The NON COMMUTATIVE GEOMETRY of the INTEGER QUANTUM HALL EFFECT

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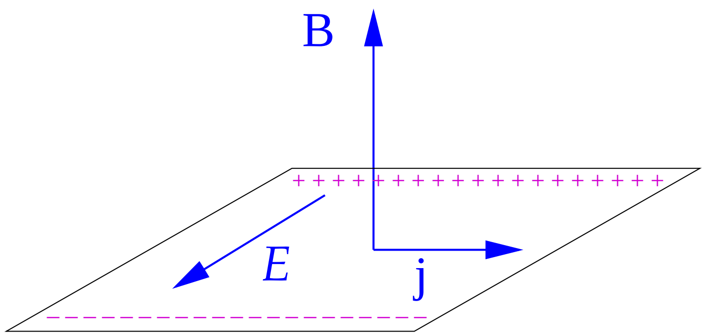
I - INTRODUCTION to the IQHE

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471.



B = magnetic field
j = current density
E = Hall electric field
n = charge carrier density

I.1)- The Classical HALL Effect:

In the stationnary state: $$e\,n\,\vec{\mathcal{E}} + \vec{j} \times \vec{B} = 0$$

$$\Rightarrow \quad \vec{j} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \vec{\mathcal{E}}\,, \qquad \sigma_H = \frac{ne}{B}\,.$$

$$\text{Units}: \frac{n}{B} = \left[\frac{1}{\text{flux}}\right]\,,\ \frac{h}{e} = [\text{flux}] \Rightarrow \nu = [1]\ .$$

$$\text{where}: \qquad \nu = \frac{nh}{eB} = \quad \textit{filling factor}\,.$$

HALL's formula

$$\sigma_H = \frac{\nu}{R_H}\,, \qquad R_H = \frac{h}{e^2} = 25\ 812.80\ \Omega\,.$$

I.2)- The (Integer) Quantum Hall Effect:

→ *Conditions of observation:*

1. Low temperatures ($\leq$ few Kelvins)
2. Large sample size ($\geq$ few μm)
3. High mobility together with large enough quenched disorder.
4. $2D$ fermion fluid.

→ *Experiments show that:*

1. Very flat plateaux at ν close to integers, namely if:

$$\sigma_H = \frac{i}{R_H} \quad i = 1, 2, 3, \cdots$$ quantization *(Von Klitzing et al.)*

2. On plateaux $\delta\sigma_H/\sigma_H$ and $\sigma_{//}/\sigma_H \leq 10^{-8}$.
   This indicates *localization*
   (Prange, Thouless, Halperin).
3. For $i \geq 2$, Coulomb interaction becomes negligible.

→ *Questions:*

1. Why is σ_H quantized ?
2. What is the rôle of localization ?

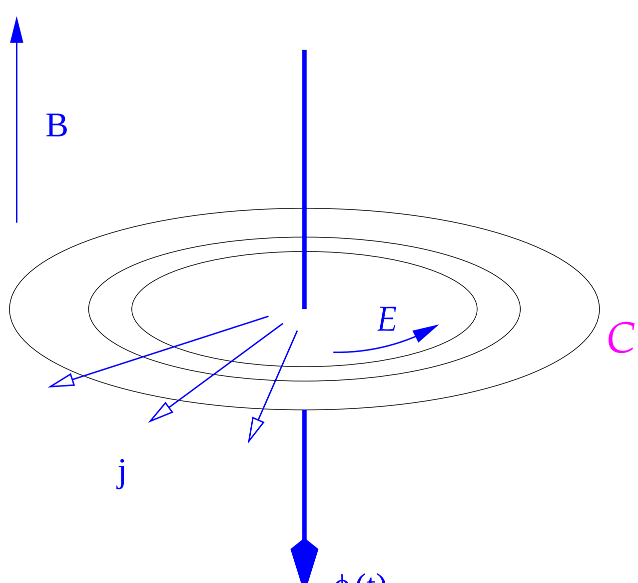
I.3)- Earlier Works:

R.B. LAUGHLIN, *Phys. Rev.* **B23**, 5632 (1981).

- Piercing the plane at x with a flux tube adiabatically varying from 0 to $\phi_0 = h/e$ forces 1 charge per filled Landau level to transfer from x to ∞.
- This adiabatic change induces a unitary tranformation on the Landau Hamiltonian (gauge transformation).
- This gives the quantization of the Hall conductance.

R.E. PRANGE, *Phys. Rev.* **B23**, 4802 (1981).
D.J. THOULESS, *J. Phys.* **C14**, 3475 (1981).
R. JOYNT, R.E. PRANGE, *Phys. Rev.* **B29**, 3303 (1984).

- Localized states do not see the adiabatic change !



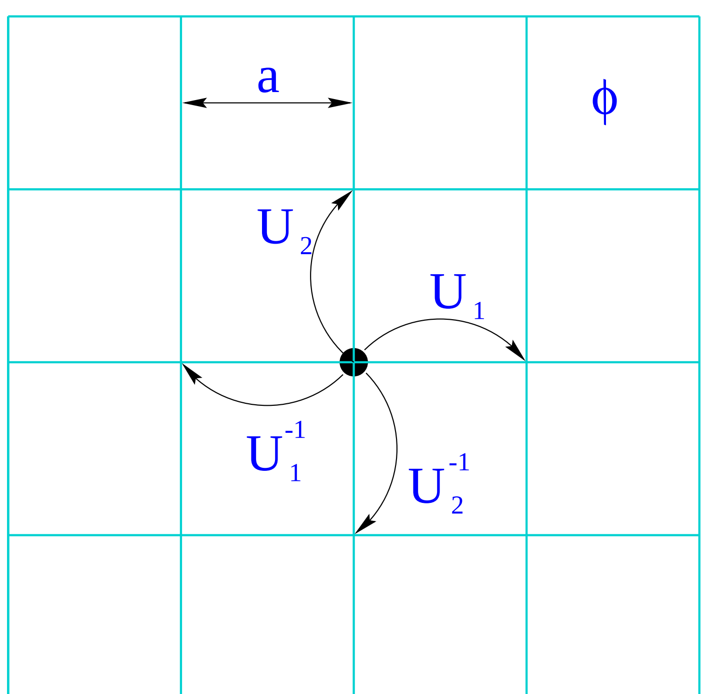
D. Thouless, M. Kohmoto, M. Nightingale, M. den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982).
J.E. Avron, R. Seiler, B. Simon, *Phys. Rev. Lett.* **51**, 51 (1983).

Harper's model: one electron on a square lattice in a uniform magnetic field. Magnetic translations U_1, U_2, satisfy:

$$U_1 U_2 = e^{2\imath\pi\alpha} U_2 U_1 \,, \qquad \alpha = \frac{\phi}{\phi_0} = \frac{Ba^2}{h/e} \,.$$

