The THEORY of APERIODIC SOLIDS

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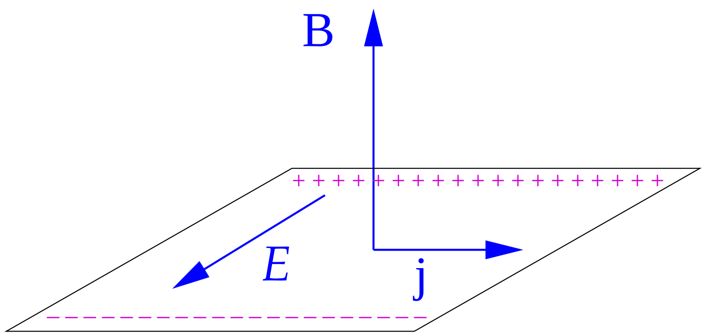
Lecture III

The INTEGER QUANTUM HALL EFFECT

I - INTRODUCTION to the IQHE

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471.



B = magnetic field
j = current density
E = Hall electric field
n = charge carrier density

I.1)- The Classical HALL Effect:

In the stationnary state: $e\,n\,\vec{\mathcal{E}} + \vec{j} \times \vec{B} = 0$

$$\Rightarrow \quad \vec{j} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \vec{\mathcal{E}}\,, \qquad \sigma_H = \frac{ne}{B}\,.$$

$$\text{Units}: \frac{n}{B} = \left[\frac{1}{\text{flux}}\right]\,, \frac{h}{e} = [\text{flux}] \Rightarrow \nu = [1]\ .$$

where : $\nu = \dfrac{nh}{eB} =$ *filling factor* .

HALL's formula

$$\sigma_H = \frac{\nu}{R_H}\,, \qquad R_H = \frac{h}{e^2} = 25\ 812.80\ \Omega\,.$$

I.2)- The (Integer) Quantum HALL Effect:

→ *Conditions of observation:*

1. Low temperatures ($\leq$ few Kelvins)
2. Large sample size ($\geq$ few μm)
3. High mobility together with large enough quenched disorder.
4. $2D$ fermion fluid.

→ *Experiments show that:*

1. Very flat plateaux at ν close to integers, namely if:

$$\sigma_H = \frac{i}{R_H} \quad i = 1, 2, 3, \cdots$$ quantization *(Von Klitzing et al.)*

2. On plateaux $\delta\sigma_H/\sigma_H$ and $\sigma_{//}/\sigma_H \leq 10^{-8}$. This indicates *localization* (Prange, Thouless, Halperin).
3. For $i \geq 2$, Coulomb interaction becomes negligible.

→ *Questions:*

1. Why is σ_H quantized ?
2. What is the rôle of localization ?

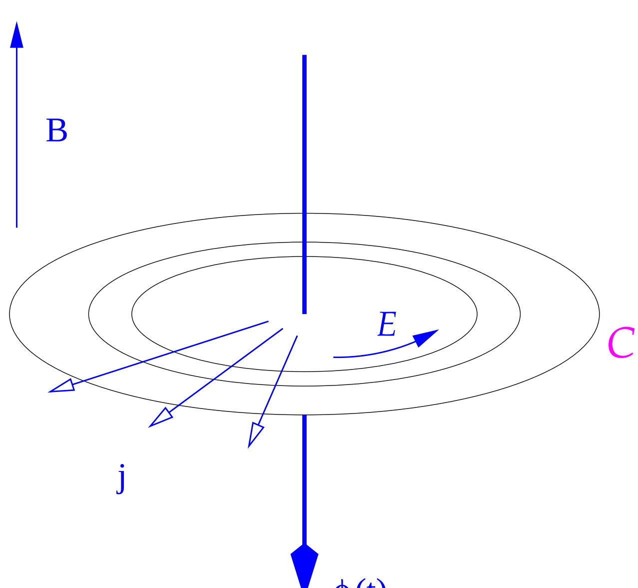
I.3)- Earlier Works:

R.B. Laughlin, *Phys. Rev.* **B23**, 5632 (1981).

- Piercing the plane at x with a flux tube adiabatically varying from 0 to $\phi_0 = h/e$ forces 1 charge per filled Landau level to transfer from x to ∞.
- This adiabatic change induces a unitary tranformation on the Landau Hamiltonian (gauge transformation).
- This gives the quantization of the Hall conductance.

R.E. Prange, *Phys. Rev.* **B23**, 4802 (1981).
D.J. Thouless, *J. Phys.* **C14**, 3475 (1981).
R. Joynt, R.E. Prange, *Phys. Rev.* **B29**, 3303 (1984).

- Localized states do not see the adiabatic change !



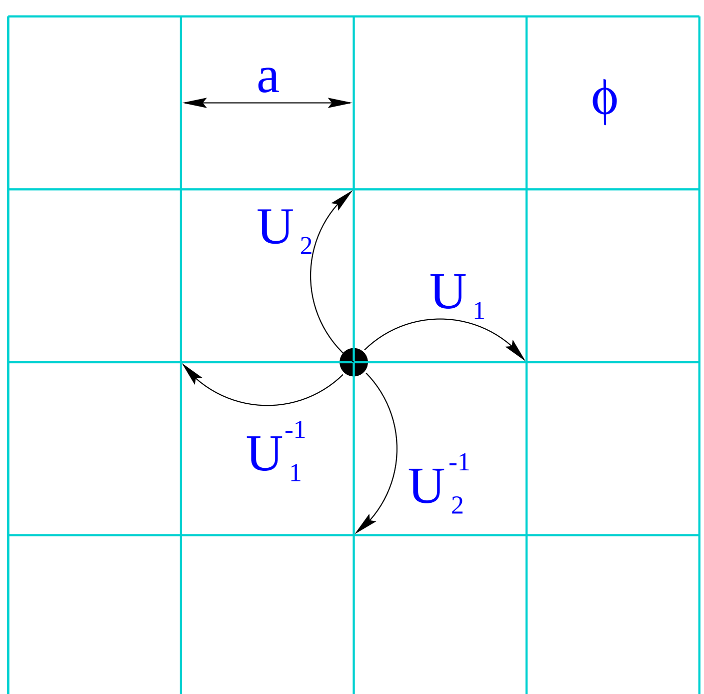
D. Thouless, M. Kohmoto, M. Nightingale, M. den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982).
J.E. Avron, R. Seiler, B. Simon, *Phys. Rev. Lett.* **51**, 51 (1983).

