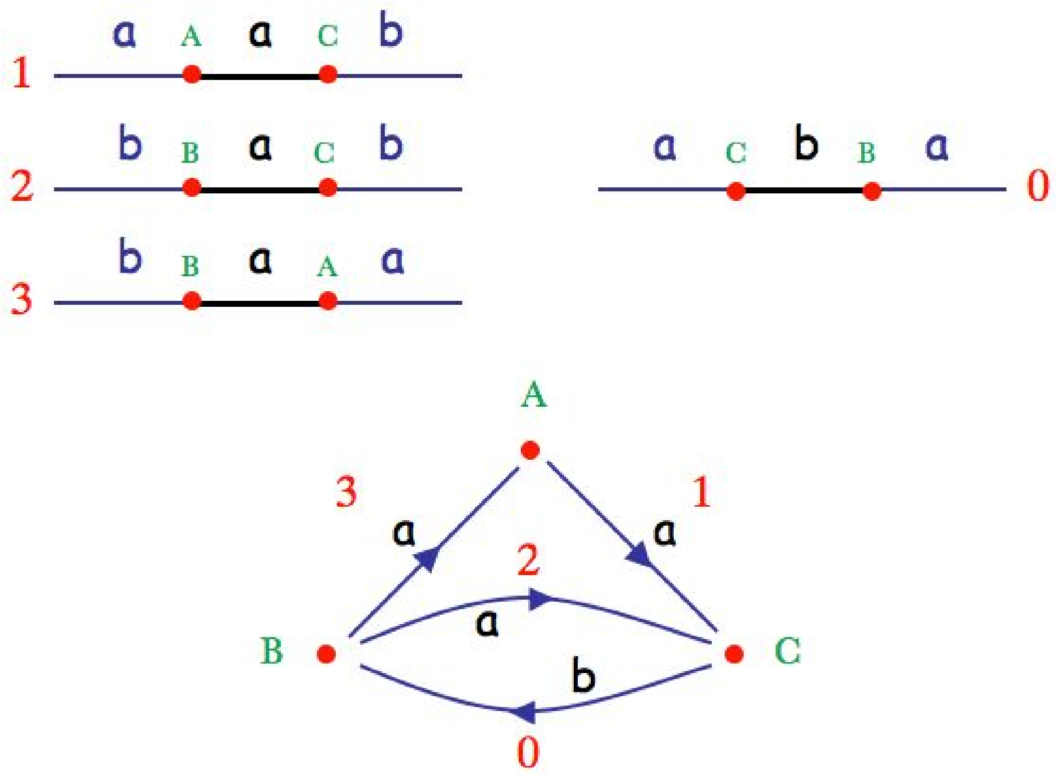
- Fibonacci: divides $\mathbb{R}$ into intervals a, b of length $1, \sigma = (\sqrt{5}-1)/2$ according to the substitution rule $a \mapsto ab\,,\ b \mapsto a$. Then $H^0 = \mathbb{Z}\,,\ H^1 = \mathbb{Z}^2$



- The Anderson-Putnam complex for the Fibonacci tiling -

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- Penrose 2D: $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^5$, $H^2 = \mathbb{Z}^8$

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- Penrose 2D:
  $H^0 = \mathbb{Z}\,,\ H^1 = \mathbb{Z}^5\,,\ H^2 = \mathbb{Z}^8$
- Chair tiling:
  $H^0 = \mathbb{Z}\,,\ H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]\,,\ H^2 = \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$

Other Cohomologies

- Longitudinal Cohomology (Connes, Moore-Schochet)
- Pattern-equivariant cohomology (Kellendonk-Putnam, Sadun)
- PV-cohomology (Savinien-Bellissard)

In maximal degree the Čech *Homology* does exists. It contains a natural *positive cone* isomorphic to the set of *positive* $\mathbb{R}^d$*-invariant measures* on the Hull (Bellissard-Benedetti-Gambaudo).

Cohomology and *K*-theory

The main topological property of the Hull (or tiling psace) is summarized in the following

Theorem 8 *(i) The various cohomologies, Čech, longitudinal, pattern-equivariant and PV, are isomorphic.*
(ii) There is a spectral sequence converging to the K-group of the Hull with page 2 given by the cohomology of the Hull.
(iii) In dimension $d \leq 3$ the K-group coincides with the cohomology.

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.

2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.

3. This dynamical system can be seen as a *lamination* or, equivalently, as the *inverse limit* of *Branched Oriented Flat Riemannian Manifolds*.

4. The Čech cohomology is equivalent to the longitudinal one, obtained by inverse limit, to the pattern-equivariant one or to the PV-cohomology. They are equivalent *Cohomologies* of the Hull. The *K-group* of the Hull can be computed through a spectral sequence with the cohomology in page 2.

5. In maximum degree, the *Homology* gives the family of *invariant measures* and the *Gap Labeling Theorem*.