Harper's Hamiltonian:

$$H_H = U_1 + U_1^{-1} + U_2 + U_2^{-1} \,.$$



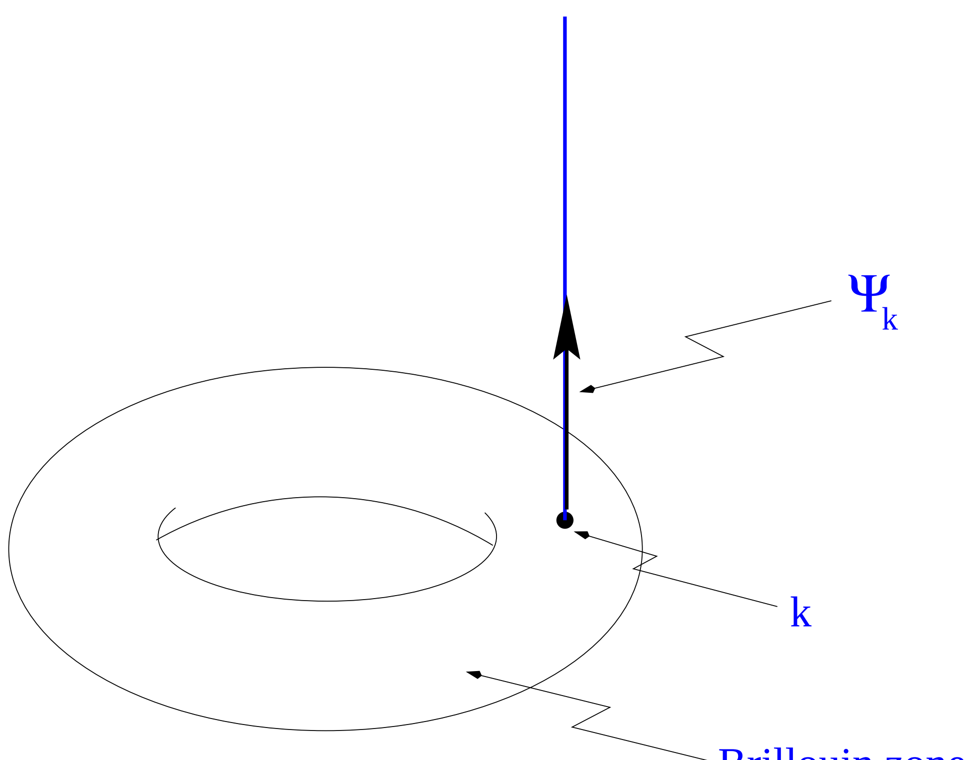
a = lattice spacing

ϕ = flux through unit cell

- If $\alpha = p/q$ then H_H is q-periodic;
- Bloch theory $\Rightarrow$ wave function Ψ depends on quasimomenta $\vec{k} = (k_1, k_2)$.
- $\vec{k} \in \mathbb{B}$ where $\mathbb{B} \approx \mathbb{T}^2$ is the *Brillouin zone*.
- Ψ defines a *line bundle* over $\mathbb{B}$.
- Non triviality controlled by the *Chern class*

$$\mathbf{Ch}(\Psi) = \frac{1}{\pi} \int_0^{2\pi} dk_1 \int_0^{2\pi} dk_2 \Im m < \frac{\partial \Psi}{\partial k_1} | \frac{\partial \Psi}{\partial k_2} >$$

- $\mathbf{Ch}(\Psi) \in \mathbb{Z}$ and is homotopy invariant.



- Assume *Fermi level* E_F lies in a gap.

- Assume N bands $E_1(\vec{k}) < \cdots < E_N(\vec{k}) < E_F < E_{N+1}(\vec{k})$ below Fermi level.

- Set $P_F = \sum_{i \leq N} |\Psi_i >< \Psi_i|$ (*Fermi projection*).

- Set $\mathbf{Ch}(P_F) = \sum_{i \leq N} \mathbf{Ch}(\Psi_i)$.

- The following holds true:

$$\mathbf{Ch}(P_F) \;=\; 2\imath\pi \int_{\mathbb{T}^2} \frac{d^2\vec{k}}{4\pi^2} \,\mathrm{Tr}\left(P_F(\vec{k}) \left[\partial_1 P_F(\vec{k}), \partial_2 P_F(\vec{k})\right] \right)$$

- Then Hall conductance is given by the *Chern-Kubo formula*

$$\sigma_H \;=\; \frac{e^2}{h} \,\mathbf{Ch}(P_F)$$

- $\Rightarrow$ Hall conductivity is *quantized* from *topological* origin.

I.4)- Difficulties with Earlier Works

1. If the magnetic flux is *irrational*
   $\Rightarrow$ no Bloch theory !
2. Disorder destroys also periodicity
   $\Rightarrow$ no Bloch theory !
3. Robustness against small disorder *suggested* from the Kubo-Chern formula,
   (see H. KUNZ, *Commun. Math. Phys.* **112**, 121 (1987).).
   But a general proof is needed.
4. How does one understand *localization* in this context ?

$\longrightarrow$ *Proposal*

1)- J. BELLISSARD, in *Lecture Notes in Phys.*, n°**153**, Springer Verlag, Berlin, Heidelberg, New York, (1982).
2)- J. BELLISSARD, in *Lecture Notes in Physics* **257**, Springer-Verlag, Berlin, Heidelberg, New York, (1986).

Use C^*-algebras and their
Non Commutative Geometry !

II - The NON COMMUTATIVE BRILLOUIN ZONE

J. Bellissard, in *From Number Theory to Physics*, Springer-Verlag, Berlin, (1992).

II.1)- The Hull of Aperiodic Media

II.1.1- A TYPICAL HAMILTONIAN

The Schrödinger Hamiltonian for an electron submitted to atomic forces is given by (ignoring interactions):

$$H = \frac{1}{2m}\left(\vec{P} - q\vec{A}(\,.\,)\right)^2 + \sum_{r=1}^{K}\sum_{y\in L_r} v_r(\,.-y)\ .$$

acting on $\mathcal{H} = L^2(\mathbb{R}^d)$.

- d : physical space dimension
- $r = 1, \ldots, K$ labels the atomic species,
- L_r : set of positions of atoms of type r,
- v_r : effective potential for valence electrons near an atom of type r,
- m and q : mass and charge of the carrier,
- $\vec{P} = -\imath\hbar\vec{\nabla}$: momentum operator,
- $\vec{A}$: magnetic vector potential.

II.1.2- Magnetic translations

- In $d = 2$, uniform magnetic field $B = \partial_1 A_2 - \partial_2 A_1$.
- Magnetic translations

$$U(\vec{a}) = e^{\frac{\imath}{\hbar} \oint_0^{\vec{a}} d\vec{s} \left(\vec{P} - q\vec{A}(\vec{s}) \right)}$$

- Weyl's commutations relations

$$U(\vec{a})\, U(\vec{b}) \;=\; e^{\imath \frac{q}{\hbar} B \vec{a} \times \vec{b}}\; U(\vec{b})\, U(\vec{a})$$

- Translation invariance of the kinetic part.

$$U(\vec{a}) \left(\vec{P} - q\vec{A}(\,.\,) \right)^2 U(\vec{a})^{-1} \;=\; \left(\vec{P} - q\vec{A}(\,.\,) \right)^2$$

- Translation of the potential

$$U(\vec{a})\, V(\,.\,)\, U(\vec{a})^{-1} \;=\; V(\,. - \vec{a})$$

II.1.3- The Hull

- The set $\{H_a = U(a)HU(a)^{-1}; a \in \mathbb{R}^2\}$ of translated of H, is endowed with the strong-resolvent topology.
- Let Ω be its closure and $\omega^{(0)}$ be the representative of H.

Definition 1 *The operator H is homogeneous if Ω is compact.*

- $(\Omega, \mathbb{R}^2)$ becomes a dynamical system, *the Hull* of H. It is topologically transitive (one dense orbit). The action is denoted by $\omega \mapsto \tau^a\omega$ $(a \in \mathbb{R}^2)$.
- If the potential V is continuous, there is a continuous function $\hat{v}$ on Ω such that if $\omega \in \Omega$ the corresponding operator H_ω is a Schrödinger operator with potential $V_\omega(x) = \hat{v}(\tau^{-x}\omega)$.
- *Covariance* $U(a)H_\omega U(a)^{-1} = H_{\tau^a\omega}$
- The observable algebra $\mathcal{A}_H$ is the C^*-algebra generated by bounded functions of the H_a's. It is related to the *twisted crossed product* $C^*(\Omega \rtimes \mathbb{R}^2, B)$.