Harper's model: one electron on a square lattice in a uniform magnetic field. Magnetic translations $U_1,\ U_2$, satisfy:

$$U_1 U_2 = e^{2\imath\pi\alpha} U_2 U_1 \ , \qquad \alpha = \frac{\phi}{\phi_0} = \frac{Ba^2}{h/e} \ .$$

Harper's Hamiltonian:

$$H_H = U_1 + U_1^{-1} + U_2 + U_2^{-1} \ .$$



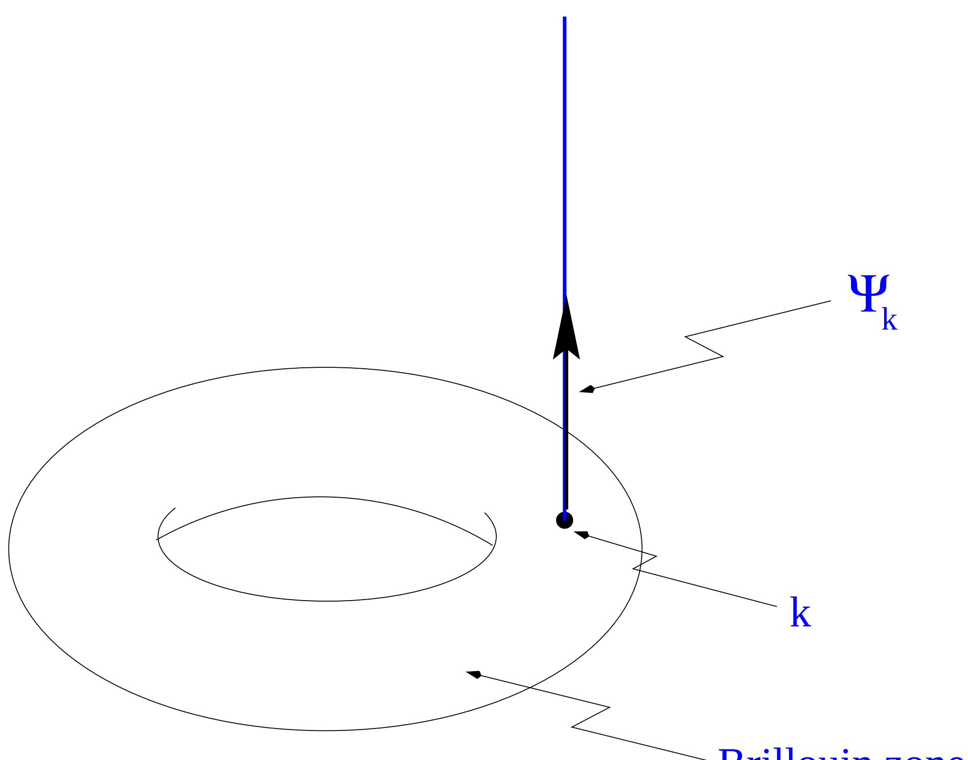
a = lattice spacing

ϕ = flux through unit cell

- If $\alpha = p/q$ then H_H is q-periodic;
- Bloch theory $\Rightarrow$ wave function Ψ depends on quasimomenta $\vec{k} = (k_1, k_2)$.
- $\vec{k} \in \mathbb{B}$ where $\mathbb{B} \approx \mathbb{T}^2$ is the *Brillouin zone.*
- Ψ defines a *line bundle* over $\mathbb{B}$.
- Non triviality controlled by the *Chern class*

$$\mathbf{Ch}(\Psi) = \frac{1}{\pi} \int_0^{2\pi} dk_1 \int_0^{2\pi} dk_2 \Im m < \frac{\partial \Psi}{\partial k_1} | \frac{\partial \Psi}{\partial k_2} >$$

- $\mathbf{Ch}(\Psi) \in \mathbb{Z}$ and is homotopy invariant.



- Assume *Fermi level* E_F lies in a gap.

- Assume N bands $E_1(\vec{k}) < \cdots < E_N(\vec{k}) < E_F < E_{N+1}(\vec{k})$ below Fermi level.

- Set $P_F = \sum_{i \leq N} |\Psi_i >< \Psi_i|$ (*Fermi projection*).

- Set $\mathbf{Ch}(P_F) = \sum_{i \leq N} \mathbf{Ch}(\Psi_i)$.

- The following holds true:

$$\mathbf{Ch}(P_F) = 2\imath\pi \int_{\mathbb{T}^2} \frac{d^2\vec{k}}{4\pi^2} \mathrm{Tr}\left(P_F(\vec{k}) \left[\partial_1 P_F(\vec{k}), \partial_2 P_F(\vec{k})\right] \right)$$

- Then Hall conductance is given by the *Chern-Kubo formula*

$$\sigma_H = \frac{e^2}{h} \mathbf{Ch}(P_F)$$

- $\Rightarrow$ Hall conductivity is *quantized* from *topological* origin.

I.4)- Difficulties with Earlier Works

1. If the magnetic flux is *irrational*
   $\Rightarrow$ no Bloch theory !
2. Disorder destroys also periodicity
   $\Rightarrow$ no Bloch theory !
3. Robustness against small disorder *suggested* from the Kubo-Chern formula,
   (see H. KUNZ, *Commun. Math. Phys.* **112**, 121 (1987).).
   But a general proof is needed.
4. How does one understand *localization* in this context ?

$\longrightarrow$ *Proposal*

1)- J. BELLISSARD, in *Lecture Notes in Phys.*, n°**153**, Springer Verlag, Berlin, Heidelberg, New York, (1982).

