RECONNECTION COHOMOLOGY OF COMPACT METRIZABLE SPACES

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Abstract. Given a compact metrizable space X a Cantorizarion is a continuous surjective map from the Cantor set onto X. If such a Cantorization is non degenerate, it gives rise to the concept of reconnection cohomology. Such a cohomology is defined and is shown to be isomorphic to the Čech cohomology of X.

Contents

Temporary Draft

This is a summary of a subject that is part of a larger work

Work supported in part by NSF grants DMS 06009565, DMS 0901514 and by the CRC701, *Spectral Structures and Topological Methods in Mathematics*, Mathematik, Universität Bielefeld, Germany.

1. Cantorization and Reconnection Cohomology

The purpose of these notes is to show that the topology of a compact space, in particular its Čech cohomology, can be reconstructed from breaking the space apart to make it a Cantor set (Cantorization) and reconnecting the pieces together. This will be done by using a new cochain complex called the *reconnection cohomology*. The main motivation comes from the description of the Hull of a uniformly discrete set that have not necessarily finite local complexity. Such sets actually are relevant in describing various materials like liquids and glasses. As it turns out, an original idea by T. Egami permits to represent such sets as graph by defining properly and mathematically the notion of chemical bond between points. This graph ia a skeleton of the configuration of the solid and it has finite local complexity just by construction. In turns, the Egami graph leads to a Cantorization of the tiling space from which the Čech cohomology can be computed, in principle.

1.1. **Cantorization.**

Definition 1. *A Cantor set is a metrizable compact, totally disconnected space without isolated point.*

By totally disconnected, it is meant that there the topology is generated by a family of closed open sets. Such sets will be called *clopen* in what follows. A typical example of Cantor set is provided by the set $\Xi = \{0,1\}^{\mathbb{N}}$ of sequences $(\epsilon_n)_{n\in\mathbb{N}}$ of zero's and one's, namely $\epsilon_n \in \{0,1\}$. The topology is the product topology, so that, given a finite word $w = (w_1, \cdots, w_n) \in \{0,1\}^n$ the set $\Xi(w) = \{\epsilon \in \Xi\,;\, \epsilon_j = w_j\,,\, 1 \leq j \leq n\}$ is both closed and open. In addition the $\Xi(w)$'s make up a basis for the topology of Ξ. As it turns out

Theorem 1 (Brouwer [3])**.** *Any Cantor set is homeomorphic to* $\{0,1\}^{\mathbb{N}}$.

Definition 2. *Let X be a metrizable compact space. A Cantorization of X is a continuous surjective map $\phi : C \to X$ where C is a Cantor set. It is non degenerate whenever for any pair of disjoint compact open sets P, Q in C, the set $\phi(P) \setminus \phi(Q)$ is nonempty.*

The simplest example of such Cantorization is provided by the representation of numbers in the interval $X = [0,1]$ by dyadic decomposition. Namely for $0 \leq x \leq 1$ there is a sequence $\epsilon = (\epsilon_n)_{n\in\mathbb{N}} \in \Xi = \{0,1\}^{\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n} = \phi(\epsilon)\,.$$

If $x = k/2^l$ for some $l \in \mathbb{N}$ and $0 \leq k \leq 2^l$, then it admits two such representations. Otherwise this representation is unique. Moreover ϕ is continuous as it is easy to show. Hence $\phi : \Xi \to [0,1]$ is a Cantorization. In much the same way, the map $\phi : \Xi \to \mathbb{S}^1$ given by $\psi(\epsilon) = e^{2\imath\pi\phi(\epsilon)}$ is a Cantorisation of the unit circle. Both Cantorizations are non degenerate.

A general method to built a Cantorization proceeds from the notion of Borel partition out of a finite open cover. Let X be a compact space and let $\mathcal{O}$ be a finite open cover. The σ-algebra of $\mathcal{O}$, denoted by $\sigma(\mathcal{O})$ is the family of all subsets of X obtained from the elements of $\mathcal{O}$ by intersection, union and complementation. In particular, since $\mathcal{O}$ is finite, all element of $\sigma(\mathcal{O})$ are Borel sets in X and $\sigma(\mathcal{O})$ is also finite. Moreover $\sigma(\mathcal{O})$ is ordered by inclusion and admits X as a maximal element and $\emptyset$ as the minimal one if there are two non intersecting open sets in $\mathcal{O}$. In addition any decreasing sequence of sets in $\sigma(\mathcal{O})$ admits its intersection as a minimum element. Therefore, thanks to the Zorn Lemma, the set of non empty elements of $\sigma(\mathcal{O})$ admits minimal elements, which will be called *atoms*. Such atoms make up a partition, denoted by $\mathfrak{P}(\mathcal{O})$ of X by Borel sets.