II.2)- The C^*-algebra $C^*(\Omega \rtimes \mathbb{R}^2, B)$

II.2.1- Definition

Endow $\mathcal{A}_0 = \mathcal{C}_c(\Omega \times \mathbb{R}^2)$ with (here $A, A' \in \mathcal{A}_0$):

1. Product

$$A \cdot A'(\omega, \vec{x}) = \int_{\vec{y}\in\mathbb{R}^2} d^2\vec{y} A(\omega, \vec{y}) A'(\tau^{-\vec{y}}\omega, \vec{x} - \vec{y}) e^{\frac{iqB}{2\hbar}\vec{x}\wedge\vec{x}}$$

2. Involution

$$A^*(\omega, \vec{x}) = \overline{A(\tau^{-\vec{x}}\omega, -\vec{x})}$$

3. A faithfull family of representations in $\mathcal{H} = L^2(\mathbb{R}^2)$

$$\pi_\omega(A)\psi(\vec{x}) = \int_{\mathbb{R}^2} d^2\vec{y} A(\tau^{-\vec{x}}\omega, \vec{y} - \vec{x}) e^{\frac{iqB}{2\hbar}\vec{y}\wedge\vec{x}} \psi(\vec{y}) \ .$$

   if $A \in \mathcal{A}_0$, $\psi \in \mathcal{H}$.

4. C^*-norm

$$\|A\| = \sup_{\omega\in\Omega} \|\pi_\omega(A)\| \ .$$

Definition 2 *The C^*-algebra $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ is the completion of $\mathcal{A}_0$ under this norm.*

II.2.2- Tight-Binding Representation

J. Bellissard, in *Lecture Notes in Physics* **257**, Springer-Verlag, Berlin, Heidelberg, New York, (1986).

1. If $\mathcal{L}$ is the original set of atomic positions, let Σ be the closure of the set $\{\tau^{-\vec{x}}\omega^{(0)} \in \Omega; \vec{x} \in \mathcal{L}\}$. Σ is a *transversal*.

2. Replace $\Omega \times \mathbb{R}^2$ by $\Gamma = \{(\omega, \vec{x}) \in \Omega \times \mathbb{R}^2; \omega \in \Sigma\,, \tau^{-\vec{x}}\omega \in \Sigma\ \}$. Γ is a *groupoid*.

3. Replace integral over $\mathbb{R}^2$ by discrete sum over $\vec{x}$.

4. Replace $\mathcal{A}_0$ by $\mathcal{C}_c(\Gamma)$, the space of continuous function with compact support on Γ. Then proceed as before to get $C^*(\Gamma, B)$.

5. $C^*(\Gamma, B)$ is unital.

6. One can restrict the original Hamiltonian H to a spectral bounded interval (in practice near the Fermi level), so as to get an *effective Hamiltonian* H_{eff} in $C^*(\Gamma, B)$. Thus H_{eff} is bounded.

II.2.3- CALCULUS

- Let $\mathbf{P}$ be an $\mathbb{R}^2$-invariant ergodic probability measure on Ω. Then set (for $A \in \mathcal{A}_0$)):

$$\mathcal{T}(A) = \int_\Omega d\mathbf{P}\, A(\omega, 0) = \overline{< 0|\pi_\omega(A) 0 >}^{dis.}$$

Then $\mathcal{T}$ extends as a *positive trace* on $\mathcal{A}$.

- $\mathcal{T}$ is a *trace per unit volume*, thanks to Birkhoff's theorem:

$$\mathcal{T}(A) = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \mathrm{Tr}(\pi_\omega(A) \restriction_\Lambda) \quad \text{a.e. } \omega$$

- A commuting set of $*$-derivations is given by

$$\partial_i A(\omega, \vec{x}) = \imath x_i A(\omega, \vec{x})$$

defined on $\mathcal{A}_0$. It satisfies $\pi_\omega(\partial_i A) = -\imath[X_i, \pi_\omega(A)]$ where $\vec{X} = (X_1, X_2)$ are the coordinates of the position operator.

II.2.4- Properties of $\mathcal{A}$

Theorem 1 *Let $\mathcal{L}$ be a periodic lattice in $\mathbb{R}^2$. If H is $\mathcal{L}$-invariant, $\mathcal{A}$ is isomorphic to $\mathcal{C}(\mathbb{B}) \otimes \mathcal{K}$, where $\mathbb{B}$ is the Brillouin zone and $\mathcal{K}$ is the C^*-algebra of compact operators.*

$\mathcal{A}$ is the non commutative analog of the space of continuous functions on the Brillouin zone : it will be called *the Non Commutative Brillouin zone*.

Theorem 2 *Let H be a homogeneous Schrödinger operator with hull Ω. Then for any $z \in \mathbb{C} \backslash \sigma(H)$ there is an element $R(z) \in \mathcal{A}$ (which is C^∞), such that*

$$\pi_\omega(R(z)) = (z\mathbf{1} - H_\omega)^{-1}$$

for all $\omega \in \Omega$.

Moreover, the spectrum of $R(z)$ is given by

$$\sigma(R(z)) = \{(z-\zeta)^{-1}; \zeta \in \Sigma\}, \ \Sigma = \cup_{\omega\in\Omega}\sigma(H_\omega)$$

II.2.5- IDoS and Shubin's formula

- Let $\mathbf{P}$ be an invariant ergodic probability on Ω. Let

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega \restriction_\Lambda \leq E\}$$

It is the *Integrated Density of states* or *IDoS*.

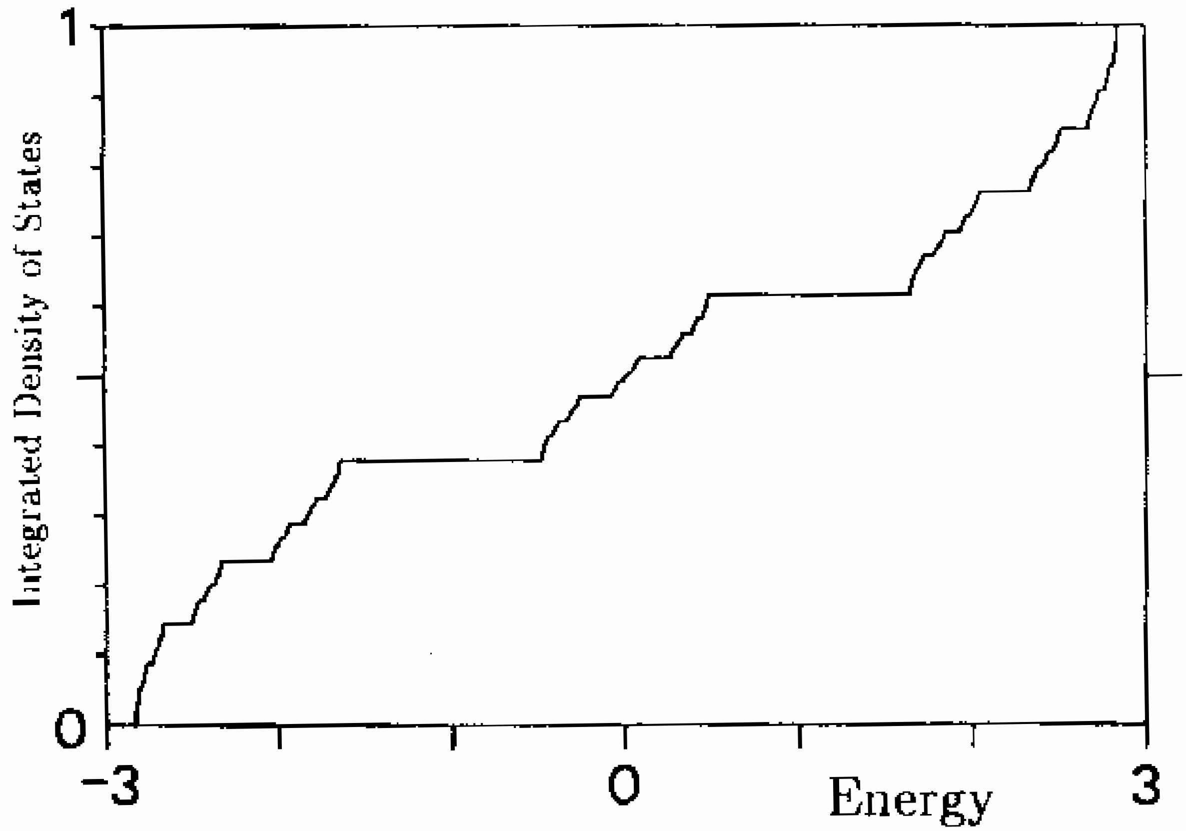
- The limit above exists $\mathbf{P}$-almost surely and