2)- J. BELLISSARD, in *Lecture Notes in Physics* **257**, Springer-Verlag, Berlin, Heidelberg, New York, (1986).

Use C^*-algebras and their
Non Commutative Geometry !

II - The NON COMMUTATIVE BRILLOUIN ZONE

J. Bellissard, in *From Number Theory to Physics*, Springer-Verlag, Berlin, (1992).

II.1)- The Hull of Aperiodic Media

II.1.1- A typical Hamiltonian

The Schrödinger Hamiltonian for an electron submitted to atomic forces is given by (ignoring interactions):

$$H = \frac{1}{2m}\left(\vec{P} - q\vec{A}(\,.\,)\right)^2 + \sum_{r=1}^{K} \sum_{y \in L_r} v_r(\,.\, - y)\ .$$

acting on $\mathcal{H} = L^2(\mathbb{R}^d)$.

- d : physical space dimension
- $r = 1, \ldots, K$ labels the atomic species,
- L_r : set of positions of atoms of type r,
- v_r : effective potential for valence electrons near an atom of type r,
- m and q : mass and charge of the carrier,
- $\vec{P} = -\imath\hbar\vec{\nabla}$: momentum operator,
- $\vec{A}$: magnetic vector potential.

II.1.2- Magnetic translations

- In $d = 2$, uniform magnetic field $B = \partial_1 A_2 - \partial_2 A_1$.
- Magnetic translations

$$U(\vec{a}) = e^{\frac{\imath}{\hbar} \oint_0^{\vec{a}} d\vec{s} \left(\vec{P} - q\vec{A}(\vec{s})\right)}$$

- Weyl's commutations relations

$$U(\vec{a})\, U(\vec{b}) \;=\; e^{\imath \frac{q}{\hbar} B \vec{a} \times \vec{b}}\; U(\vec{b})\, U(\vec{a})$$

- Translation invariance of the kinetic part.

$$U(\vec{a})\, \left(\vec{P} - q\vec{A}(\,.\,)\right)^2 U(\vec{a})^{-1} \;=\; \left(\vec{P} - q\vec{A}(\,.\,)\right)^2$$

- Translation of the potential

$$U(\vec{a})\, V(\,.\,)\, U(\vec{a})^{-1} \;=\; V(\,. - \vec{a})$$

II.1.3- The Hull

- The set $\{H_a = U(a)HU(a)^{-1}; a \in \mathbb{R}^2\}$ of translated of H, is endowed with the strong-resolvent topology.
- Let Ω be its closure and $\omega^{(0)}$ be the representative of H.

Definition 1 *The operator H is homogeneous if Ω is compact.*

- $(\Omega, \mathbb{R}^2)$ becomes a dynamical system, *the Hull* of H. It is topologically transitive (one dense orbit). The action is denoted by $\omega \mapsto \tau^a\omega$ $(a \in \mathbb{R}^2)$.
- If the potential V is continuous, there is a continuous function $\hat{v}$ on Ω such that if $\omega \in \Omega$ the corresponding operator H_ω is a Schrödinger operator with potential $V_\omega(x) = \hat{v}(\tau^{-x}\omega)$.
- *Covariance* $U(a)H_\omega U(a)^{-1} = H_{\tau^a\omega}$
- The observable algebra $\mathcal{A}_H$ is the C^*-algebra generated by bounded functions of the H_a's. It is related to the *twisted crossed product* $C^*(\Omega \rtimes \mathbb{R}^2, B)$.

II.2)- The C^*-algebra $C^*(\Omega \rtimes \mathbb{R}^2, B)$

II.2.1- Definition

Endow $\mathcal{A}_0 = \mathcal{C}_c(\Omega \times \mathbb{R}^2)$ with (here $A, A' \in \mathcal{A}_0$):

1. Product
$$A \cdot A'(\omega, \vec{x}) = \int_{\vec{y} \in \mathbb{R}^2} d^2\vec{y} A(\omega, \vec{y}) A'(\tau^{-\vec{y}}\omega, \vec{x} - \vec{y}) e^{\frac{\imath q B}{2\hbar} \vec{x} \wedge \vec{x}}$$

2. Involution
$$A^*(\omega, \vec{x}) = \overline{A(\tau^{-\vec{x}}\omega, -\vec{x})}$$

3. A faithfull family of representations in $\mathcal{H} = L^2(\mathbb{R}^2)$
$$\pi_\omega(A)\psi(\vec{x}) = \int_{\mathbb{R}^2} d^2\vec{y} A(\tau^{-\vec{x}}\omega, \vec{y} - \vec{x}) e^{\frac{\imath q B}{2\hbar} \vec{y} \wedge \vec{x}} \psi(\vec{y}) \; .$$

   if $A \in \mathcal{A}_0$, $\psi \in \mathcal{H}$.

4. C^*-norm
$$\|A\| = \sup_{\omega \in \Omega} \|\pi_\omega(A)\| \; .$$

Definition 2 *The C^*-algebra $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ is the completion of $\mathcal{A}_0$ under this norm.*

II.2.2- Tight-Binding Representation

J. Bellissard, in *Lecture Notes in Physics* **257**, Springer-Verlag, Berlin, Heidelberg, New York, (1986).

1. If $\mathcal{L}$ is the original set of atomic positions, let Σ be the closure of the set $\{\tau^{-\vec{x}}\omega^{(0)} \in \Omega; \vec{x} \in \mathcal{L}\}$.
   Σ is a *transversal*.

2. Replace $\Omega \times \mathbb{R}^2$ by $\Gamma = \{(\omega, \vec{x}) \in \Omega \times \mathbb{R}^2; \omega \in \Sigma\,, \tau^{-\vec{x}}\omega \in \Sigma\ \}$. Γ is a *groupoid*.

3. Replace integral over $\mathbb{R}^2$ by discrete sum over $\vec{x}$.

4. Replace $\mathcal{A}_0$ by $\mathcal{C}_c(\Gamma)$, the space of continuous function with compact support on Γ. Then proceed as before to get $C^*(\Gamma, B)$.