$$\mathcal{N}(E) = \mathcal{T}(\chi(H \leq E)) \qquad \text{(Shubin, 1976)}$$

$\chi(H \leq E)$ is the eigenprojector of H in $\mathcal{L}^\infty(\mathcal{A})$.

- $\mathcal{N}$ is non decreasing, non negative and constant on gaps. $\mathcal{N}(E) = 0$ for $E < \inf \Sigma$. For $E \to \infty$ $\mathcal{N}(E) \sim \mathcal{N}_0(E)$ where $\mathcal{N}_0$ is the IDoS of the free case (namely $V = 0$).

- $d\mathcal{N}/dE = n_{\text{DOS}}$ defines a Stieljes measure called the *Density of States* or *DOS*.



- AN EXAMPLE OF IDOS -

II.2.6- STATES

We consider states on $\mathcal{A}$ of the form

$$A \in \mathcal{A} \to \mathcal{T}\{\rho A\} ,$$

with $\rho \geq 0$ and $\mathcal{T}\{\rho\} = n$ if n is the charge carrier density. Then

$$\rho \in L^1(\mathcal{A}, \mathcal{T})$$

THE *Fermi-Dirac* STATE:

describes equilibrium of a fermion gas of independent particles at inverse temperature $\beta = 1/k_{\mathrm{B}}T$ and chemical potential μ:

$$\rho_{\beta,\mu} = \frac{1}{\mathbf{1} + e^{\beta(H-\mu)}}$$

μ is fixed by the normalization condition

$$\mathcal{T}\{\rho_{\beta,\mu}\} = n .$$

II.3)- To Summarize

1. The C^*-algebra $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ is a Non Commutative analog of the space of continuous functions over the Brillouin zone $\mathbb{B}$ if the lattice of atoms is no longer periodic, or if there is a magnetic field.
2. A groupoid Γ associated to the discrete set of atomic positions, gives rise to tight-binding models.
3. Calculus on $\mathcal{A}$ is available and generalizes the usual calculus on $\mathbb{B}$.
4. Textbook formulæ valid for perfect crystals can be easily generalized using this calculus. If P_F is the zero temperature limit of the *Fermi-Dirac* state, constrained by $\mathcal{T}(P_F) = n$, the expression

$$\mathbf{Ch}(P_F) = 2\imath\pi\mathcal{T}\left(P_F\left[\partial_1 P_F, \partial_2 P_F\right]\right)$$

is valid at least if $E_F = \mu \restriction_{T=0}$ belongs to a gap of the energy spectrum.

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III - The FOUR TRACE WAY

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471.

III.1)- The Kubo Formula

III.1.1- Background

- The *(non dissipative) current* is

$$\vec{J} = q\frac{d\vec{X}}{dt} = \frac{\imath q}{\hbar}[H, \vec{X}] = \frac{q}{\hbar}\vec{\nabla} H$$

- The *thermal average* of $A \in \mathcal{A}$

$$< A >_{\beta,\mu} = \mathcal{T}\,(A\,\rho_{\beta,\mu})$$

- The *Liouville operator* acts on $\mathcal{A}$

$$\mathcal{L}_H = \frac{\imath}{\hbar}[H, \,.\,]$$

- A *dissipative evolution* requires an operator C acting on $\mathcal{A}$ such that $exp\{-tC\} : \mathcal{A} \mapsto \mathcal{A}$ is a *completely positive contraction semigroup*. C has the dimension of $[\text{time}]^{-1}$. The (dissipative) evolution, with a uniform electric field, is given by the *Master Equation*:

$$\frac{dA}{dt} = \mathcal{L}_H(A) + \frac{q}{\hbar}\vec{\mathcal{E}}.\vec{\nabla} A - C(A)$$

III.1.2- Linear Response Theory

- The thermal averaged current satifies:

$$\vec{j} = < q \frac{d\vec{X}}{dt} >_{\beta,\mu} = \sigma \vec{\mathcal{E}} + O(\vec{\mathcal{E}}^2)$$

- The 2×2 matrix σ is the *conductivity tensor*. It is given by *Kubo's formula*

$$\sigma_{ij} = \frac{q^2}{\hbar} \mathcal{T} \left(\partial_j \rho_{\beta,\mu} \frac{1}{\hbar C - \hbar \mathcal{L}_H} (\partial_i H) \right)$$

- C usually depends on T so that as $T \downarrow 0$, $C \downarrow 0$.

- We have $\lim_{T \downarrow 0} \rho_{\beta,\mu} = P_F$.

Theorem 3 *Let assume*

1. *The Fermi level E_F is not a discontinuity point of the DOS of H.*
2. $\lim_{T\downarrow 0} C = 0$.
3. *P_F is Sobolev differentiable:* $\mathcal{T}\left\{(\vec{\nabla} P_F)^2\right\} < \infty$.

Then, as $T \downarrow 0$, the conductivity tensor converges to

$$\sigma_{ij} = \frac{q^2}{h} 2\imath\pi \, \mathcal{T}\left(P_F \left[\partial_i P_F \, , \, \partial_j P_F \right]\right) .$$

In particular the direct conductivity vanishes and

$$\sigma_{12} = \sigma_H = \frac{q^2}{h}\mathbf{Ch}(P_F)$$

III.2)- The Four Traces

- On every Hilbert space $\mathcal{H}$, *the usual trace* is denoted by Tr.
- In $\mathcal{A}$ we have the *trace per unit volume* $\mathcal{T}$, associated to a translation invariant probability measure $\mathbf{P}$ on the Hull.

III.2.1- Dixmier's Traces

J. Dixmier, *C.R.A.S.*, 1107 (1966).

- On a Hilbert space $\mathcal{H}$, $\mathcal{L}^p(\mathcal{H})$ denotes the *Schatten ideal* of those compact operator on $\mathcal{H}$ such that $\mathrm{Tr}(|T|^p) < \infty$.
- Given T a compact operator on $\mathcal{H}$, let $\mu_0 \geq \cdots \geq \mu_n \geq \ldots \geq 0$ be its singular values (eigenvalues of $|T|$) labelled in decreasing order. Set

$$\|T\|_{p+} = \left(\limsup_{n \in \mathbb{N}} \frac{1}{\ln N} \sum_{n=0}^{N-1} \mu_n^p \right)^{1/p}$$

- The set of $\{T\,;\, \|T\|_{p+} < \infty\}$ is denoted by $\mathcal{L}^{p+}(\mathcal{H})$. This a *Mačaev ideal*.

Theorem 4 *Set* $\mathcal{L}^{p-}(\mathcal{H}) = \{T \text{compact}; \|T\|_{p+} = 0\}$.

1. $\mathcal{L}^{p-}(\mathcal{H})$ *and* $\mathcal{L}^{p+}(\mathcal{H})$ *are two-sided ideals in* $\mathcal{L}(\mathcal{H})$.
2. *For* $p < p' \in [0, \infty)$,
$$\mathcal{L}^{p}(\mathcal{H}) \subset \mathcal{L}^{p-}(\mathcal{H}) \subset \mathcal{L}^{p+}(\mathcal{H}) \subset \mathcal{L}^{p'}(\mathcal{H})$$
3. $\|T\|_{p+}$ *is a seminorm making* $\mathcal{L}^{p+}(\mathcal{H})/\mathcal{L}^{p-}(\mathcal{H})$ *a Banach space.*

- Given a euclidean invariant mean M on $\mathbb{R}$, one can define a linear form Lim_M on $\ell^\infty(\mathbb{N})$ such that
(i) $\mathrm{Lim}_M(a_0, a_1, a_2, \cdots) = \mathrm{Lim}_M(a_1, a_2, a_3, \cdots)$,
(ii) $\mathrm{Lim}_M(a_0, a_1, a_2, \cdots) = \mathrm{Lim}_M(a_0, a_0, a_1, a_1, \cdots)$,
(iii) if $a \in \ell^\infty(\mathbb{N})$ converges, $\mathrm{Lim}_M(a) = \lim_{n \to \infty} a_n$.
- The *Dixmier trace* associated to M is given by
$$\mathrm{Tr}_{\mathrm{Dix}}(T) = \mathrm{Lim}_M \left(\frac{1}{\ln N} \sum_{n=0}^{N-1} \mu_n \right) .$$
if $T \in \mathcal{L}^{1+}(\mathcal{H})$ is positive.
- $\mathrm{Tr}_{\mathrm{Dix}}$ can be extended as a positive continuous linear form on $\mathcal{L}^{1+}(\mathcal{H})$ vanishing on $\mathcal{L}^{1-}(\mathcal{H})$ such that
$$\mathrm{Tr}_{\mathrm{Dix}}(UTU^{-1}) = \mathrm{Tr}_{\mathrm{Dix}}(T), \quad \mathrm{Tr}_{\mathrm{Dix}}(ST) = \mathrm{Tr}_{\mathrm{Dix}}(TS)$$
for $U \in \mathcal{L}(\mathcal{H})$ unitary and $S, T \in \mathcal{L}^{1+}(\mathcal{H})$.

III.2.2- Graded Trace and Fredholm Module

M. Atiyah, *K-Theory*, (Benjamin, New York, 1967).
A. Connes, *Publ. IHES*, **62**, 257 (1986).

- Set $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2$ with $\mathcal{H} = L^2(\mathbb{R}^2)$. The *grading operator* G is

$$G = \begin{pmatrix} +\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

- $T \in \mathcal{L}(\hat{\mathcal{H}})$ has *degree* 0 if $GT - TG = 0$
  $T \in \mathcal{L}(\hat{\mathcal{H}})$ has *degree* 1 if $GT + TG = 0$.

- The *graded commutator* is given by

$$[T, T']_S = TT' - (-)^{d^\circ T . d^\circ T'} T'T$$

- A degree 1 operator F is defined by

$$F = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}$$

  where $u = X/|X|$ and $X = X_1 + \imath X_2$ is the position operator. Then $F = F^*$, $F^2 = \mathbf{1}$.

- A *differential* d with $d^2 = 0$ is given by

$$dT = [F, T]_S$$

- The *Leibniz rule* becomes

$$d(TT') = dT\,T' + (-)^{d^\circ T} T\,dT'$$

- A *graded trace* is defined as

$$\boxed{\mathrm{Tr}_S(T) = \frac{1}{2}\mathrm{Tr}(GFdT)}$$

if $dT \in \mathcal{L}^1(\hat{\mathcal{H}})$.

- Tr_S is *linear* and satisfies

$$dT\,,\; dT' \in \mathcal{L}^1(\hat{\mathcal{H}}), \Rightarrow \mathrm{Tr}_S([T, T']_S) = 0$$

- **Note:**

1. Tr_S *is not positive* in general.
2. $u = X/|X|$ coincides precisely with the singular gauge transformation corresponding to piercing the plane adiabatically with one flux quantum.
   J.E. Avron, R. Seiler, B. Simon, *Commun. Math. Phys.*, **159**, 399 (1994).

III.3)- Connes Formulæ

III.3.1- First Connes Formula

- Let $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ acts on $\hat{\mathcal{H}}$ by $\hat{\pi}_\omega = \pi_\omega \otimes \mathrm{id}$ through degree 0 elements.

- First Connes Formula : for $A \in \mathcal{A}_0$ and $\mathbf{P}$-almost all ω's:

$$\mathcal{T}\left(|\vec{\nabla} A|^2\right) = \frac{1}{\pi} \mathrm{Tr}_{\mathrm{Dix}}\left(|d\hat{\pi}_\omega(A)|^2\right)$$

- Let $\mathcal{S}$ be the *Non Commutative Sobolev space* namely the Hilbert space generated by $A \in \mathcal{A}_0$ such that $\mathcal{T}(|A|^2 + |\vec{\nabla} A|^2) < \infty$. Then

$$A \in \mathcal{S} \Rightarrow d\hat{\pi}_\omega(A) \in \mathcal{L}^{2+}(\hat{\mathcal{H}})$$

- Also $d\hat{\pi}_\omega(A)$ is compact for any $A \in \mathcal{A}$.

III.3.2- A Cyclic 2-cocycle

- For A_0 , A_1 , $A_2 \in \mathcal{A}_0$, a *cyclic 2-cocycle* is defined by

$$\mathcal{T}_2(A_0, A_1, A_2) = 2\imath\hat{\pi}\mathcal{T}(A_0\partial_1 A_1\partial_2 A_2 - A_0\partial_2 A_1\partial_1 A_2)$$

  This trilinear form extends by continuity to $\mathcal{S}$.

- $\mathcal{T}_2$ is *cyclic*

$$\mathcal{T}_2(A_0, A_1, A_2) = \mathcal{T}_2(A_2, A_0, A_1)$$

- $\mathcal{T}_2$ is *Hochschild closed*

$$\begin{aligned} 0 &= (b\mathcal{T}_2)(A_0, A_1, A_2, A_3) \equiv \\ &\mathcal{T}_2(A_0A_1, A_2, A_3) - \mathcal{T}_2(A_0, A_1A_2, A_3) \\ &+\mathcal{T}_2(A_0, A_1, A_2A_3) - \mathcal{T}_2(A_3A_0, A_1, A_2) \end{aligned}$$

- Second Connes Formula : for $A_i \in \mathcal{A}_0$:

$$\boxed{\mathcal{T}_2(A_0, A_1, A_2) = \int_\Omega d\mathbf{P}\, \mathrm{Tr}_S\left(\hat{\pi}_\omega(A_0)d\hat{\pi}_\omega(A_1)d\hat{\pi}_\omega(A_2)\right)}$$

III.4)- Quantization of Hall conductivity

Recall that at $T = 0$, the Hall conductivity becomes

$$\sigma_H = \frac{e^2}{\hbar}\mathbf{Ch}(P_F)$$

III.4.1- Fredholm Index

- FACT 1 : let P be a projection on $\mathcal{H}$ and $\hat{P} = P \otimes \mathbf{1}_2$. If $d\hat{P} \in \mathcal{L}^3(\mathcal{H})$ then PuP is *Fredholm* on $P\mathcal{H}$ and

$$\mathrm{Tr}_S\left(\hat{P}d\hat{P}d\hat{P}\right) = \mathrm{Ind}\left(PuP \restriction_{P\mathcal{H}}\right) \in \mathbb{Z}$$

- FACT 2 : $d\hat{P} \in \mathcal{L}^3(\mathcal{H}) \iff (uPu^* - P) \in \mathcal{L}^3(\mathcal{H})$ and

$$\mathrm{Ind}\left(PuP \restriction_{P\mathcal{H}}\right) = \mathrm{Tr}\left((uPu^* - P)^{2n+1}\right) \quad \forall n \geq 1$$

- Thus $\mathrm{Ind}(PuP \restriction_{P\mathcal{H}})$ *measures the increase of the dimension of* $P\mathcal{H}$ *after applying* u.