5. $C^*(\Gamma, B)$ is unital.

6. One can restrict the original Hamiltonian H to a spectral bounded interval (in practice near the Fermi level), so as to get an *effective Hamiltonian* H_{eff} in $C^*(\Gamma, B)$. Thus H_{eff} is bounded.

II.2.3- Calculus

- Let $\mathbf{P}$ be an $\mathbb{R}^2$-invariant ergodic probability measure on Ω. Then set (for $A \in \mathcal{A}_0$)):

$$\mathcal{T}(A) = \int_\Omega d\mathbf{P} A(\omega, 0) = \overline{< 0|\pi_\omega(A)0 >}^{dis.}$$

Then $\mathcal{T}$ extends as a *positive trace* on $\mathcal{A}$.

- $\mathcal{T}$ is a *trace per unit volume*, thanks to Birkhoff's theorem:

$$\mathcal{T}(A) = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \mathrm{Tr}(\pi_\omega(A) \restriction_\Lambda) \quad \text{a.e. } \omega$$

- A commuting set of $*$-derivations is given by

$$\partial_i A(\omega, \vec{x}) = \imath x_i A(\omega, \vec{x})$$

defined on $\mathcal{A}_0$. It satisfies $\pi_\omega(\partial_i A) = -\imath[X_i, \pi_\omega(A)]$ where $\vec{X} = (X_1, X_2)$ are the coordinates of the position operator.

II.2.4- Properties of $\mathcal{A}$

Theorem 1 *Let $\mathcal{L}$ be a periodic lattice in $\mathbb{R}^2$. If H is $\mathcal{L}$-invariant, $\mathcal{A}$ is isomorphic to $\mathcal{C}(\mathbb{B}) \otimes \mathcal{K}$, where $\mathbb{B}$ is the Brillouin zone and $\mathcal{K}$ is the C^*-algebra of compact operators.*

$\mathcal{A}$ is the non commutative analog of the space of continuous functions on the Brillouin zone : it will be called *the Non Commutative Brillouin zone*.

Theorem 2 *Let H be a homogeneous Schrödinger operator with hull Ω. Then for any $z \in \mathbb{C} \backslash \sigma(H)$ there is an element $R(z) \in \mathcal{A}$ (which is C^∞), such that*

$$\pi_\omega(R(z)) = (z\mathbf{1} - H_\omega)^{-1}$$

for all $\omega \in \Omega$.

Moreover, the spectrum of $R(z)$ is given by

$$\sigma(R(z)) = \{(z - \zeta)^{-1}; \zeta \in \Sigma\}, \ \Sigma = \cup_{\omega \in \Omega} \sigma(H_\omega)$$

II.2.5- IDoS and Shubin's formula

- Let $\mathbf{P}$ be an invariant ergodic probability on Ω. Let

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega \restriction_\Lambda \leq E\}$$

It is the *Integrated Density of states* or *IDoS*.

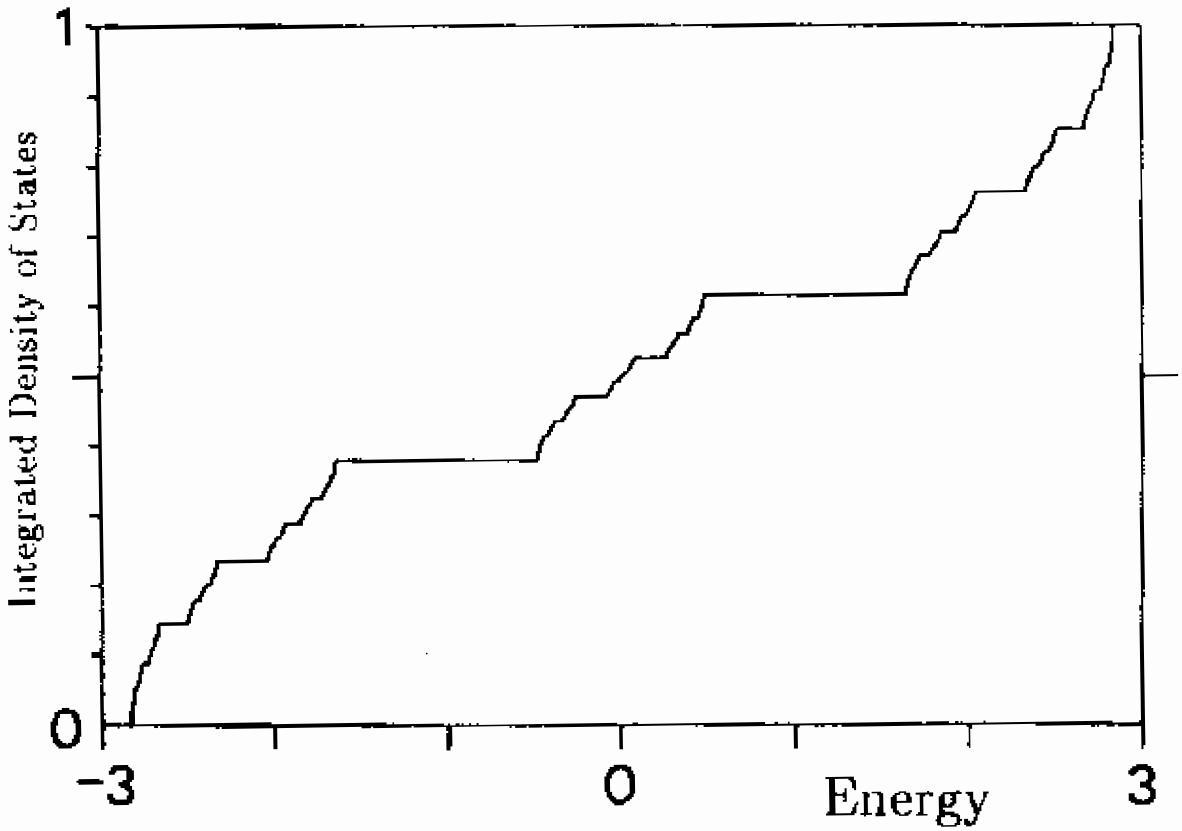
- The limit above exists $\mathbf{P}$-almost surely and

$$\mathcal{N}(E) = \mathcal{T}(\chi(H \leq E)) \qquad \text{(Shubin, 1976)}$$

$\chi(H \leq E)$ is the eigenprojector of H in $\mathcal{L}^\infty(\mathcal{A})$.