III.4.2- $\mathbf{Ch}(P_F)$ IS AN INTEGER

- ASSUME : $P_F \in \mathcal{S}$.
  Then $d\hat{\pi}_\omega(P_F) \in \mathcal{L}^{2+}(\mathcal{H}) \subset \mathcal{L}^3(\mathcal{H})$ (1st Connes formula).
- By the 2nd Connes formula we get

$$\mathbf{Ch}(P_F) = \int_\Omega d\mathbf{P}\, \mathrm{Tr}_S\left(\hat{\pi}_\omega(P_F) d\hat{\pi}_\omega(P_F) d\hat{\pi}_\omega(P_F)\right)$$

  The r.h.s. is the disordered average of

$$n(\omega) = \mathrm{Ind}(\pi_\omega(P_F) u \pi_\omega(P_F) \restriction_{\pi_\omega(P_F)\mathcal{H}}) \in \mathbb{Z}$$

- By *covariance* one gets

$$n(\tau^{\vec{x}}\omega) = n(\omega) \quad \mathbf{P}\text{-almost all } \omega \text{ and } \vec{x} \in \mathbb{R}^2$$

- Since $\mathbf{P}$ is *invariant ergodic*, $n(\omega)$ is almost surely constant so that

$$P_F \in \mathcal{S} \Rightarrow \mathbf{Ch}(P_F) \in \mathbb{Z}$$

  and $\mathbf{Ch}(P_F)$ measures the number of states created if one applies u namely the *Laughlin singular gauge transformation* ! This is indeed the number of charges sent at ∞.

III.4.3- Existence of Plateaux

- The Fermi level E_F is defined as the limit as $T \downarrow 0$ of the chemical potential μ, constrained to

$$\mathcal{T}(\rho_{\beta,\mu}) = n$$

  where n is the charge carrier density.

- Experimentally one can change E_F either by changing the magnetic field B or by changing n.
  Both ways are used in practice.

- Remark that $P \in \mathcal{S} \mapsto \mathbf{Ch}(P) \in \mathbb{Z}$ is continuous thanks to Connes formulæ.

- Since $P_F = \chi(H \leq E_F)$, if we assume that the map $E_F \in (E_-, E_+) \mapsto P_F \in \mathcal{S}$ is continuous (for the Sobolev norm), then $\mathbf{Ch}(P_F)$ stay constant for E_F in the interval (E_-, E_+) !
  This is the mechanism through which plateaux occur in the Hall conductivity.

- In the next section we will see that the condition $P_F \in \mathcal{S}$ is a consequence of the *existence of localized states around the Fermi level*.

IV - LOCALIZATION and TRANSPORT

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471.
J. Bellissard, H. Schulz-Baldes, *Rev. Math. Phys.*, **10**, 1-46 (1998).
J. Bellissard, H. Schulz-Baldes, *J. Stat. Phys.*, **91**, (1998), 991-1026.
J. Bellissard, D. Spehner, work in progress.

IV.1)- Localization Theory

IV.1.1- Definitions

- The DOS of $H = H^*$ affiliated to $\mathcal{A}$, was defined as the Stieljies-Lebesgue measure
$$d\mathcal{N}(E) = d\mathcal{T}(P_E)$$
with $P_E = \chi(H \leq E)$.
For $\Delta \subset \mathbb{R}$ borelian, we set $P_\Delta = \chi(H \in \Delta)$.

- The non dissipative current is given by $\vec{J} = q/\hbar \vec{\nabla} H$.

- The *current-current correlation* is the measure m defined on $\mathbb{R} \times \mathbb{R}$ by:
$$\int_{\mathbb{R} \times \mathbb{R}} m(dE, dE') f(E) g(E') = \mathcal{T}\left(f(H)\, \vec{\nabla} H\, g(H)\, \vec{\nabla} H \right)$$
for f, g continuous functions with compact support on $\mathbb{R}$. In physicists notations (ignoring $q/\hbar$)
$$m(dE, dE') \text{``=''} | < E|\vec{J}|E' > |^2$$

- If H_ω is the representative of H associated to $\omega \in \Omega$ we set
$$\vec{X}_\omega(t) = e^{\imath \frac{t}{\hbar} H_\omega} \vec{X} e^{-\imath \frac{t}{\hbar} H_\omega}$$

IV.1.2- Localization Length

- Given $\Delta \subset \mathbb{R}$ borelian, the *average localization length* $\ell(\Delta)$ *of states with energy within* Δ is defined through the following steps

  1. Project the initial state $|\vec{x}>$ on Δ: $\pi_\omega(P_\Delta)|\vec{x}>$.
  2. Measure the distance it goes during time t by applying $(\vec{X}_\omega(t) - \vec{X})\pi_\omega(P_\Delta)|\vec{x}>$
  3. Square it to get the quantum average,
     average over time
     average over disorder to get

$$L_\Delta(t)^2 = \frac{1}{t}\int_0^t \frac{ds}{s}\int_\Omega d\mathbf{P} <\vec{x}|\pi_\omega(P_\Delta)(\vec{X}_\omega(t) - \vec{X})^2\pi_\omega(P_\Delta)|\vec{x}>$$

  4. Then

$$\ell(\Delta) = \limsup_{t \to \infty} L_\Delta(t)$$

- Fact :

$$L_\Delta(t)^2 = \frac{1}{t}\int_0^t \frac{ds}{s}\mathcal{T}\left(|\vec{\nabla}e^{-\imath\frac{t}{\hbar}H}|^2 P_\Delta\right)$$

Thus the localization length is *algebraic* and independent of the representation of the Hamiltonian !

IV.1.3- Localization: Results

Assume $\ell(\Delta) < \infty$:

1. The *spectrum* of H_ω in Δ is *pure point* almost surely w.r.t. ω: *all states in Δ are localized*.

2. There is an $\mathcal{N}$-measurable function ℓ on Δ such that for any $\Delta' \subset \Delta$ borelian,

$$\ell(\Delta') = \int_{\Delta'} d\mathcal{N}(E)\ell(E)^2$$

   $\ell(E)$ is the *localization length at energy* E

3. One has

$$\ell(\Delta') = \int_\Omega d\mathbf{P} \int_{\mathbb{R}^2} d^2\vec{x}\ |\vec{x}|^2 \sum_{E \in \sigma_{pp}(H_\omega) \cap \Delta'} \left| < 0|P_{\{E\},\omega}|\vec{x} > \right|^2$$

   where $P_{\{E\},\omega}$ is the eigenprojection of H_ω on the energy E.