- $\mathcal{N}$ is non decreasing, non negative and constant on gaps. $\mathcal{N}(E) = 0$ for $E < \inf \Sigma$. For $E \to \infty$ $\mathcal{N}(E) \sim \mathcal{N}_0(E)$ where $\mathcal{N}_0$ is the IDoS of the free case (namely $V = 0$).

- $d\mathcal{N}/dE = n_{\text{DOS}}$ defines a Stieljes measure called the *Density of States* or *DOS*.



- An example of IDoS -

II.2.6- States

We consider states on $\mathcal{A}$ of the form

$$A \in \mathcal{A} \rightarrow \mathcal{T}\{\rho A\} \, ,$$

with $\rho \geq 0$ and $\mathcal{T}\{\rho\} = n$ if n is the charge carrier density. Then

$$\rho \in L^1(\mathcal{A}, \mathcal{T})$$

The *Fermi-Dirac* state:

describes equilibrium of a fermion gas of independent particles at inverse temperature $\beta = 1/k_{\mathrm{B}}T$ and chemical potential μ:

$$\rho_{\beta,\mu} \; = \; \frac{1}{\mathbf{1} + e^{\beta(H-\mu)}}$$

μ is fixed by the normalization condition

$$\mathcal{T}\{\rho_{\beta,\mu}\} = n \; .$$

II.3)- To Summarize

1. The C^*-algebra $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ is a Non Commutative analog of the space of continuous functions over the Brillouin zone $\mathbb{B}$ if the lattice of atoms is no longer periodic, or if there is a magnetic field.
2. A groupoid Γ associated to the discrete set of atomic positions, gives rise to tight-binding models.
3. Calculus on $\mathcal{A}$ is available and generalizes the usual calculus on $\mathbb{B}$.
4. Textbook formulæ valid for perfect crystals can be easily generalized using this calculus. If P_F is the zero temperature limit of the *Fermi-Dirac* state, constrained by $\mathcal{T}(P_F) = n$, the expression

$$\mathbf{Ch}(P_F) = 2\imath\pi\mathcal{T}\left(P_F\left[\partial_1 P_F, \partial_2 P_F\right]\right)$$

is valid at least if $E_F = \mu \restriction_{T=0}$ belongs to a gap of the energy spectrum.

III - The FOUR TRACE WAY

J. BELLISSARD, H. SCHULZ-BALDES, A. VAN ELST, *J. Math. Phys.*, **35**, (1994), 5373-5471.

III.1)- The Kubo Formula

III.1.1- Background

- The *(non dissipative) current* is

$$\vec{J} = q\,\frac{d\vec{X}}{dt} = \frac{\imath q}{\hbar}[H, \vec{X}] = \frac{q}{\hbar}\vec{\nabla} H$$

- The *thermal average* of $A \in \mathcal{A}$

$$< A >_{\beta,\,\mu} = \mathcal{T}\,(A\,\rho_{\beta,\,\mu})$$

- The *Liouville operator* acts on $\mathcal{A}$

$$\mathcal{L}_H = \frac{\imath}{\hbar}[H,\,.\,]$$

- A *dissipative evolution* requires an operator C acting on $\mathcal{A}$ such that $exp\{-tC\} : \mathcal{A} \mapsto \mathcal{A}$ is a *completely positive contraction semigroup*. C has the dimension of $[\text{time}]^{-1}$. The (dissipative) evolution, with a uniform electric field, is given by the *Master Equation*:

$$\frac{dA}{dt} = \mathcal{L}_H(A) + \frac{q}{\hbar}\vec{\mathcal{E}}.\vec{\nabla} A - C(A)$$

III.1.2- Linear Response Theory

- The thermal averaged current satifies:

$$\vec{j} = < q \frac{d\vec{X}}{dt} >_{\beta,\mu} = \sigma \vec{\mathcal{E}} + O(\vec{\mathcal{E}}^2)$$

- The 2×2 matrix σ is the *conductivity tensor*. It is given by *Kubo's formula*

$$\sigma_{ij} = \frac{q^2}{\hbar} \mathcal{T} \left(\partial_j \rho_{\beta,\mu} \frac{1}{\hbar C - \hbar \mathcal{L}_H} (\partial_i H) \right)$$

- C usually depends on T so that as $T \downarrow 0$, $C \downarrow 0$.

- We have $\lim_{T \downarrow 0} \rho_{\beta,\mu} = P_F$.

Theorem 3 *Let assume*

1. *The Fermi level E_F is not a discontinuity point of the DOS of H.*
2. $\lim_{T \downarrow 0} C = 0$.
3. *P_F is Sobolev differentiable:* $\mathcal{T}\left\{(\vec{\nabla} P_F)^2\right\} < \infty$.

Then, as $T \downarrow 0$, the conductivity tensor converges to

$$\boxed{\sigma_{ij} = \frac{q^2}{h} 2\imath\pi\, \mathcal{T}\left(P_F \left[\partial_i P_F \,,\, \partial_j P_F\right]\right) .}$$

In particular the direct conductivity vanishes and

$$\boxed{\sigma_{12} = \sigma_H = \frac{q^2}{h}\mathbf{Ch}(P_F)}$$

III.2)- The Four Traces

- On every Hilbert space $\mathcal{H}$, *the usual trace* is denoted by Tr.
- In $\mathcal{A}$ we have the *trace per unit volume* $\mathcal{T}$, associated to a translation invariant probability measure $\mathbf{P}$ on the Hull.