4. One also gets:

$$\ell(\Delta') = 2 \int_{\Delta' \times \mathbb{R}} \frac{m(dE, dE')}{|E - E'|^2}$$

5. If $[E_0, E_1] \subset \Delta$

$$\|P_{E_1} - P_{E_0}\|_{\mathcal{S}} \leq \int_{E_0}^{E_1} (1 + \ell^2) d\mathcal{N}$$

IV.1.4- Existence of Plateaux

- From the previous results $P_F \in \mathcal{S}$ as long as E_F belongs to a region of localized states. Thus *localization* ⇒ *existence of plateaux* for the Hall conductivity.
- From previous results by Fröhlich & Spencer, Aizenman & Molčanov, the localization length is finite at high disorder for the *Anderson model*.
- More recent results by Combes & Hislop, W.M. Wang, the same is true for the *Landau Hamiltonian with a random potential*, at least $O(B^{-\infty})$-away from the Landau levels.

IV.2)- Kinetic Theory for Electronic Transport

IV.2.1- Mott's Variable Range Hopping

B. Shklovskii, A. L. Effros, *Electronic Properties of Doped Semiconductors*, (Springer-Verlag, Berlin, 1984).

- Strongly localized regime, dimension d
- Low electronic DOS, Low temperature
- Absorption-emission of a phonon of energy ε

$$Prob \propto e^{-\varepsilon/k_B T}$$

- Tunnelling probability at distance r

$$Prob \propto e^{-r/\xi}$$

- Density of state at Fermi level n_F

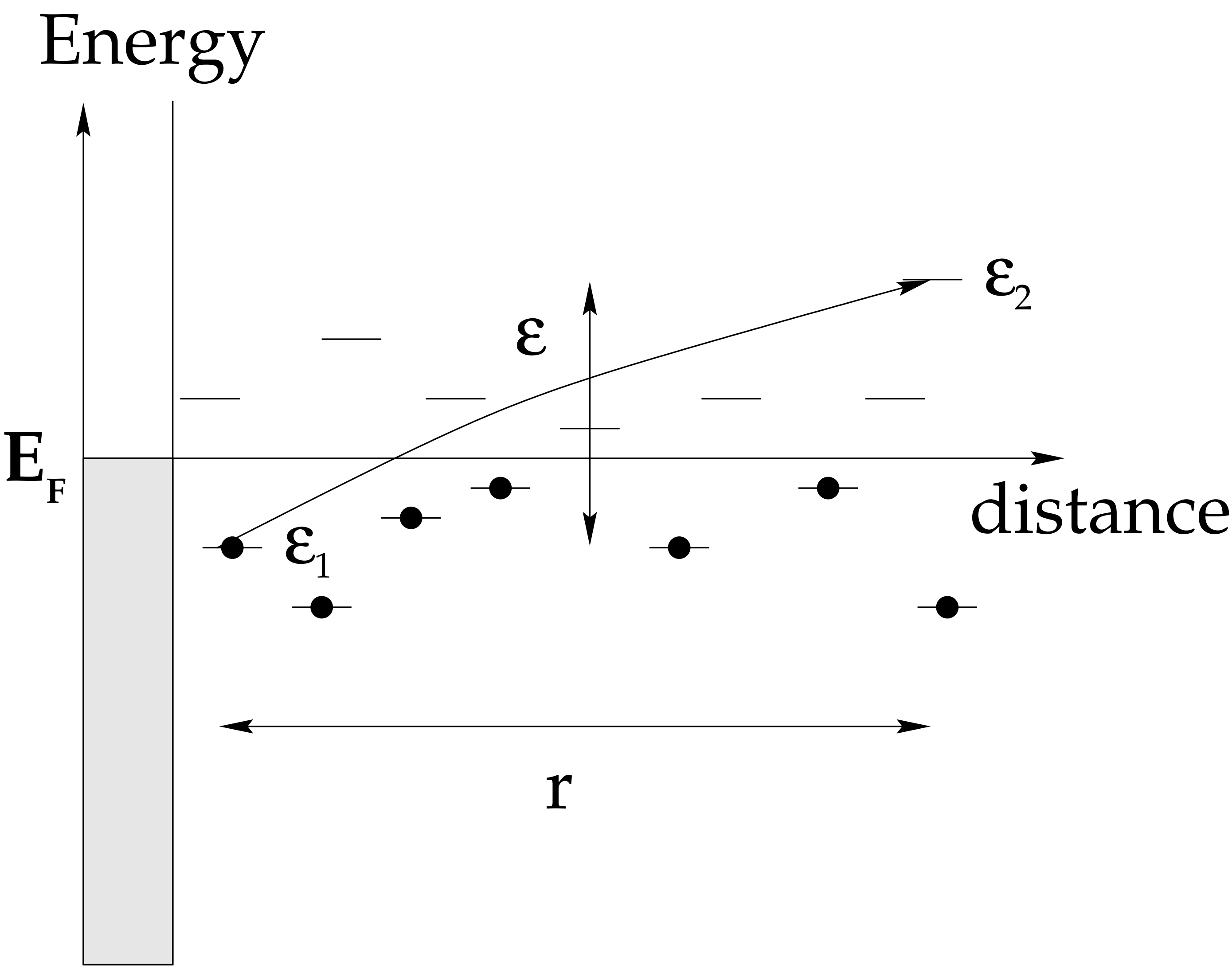
$$\varepsilon \, n_F \, r^d \approx 1$$

Optimizing, the conductivity satisfies

$$\sigma \propto e^{-(T_0/T)^{1/d+1}} \qquad \textit{Mott's law}$$

Optimal energy $\varepsilon_{opt} \sim T^{d/(d+1)} \gg T$

Optimal distance $r_{opt} \sim 1/T^{1/(d+1)} \gg \xi$



- MOTT'S VARIABLE RANGE HOPPING -

IV.2.2- The Drude model

Drude, (1900).

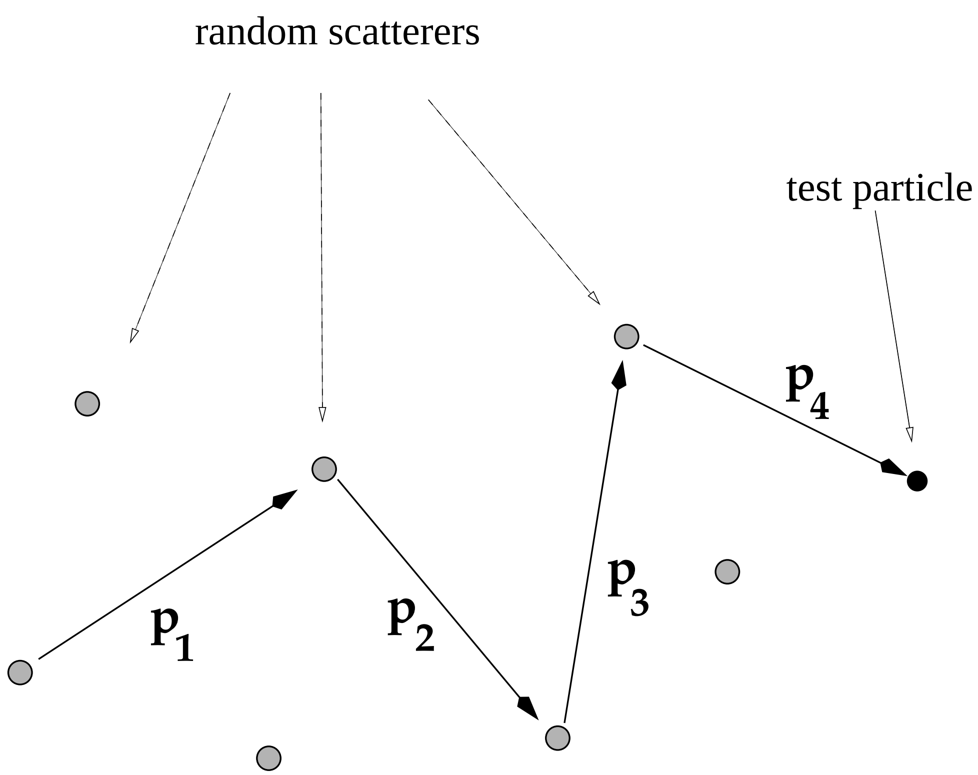
- Electrons in a metal are free classical particles of mass m_* and charge q.
- Electron density is n.
- Collisions occur at random poissonian times $\cdots < t_{-1} < t_0 < \cdots < t_{n+1} < \cdots$ with

$$< t_{n+1} - t_n >= \tau_{rel}$$

- If p_n is the electron momentum between times t_n and t_{n+1}, then the $p_{n+1} - p_n$'s are *independent random variables* distributed according to the *Maxwell distribution* at temperature T.