III.2.1- Dixmier's Traces

J. Dixmier, *C.R.A.S.*, 1107 (1966).

- On a Hilbert space $\mathcal{H}$, $\mathcal{L}^p(\mathcal{H})$ denotes the *Schatten ideal* of those compact operator on $\mathcal{H}$ such that $\mathrm{Tr}(|T|^p) < \infty$.
- Given T a compact operator on $\mathcal{H}$, let $\mu_0 \geq \cdots \geq \mu_n \geq \ldots \geq 0$ be its singular values (eigenvalues of $|T|$) labelled in decreasing order. Set

$$\|T\|_{p+} = \left(\limsup_{n \in \mathbb{N}} \frac{1}{\ln N} \sum_{n=0}^{N-1} \mu_n^p \right)^{1/p}$$

- The set of $\{T\,;\, \|T\|_{p+} < \infty\}$ is denoted by $\mathcal{L}^{p+}(\mathcal{H})$. This a *Mačaev ideal*.

Theorem 4 *Set* $\mathcal{L}^{p-}(\mathcal{H}) = \{T \text{compact}; \|T\|_{p+} = 0\}$.

1. $\mathcal{L}^{p-}(\mathcal{H})$ *and* $\mathcal{L}^{p+}(\mathcal{H})$ *are two-sided ideals in* $\mathcal{L}(\mathcal{H})$.
2. *For* $p < p' \in [0, \infty)$,
$$\mathcal{L}^{p}(\mathcal{H}) \subset \mathcal{L}^{p-}(\mathcal{H}) \subset \mathcal{L}^{p+}(\mathcal{H}) \subset \mathcal{L}^{p'}(\mathcal{H})$$
3. $\|T\|_{p+}$ *is a seminorm making* $\mathcal{L}^{p+}(\mathcal{H})/\mathcal{L}^{p-}(\mathcal{H})$ *a Banach space.*

- Given a euclidean invariant mean M on $\mathbb{R}$, one can define a linear form Lim_M on $\ell^\infty(\mathbb{N})$ such that
(i) $\mathrm{Lim}_M(a_0, a_1, a_2, \cdots) = \mathrm{Lim}_M(a_1, a_2, a_3, \cdots)$,
(ii) $\mathrm{Lim}_M(a_0, a_1, a_2, \cdots) = \mathrm{Lim}_M(a_0, a_0, a_1, a_1, \cdots)$,
(iii) if $a \in \ell^\infty(\mathbb{N})$ converges, $\mathrm{Lim}_M(a) = \lim_{n \to \infty} a_n$.
- The *Dixmier trace* associated to M is given by
$$\mathrm{Tr}_{\mathrm{Dix}}(T) = \mathrm{Lim}_M \left(\frac{1}{\ln N} \sum_{n=0}^{N-1} \mu_n \right) .$$
if $T \in \mathcal{L}^{1+}(\mathcal{H})$ is positive.
- $\mathrm{Tr}_{\mathrm{Dix}}$ can be extended as a positive continuous linear form on $\mathcal{L}^{1+}(\mathcal{H})$ vanishing on $\mathcal{L}^{1-}(\mathcal{H})$ such that
$$\mathrm{Tr}_{\mathrm{Dix}}(UTU^{-1}) = \mathrm{Tr}_{\mathrm{Dix}}(T), \quad \mathrm{Tr}_{\mathrm{Dix}}(ST) = \mathrm{Tr}_{\mathrm{Dix}}(TS)$$
for $U \in \mathcal{L}(\mathcal{H})$ unitary and $S, T \in \mathcal{L}^{1+}(\mathcal{H})$.

III.2.2- Graded Trace and Fredholm Module

M. Atiyah, *K-Theory*, (Benjamin, New York, 1967).
A. Connes, *Publ. IHES*, **62**, 257 (1986).

- Set $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}^2$ with $\mathcal{H} = L^2(\mathbb{R}^2)$. The *grading operator* G is

$$G = \begin{pmatrix} +\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

- $T \in \mathcal{L}(\hat{\mathcal{H}})$ has *degree* 0 if $GT - TG = 0$
  $T \in \mathcal{L}(\hat{\mathcal{H}})$ has *degree* 1 if $GT + TG = 0$.

- The *graded commutator* is given by

$$[T, T']_S = TT' - (-)^{d^\circ T . d^\circ T'} T'T$$

- A degree 1 operator F is defined by

$$F = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}$$

  where $u = X/|X|$ and $X = X_1 + \imath X_2$ is the position operator. Then $F = F^*$, $F^2 = \mathbf{1}$.

- A *differential d* with $d^2 = 0$ is given by

$$dT = [F, T]_S$$

- The *Leibniz rule* becomes

$$d(TT') = dT\,T' + (-)^{d^\circ T} T\,dT'$$

- A *graded trace* is defined as

$$\boxed{\mathrm{Tr}_S(T) = \frac{1}{2}\mathrm{Tr}(GF dT)}$$

if $dT \in \mathcal{L}^1(\hat{\mathcal{H}})$.

- Tr_S is *linear* and satisfies

$$dT\,,\, dT' \in \mathcal{L}^1(\hat{\mathcal{H}}), \Rightarrow \mathrm{Tr}_S([T, T']_S) = 0$$

- **Note:**
  1. Tr_S *is not positive* in general.
  2. $u = X/|X|$ coincides precisely with the singular gauge transformation corresponding to piercing the plane adiabatically with one flux quantum.

     J.E. Avron, R. Seiler, B. Simon, *Commun. Math. Phys.*, **159**, 399 (1994).

III.3)- Connes Formulæ

III.3.1- First Connes Formula

- Let $\mathcal{A} = C^*(\Omega \rtimes \mathbb{R}^2, B)$ acts on $\hat{\mathcal{H}}$ by $\hat{\pi}_\omega = \pi_\omega \otimes \mathrm{id}$ through degree 0 elements.

- First Connes Formula : for $A \in \mathcal{A}_0$ and **P**-almost all ω's:

$$\mathcal{T}\left(|\vec{\nabla} A|^2\right) = \frac{1}{\pi} \mathrm{Tr}_{\mathrm{Dix}}\left(|d\hat{\pi}_\omega(A)|^2\right)$$

- Let $\mathcal{S}$ be the *Non Commutative Sobolev space* namely the Hilbert space generated by $A \in \mathcal{A}_0$ such that $\mathcal{T}(|A|^2 + |\vec{\nabla} A|^2) < \infty$. Then

$$A \in \mathcal{S} \Rightarrow d\hat{\pi}_\omega(A) \in \mathcal{L}^{2+}(\hat{\mathcal{H}})$$

- Also $d\hat{\pi}_\omega(A)$ is compact for any $A \in \mathcal{A}$.

III.3.2- A Cyclic 2-cocycle

- For A_0 , A_1 , $A_2 \in \mathcal{A}_0$, a *cyclic 2-cocycle* is defined by

$$\mathcal{T}_2(A_0, A_1, A_2) = 2\imath\hat{\pi}\mathcal{T}(A_0\partial_1 A_1 \partial_2 A_2 - A_0\partial_2 A_1\partial_1 A_2)$$

  This trilinear form extends by continuity to $\mathcal{S}$.

- $\mathcal{T}_2$ is *cyclic*

$$\mathcal{T}_2(A_0, A_1, A_2) = \mathcal{T}_2(A_2, A_0, A_1)$$

- $\mathcal{T}_2$ is *Hochschild closed*

$$0 = (b\mathcal{T}_2)(A_0, A_1, A_2, A_3) \equiv \mathcal{T}_2(A_0A_1, A_2, A_3) - \mathcal{T}_2(A_0, A_1A_2, A_3) + \mathcal{T}_2(A_0, A_1, A_2A_3) - \mathcal{T}_2(A_3A_0, A_1, A_2)$$

- Second Connes Formula : for $A_i \in \mathcal{A}_0$:

$$\boxed{\mathcal{T}_2(A_0, A_1, A_2) = \int_\Omega d\mathbf{P}\, \mathrm{Tr}_S\left(\hat{\pi}_\omega(A_0) d\hat{\pi}_\omega(A_1) d\hat{\pi}_\omega(A_2)\right)}$$

III.4)- Quantization of Hall conductivity

Recall that at $T=0$, the Hall conductivity becomes

$$\sigma_H = \frac{e^2}{\hbar}\mathbf{Ch}(P_F)$$

III.4.1- Fredholm Index

- Fact 1 : let P be a projection on $\mathcal{H}$ and $\hat{P} = P \otimes \mathbf{1}_2$. If $d\hat{P} \in \mathcal{L}^3(\mathcal{H})$ then PuP is *Fredholm* on $P\mathcal{H}$ and

$$\mathrm{Tr}_S\left(\hat{P}d\hat{P}d\hat{P}\right) = \mathrm{Ind}\left(PuP\restriction_{P\mathcal{H}}\right) \in \mathbb{Z}$$

- Fact 2 : $d\hat{P} \in \mathcal{L}^3(\mathcal{H}) \iff (uPu^* - P) \in \mathcal{L}^3(\mathcal{H})$ and

$$\mathrm{Ind}\left(PuP\restriction_{P\mathcal{H}}\right) = \mathrm{Tr}\left((uPu^* - P)^{2n+1}\right) \quad \forall n \geq 1$$

- Thus $\mathrm{Ind}(PuP\restriction_{P\mathcal{H}})$ *measures the increase of the dimension of* $P\mathcal{H}$ *after applying* u.

III.4.2- $\mathbf{Ch}(P_F)$ IS AN INTEGER

- ASSUME : $P_F \in \mathcal{S}$.
  Then $d\hat{\pi}_\omega(P_F) \in \mathcal{L}^{2+}(\mathcal{H}) \subset \mathcal{L}^3(\mathcal{H})$ (1st Connes formula).
- By the 2nd Connes formula we get

$$\mathbf{Ch}(P_F) = \int_\Omega d\mathbf{P} \, \mathrm{Tr}_S \left(\hat{\pi}_\omega(P_F) d\hat{\pi}_\omega(P_F) d\hat{\pi}_\omega(P_F)\right)$$

  The r.h.s. is the disordered average of

$$n(\omega) = \mathrm{Ind}(\pi_\omega(P_F) u \pi_\omega(P_F) \restriction_{\pi_\omega(P_F)\mathcal{H}}) \in \mathbb{Z}$$

- By *covariance* one gets

$$n(\tau^{\vec{x}}\omega) = n(\omega) \quad \mathbf{P}\text{-almost all } \omega \text{ and } \vec{x} \in \mathbb{R}^2$$

- Since $\mathbf{P}$ is *invariant ergodic*, $n(\omega)$ is almost surely constant so that

$$P_F \in \mathcal{S} \Rightarrow \mathbf{Ch}(P_F) \in \mathbb{Z}$$

  and $\mathbf{Ch}(P_F)$ measures the number of states created if one applies u namely the *Laughlin singular gauge transformation* ! This is indeed the number of charges sent at ∞.

III.4.3- Existence of Plateaux

- The Fermi level E_F is defined as the limit as $T \downarrow 0$ of the chemical potential μ, constrained to
$$\mathcal{T}(\rho_{\beta,\mu}) = n$$
where n is the charge carrier density.

- Experimentally one can change E_F either by changing the magnetic field B or by changing n.
Both ways are used in practice.

- Remark that $P \in \mathcal{S} \mapsto \mathbf{Ch}(P) \in \mathbb{Z}$ is continuous thanks to Connes formulæ.

- Since $P_F = \chi(H \leq E_F)$, if we assume that the map $E_F \in (E_-, E_+) \mapsto P_F \in \mathcal{S}$ is continuous (for the Sobolev norm), then $\mathbf{Ch}(P_F)$ stay constant for E_F in the interval (E_-, E_+) !
This is the mechanism through which plateaux occur in the Hall conductivity.