Then the conductivity follows the *Drude formula*

$$\sigma = \frac{q^2 n}{m_*} \tau_{rel}$$



- The Drude Kinetic Model -

IV.2.2- Quantum Drude model

D. Spehner, J. Bellissard, in progress.

1. Replace the classical motion by the quantum one in the homogeneous system with Hamiltonian H.
2. Replace collisions by *quantum jumps* indexed by r at random poissonian times
$\cdots < t_{-1}^{(r)} < t_0^{(r)} < \cdots < t_{n+1}^{(r)} < \cdots$ with
$$< t_{n+1}^{(r)} - t_n^{(r)} >= \Gamma_r^{-1}$$
3. At collision of type r the *density matrix* changes
$$\rho \mapsto \mathcal{K}_r^*(\rho)$$
for some operator $\mathcal{K}_r^*$.
4. The *dissipation operator* is
$$C^*(\rho) \;=\; \sum_r \Gamma_r \left(\rho - \mathcal{K}_r^*(\rho)\right)$$
5. Assume:
   (a) If $\rho \in L^1(\mathcal{A}, \mathcal{T})$ then $C^*(\rho) \in L^1(\mathcal{A}, \mathcal{T})$,
   (b) $\exp\{-C^*\}(\rho) \geq 0$ if $\rho \geq 0$,
   (c) $\exp\{-C^*\}$ leaves $\mathcal{T}$ invariant.

IV.2.2- The Master Equation

- The electronic Hamiltonian with electric field is

$$H_{\omega,\vec{\mathcal{E}}} = H_\omega - \frac{q}{\hbar}\vec{\mathcal{E}}.\vec{X}$$

- The evolution of the density matrix is given by

$$\rho(t) = \eta^*_{t-t_n} \circ \mathcal{K}^*_{r_n} \circ \eta^*_{t_n-t_{n-1}} \circ \cdots \circ \mathcal{K}^*_{r_1} \circ \eta^*_{t_1}(\rho)$$

if

1. $t_0 \leq 0 < t_1 < \cdots < t_n \leq t < t_{n+1}$ are the collision times and $r_1, \cdots, r_n$ the corresponding quantum jumps.
2. η^*_t is the action on states of the quantum evolution associated to $H_{\omega,\vec{\mathcal{E}}}$.

Theorem 5 *The collision average evolution of an initial state $\rho \in L^1(\mathcal{A}, \mathcal{T})$ is given by :*

$$\frac{d\langle\rho\rangle}{dt} = \left\{-\mathcal{L}_H + \frac{q}{\hbar}\vec{\mathcal{E}} \cdot \vec{\nabla} - C^*\right\}\langle\rho\rangle$$

IV.2.3- The Conductivity

Theorem 6 *The (time, thermal, quantum) averaged current follows linear response theory. The corresponding AC-conductivity tensor at frequency ω_0 is given by the Kubo formula:*

$$\sigma_{i,j} = \frac{q}{\hbar}\mathcal{T}\left\{\partial_i(\rho_{\beta,\mu})\,(C - \mathcal{L}_H - \imath\,\omega_0)^{-1}\,(J_j)\right\}$$

where $\rho_{\beta,\mu}$ is the (Fermi-Dirac) equilibrium state at inverse temperature β and chemical potential μ and C is the dual action of C^* on $\mathcal{A}$.

□

The *Relaxation Time Approximation* (RTA), consists in setting $C^*(\rho) = \rho_{\beta,\mu}$ *(immediate return to equilibrium after each collision)* namely $C = \mathbf{1}/\tau$ where τ is the *relaxation time*.

In the limit of zero dissipation $\tau \to \infty$, the conductivity tests the spectral properties of $\mathcal{L}_H$ near $\imath\omega_0$ on the imaginary axis.

IV.3)- Why are Hall Plateaux so Flat ?

IV.3.1- Deviations from Ideal Hall System

The theorems concerning the Hall conductance quantization requires the following conditions

- The sample has infinite area in space.
- The electric field is vanishingly small.
- The temperature vanishes.
- The collision operator C vanishes at zero temperature.

Questions : Can one estimate the error when we are away from these conditions ?

1. Can one estimate the error when we are away from these conditions ?
2. If Yes, can one explain the accuracy of the plateaux ?

Results :

- It is possible to show that the accuracy of plateaux is limited only by the *dissipation mechanisms* in practice.
  The size of the sample, the electric field, the temperature can be arranged so that they do not contribute.

- An estimate of the dissipation based upon the RTA gives the following estimate

$$\frac{\delta\sigma_H}{\sigma_H} \leq const \cdot \nu \, \frac{e}{h} \, \frac{\ell^2}{\mu_c}$$

  where ν is the filling factor, ℓ is the *localization length* (typically of the order of 100Å) and μ_c is the *mobility* of the sample.
  Putting realistic numbers in it leads to

$$\frac{\delta\sigma_H}{\sigma_H} \leq 10^{-4}$$

  far from 10^{-8} that are observed !

- The origin of this discrepancy is due to *Mott's variable range hopping*.
  D. Polyakov, B. Shklovskii, *Phys. Rev.*, **B48**, 11167, (1993).

IV.3.1- A Quantum Jump Model

- At very low temperature, conduction electrons are confined in the *impurity band* of the semiconductor. A tight-binding approximation suffices.
- Let $\mathcal{L}_\omega$ be the *impurity lattice*. Wave functions belong to $\mathcal{H}_\omega = \ell^2(\mathcal{L}_\omega)$. Jumps are labelled by pairs $x \mapsto y$ of sites in $\mathcal{L}_\omega$.
- Whenever the jump $x \mapsto y$ occurs, the wave function changes according to

$$\psi \mapsto W_{x \mapsto y}\psi = (\mathbf{1} + (|y> - |x>) < x|)\psi$$

- Following the Mott argument one assumes that the probability of such process is

$$\Gamma_{x \mapsto y} \propto e^{-\left(\frac{|V(x)-V(y)|}{k_B T} + \frac{|x-y|}{\xi}\right)}$$

where $V(x) = V_\omega(x)$ is the (random) potential energy seen by the electron at site x, and ξ is the localization length.

Claim : *(in progress)* this model leads to a correction term

$$\frac{\delta\sigma_H}{\sigma_H} \leq const \cdot e^{-\left(\frac{T_0}{T}\right)^{1/3}} \nu \frac{e}{h} \frac{\ell^2}{\mu_c}$$

according to Mott's law.

Remark :

1. Shklovskii & Efros have argued that the DOS must vanish linearly at the Fermi level, due to Coulomb interaction.
2. It should be possible to design the Hamiltonian to take this remark into account.
3. If the DOS vanishes at the Fermi level, the exponent $1/3$ changes into $1/2$ which is observed in experiments