- In the next section we will see that the condition $P_F \in \mathcal{S}$ is a consequence of the *existence of localized states around the Fermi level*.

IV - LOCALIZATION and TRANSPORT

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IV.1)- Localization Theory

IV.1.1- Definitions

- The DOS of $H = H^*$ affiliated to $\mathcal{A}$, was defined as the Stieljies-Lebesgue measure
$$d\mathcal{N}(E) = d\mathcal{T}(P_E)$$
with $P_E = \chi(H \leq E)$.
For $\Delta \subset \mathbb{R}$ borelian, we set $P_\Delta = \chi(H \in \Delta)$.
- The non dissipative current is given by $\vec{J} = q/\hbar\vec{\nabla} H$.
- The *current-current correlation* is the measure m defined on $\mathbb{R} \times \mathbb{R}$ by:
$$\int_{\mathbb{R}\times\mathbb{R}} m(dE, dE')f(E)g(E') = \mathcal{T}\left(f(H)\,\vec{\nabla}H\, g(H)\,\vec{\nabla}H\right)$$
for f, g continuous functions with compact support on $\mathbb{R}$. In physicists notations (ignoring $q/\hbar$)
$$m(dE, dE')\text{“}=\text{”}| < E|\vec{J}|E' > |^2$$
- If H_ω is the representative of H associated to $\omega \in \Omega$ we set
$$\vec{X}_\omega(t) = e^{\imath\frac{t}{\hbar}H_\omega}\vec{X}e^{-\imath\frac{t}{\hbar}H_\omega}$$

IV.1.2- Localization Length

- Given $\Delta \subset \mathbb{R}$ borelian, the *average localization length* $\ell(\Delta)$ *of states with energy within* Δ is defined through the following steps

  1. Project the initial state $|\vec{x}>$ on Δ: $\pi_\omega(P_\Delta)|\vec{x}>$.
  2. Measure the distance it goes during time t by applying $(\vec{X}_\omega(t) - \vec{X})\pi_\omega(P_\Delta)|\vec{x}>$
  3. Square it to get the quantum average,
     average over time
     average over disorder to get

$$L_\Delta(t)^2 = \frac{1}{t}\int_0^t \frac{ds}{s}\int_\Omega d\mathbf{P} < \vec{x}|\pi_\omega(P_\Delta)(\vec{X}_\omega(t) - \vec{X})^2\pi_\omega(P_\Delta)|\vec{x}>$$

  4. Then

$$\ell(\Delta) = \limsup_{t\to\infty} L_\Delta(t)$$

- Fact :

$$L_\Delta(t)^2 = \frac{1}{t}\int_0^t \frac{ds}{s}\mathcal{T}\left(|\vec{\nabla}e^{-\imath\frac{t}{\hbar}H}|^2 P_\Delta\right)$$

Thus the localization length is *algebraic* and independent of the representation of the Hamiltonian !

IV.1.3- Localization: Results

Assume $\ell(\Delta) < \infty$:

1. The *spectrum* of H_ω in Δ is *pure point* almost surely w.r.t. ω: *all states in Δ are localized*.
2. There is an $\mathcal{N}$-measurable function ℓ on Δ such that for any $\Delta' \subset \Delta$ borelian,
$$\ell(\Delta') = \int_{\Delta'} d\mathcal{N}(E)\ell(E)^2$$
$\ell(E)$ is the *localization length at energy* E
3. One has
$$\ell(\Delta') = \int_\Omega d\mathbf{P} \int_{\mathbb{R}^2} d^2\vec{x}\ |\vec{x}|^2 \sum_{E \in \sigma_{pp}(H_\omega) \cap \Delta'} \left|< 0|P_{\{E\},\omega}|\vec{x} >\right|^2$$
where $P_{\{E\},\omega}$ is the eigenprojection of H_ω on the energy E.
4. One also gets:
$$\ell(\Delta') = 2 \int_{\Delta' \times \mathbb{R}} \frac{m(dE, dE')}{|E - E'|^2}$$
5. If $[E_0, E_1] \subset \Delta$
$$\|P_{E_1} - P_{E_0}\|_{\mathcal{S}} \leq \int_{E_0}^{E_1} (1 + \ell^2) d\mathcal{N}$$

IV.1.4- Existence of Plateaux

- From the previous results $P_F \in \mathcal{S}$ as long as E_F belongs to a region of localized states. Thus *localization* $\Rightarrow$ *existence of plateaux* for the Hall conductivity.
- From previous results by Fröhlich & Spencer, Aizenman & Molčanov, the localization length is finite at high disorder for the *Anderson model*.
- More recent results by Combes & Hislop, W.M. Wang, the same is true for the *Landau Hamiltonian with a random potential*, at least $O(B^{-\infty})$-away from the Landau levels.

IV.2)- Why are Hall Plateaux so Flat ?

The theorems concerning the Hall conductance quantization requires the following conditions

- The sample has infinite area in space.
- The electric field is vanishingly small.
- The temperature vanishes.
- The collision operator C vanishes at zero temperature.

QUESTIONS : Can one estimate the error when we are away from these conditions ?

1. Can one estimate the error when we are away from these conditions ?
2. If Yes, can one explain the accuracy of the plateaux ?

RESULTS :

- It is possible to show that the accuracy of plateaux is limited only by the *dissipation mechanisms* in practice.
  The size of the sample, the electric field, the temperature can be arranged so that they do not contribute.

- An estimate of the dissipation based upon the RTA gives the following estimate

$$\frac{\delta\sigma_H}{\sigma_H} \leq const \cdot \nu \frac{e}{h} \frac{\ell^2}{\mu_c}$$

  where ν is the filling factor, ℓ is the *localization length* (typically of the order of 100Å) and μ_c is the *mobility* of the sample.
  Putting realistic numbers in it leads to

$$\frac{\delta\sigma_H}{\sigma_H} \leq 10^{-4}$$

  far from 10^{-8} that are observed !

- The origin of this discrepancy is due to *Mott's variable range hopping*.

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