Lemma 1. *Let $\mathcal{O}$ be a finite open cover of X. If $\mathcal{A} \subset \mathcal{O}$, let $B_{\mathcal{A}} = \bigcap_{U\in\mathcal{A}} U \cap \bigcap_{V\in\mathcal{A}^c} V^c$ where V^c denotes the complement of V is X, and $\mathcal{A}^c$ the complement of $\mathcal{A}$ in $\mathcal{O}$. Let then $R(\mathcal{O})$ be the set of $\mathcal{A} \subset \mathcal{O}$ for which $B_{\mathcal{A}} \neq \emptyset$. Then the map $\mathcal{A} \in R(\mathcal{O}) \mapsto B_{\mathcal{A}} \in \sigma(\mathcal{O})$ is a bijection onto the atoms of $\sigma(\mathcal{O})$.*

Proof: Clearly $B_{\mathcal{A}}$ belongs to $\sigma(\mathcal{O})$. In addition if $\mathcal{A}, \mathcal{B}$ are two distinct subsets of $\mathcal{O}$, there is $W \in \mathcal{B}$ that is not in $\mathcal{A}$, therefore $B_{\mathcal{B}} \subset W$ while $B_{\mathcal{A}} \subset W^c$, so that $B_{\mathcal{A}} \cap B_{\mathcal{B}} = \emptyset$. If now $x \in X$, let $\mathcal{A}$ be the set of open sets in $\mathcal{O}$ containing x. If $V \notin \mathcal{A}$ it follows that $x \notin V$. Hence $x \in B_{\mathcal{A}}$. Therefore the union $\bigcup_{\mathcal{A}\subset\mathcal{O}} B_{\mathcal{A}} = X$. It follows that this family is a partition. For each open set $U \in \mathcal{O}$ either $U \in \mathcal{A}$, in which case $B_{\mathcal{A}} \subset U$ or $U \notin \mathcal{A}$ the $B_{\mathcal{A}} \subset U^c$. If $R(U)$ denotes the set of $\mathcal{A} \subset \mathcal{O}$ containing U, it follows that $U = \bigcup_{\mathcal{A}\in R(U)} B_{\mathcal{A}}$. From then it follows that each nonempty $B_{\mathcal{A}}$ is minimal. □

The main problem in this construction is that the Borel partition $\mathfrak{P}(\mathcal{O})$ might contain atoms with empty interior. This is the motivation for the following definition

Definition 3. *A finite open cover of the metrizable compact space X is full whenever any elements $U \in \mathcal{O}$ is the interior of its closure.*

Proposition 1. *(i) Given any finite open cover $\mathcal{O}$ of the metrizable compact space X, the family $\mathcal{O}'$ obtained by replacing each $U \in \mathcal{O}$ by the interior of its closure is a full finite open cover of X.*
(ii) If $\mathcal{O}$ is a full finite open cover of X, then each atom of its Borel partition $\mathfrak{P}(\mathcal{O})$ has a nonempty interior.

Proof: (i) Let $\mathcal{O}$ be a finite open cover. For each $U \in \mathcal{O}$ let U' be the interior of $\overline{U}$. Then $U \subset U'$ and the set $\mathcal{O}' = \{U' ; U \in \mathcal{O}\}$ is a full finite cover.

(ii) Let now $\mathcal{O}$ be full and let B be an atom of the partition $\mathfrak{P}(\mathcal{O})$. By definition B is not empty. Thanks to the Lemma 1, it follows that there is $\mathcal{A} \subset \mathcal{O}$ such that $B = B_{\mathcal{A}}$. Let U be the intersection of the open sets in $\mathcal{A}$. Let $V_1, \cdots, V_n$ be the elements of $\mathcal{A}^c$. Then $U \setminus V_1$ has a non empty interior. For if not, then $U \setminus V_1$ would be contained in the frontier of V so that $U \subset \overline{V_1}$. Since V_1 coincides with the interior of $\overline{V_1}$ it follows that $U \subset V_1$. This implies $U \setminus V_1 = \emptyset$ so that $B = \emptyset$, a contradiction. Let then U_1 be the interior of $U \setminus V_1$. The same argument works for U_1 versus V_2. This defines, inductively a decreasing family of nonempty open sets $U_1, \cdots, U_n$, so that U_n is finally the interior of B. □

To build a Cantorization, it is necessary to get refinements of the Borel partition. This can be done through the notion of *refining sequence*.

Definition 4. *A resolving sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$ is a sequence of full finite open covers for which, given any open set $O \subset X$ and any $x \in O$, there is an $N \in \mathbb{N}$ such that for each $n \geq N$ there is $x \in U_n \in \mathcal{O}_n$ with $\overline{U_n} \subset O$.*

In particular a resolving sequence separates the points of X. For if $x \neq y$, there are open neighborhood O_x , O_y of x and y respectively, such that $\overline{O_x} \cap \overline{O_y} = \emptyset$. Therefore for n large enough there are $U_n, V_n \in \mathcal{O}_n$ such that $x \in U_n \subset O_x$ and $y \in V_n \subset O_y$, so that $\overline{U_n} \cap \overline{V_n} = \emptyset$.

Given an resolving sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$, let $\mathfrak{P}_n$ be the partition generated by the open cover $\bigcup_{j=1}^{n} \mathcal{O}_j$. By convention $\mathfrak{P}_0 = \{X\}$. It follows immediately that each element in $\mathfrak{P}_n$ is partitioned by elements in $\mathfrak{P}_{n+1}$. This allows to defined a rooted tree graph $\mathcal{T}$ as follows: (i) elements of $\mathfrak{P}_n$ are the vertices at generation n, (ii) so that $\{X\}$ is the only vertex at generation 0 and is called the *root*, (iii) an edge of generation $n+1$ is a pair (B, B') where $B \in \mathfrak{P}_n$, $B' \in \mathfrak{P}_{n+1}$ and $B' \subset B$ (by construction $B' \neq \emptyset$). In such a case if B is called the father of B' while B' is called a child of B. Then a *descendent* of B is an atom $B" \in \mathfrak{P}_m$, for $m \geq n+1$ such that $B" \subset B$.

Definition 5 (see [6, 8])**.** *The tree $\mathcal{T}$ will be called the Michon tree of the resolving sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$.*

The boundary $\partial\mathcal{T}$ of the previous tree is the set of infinite paths starting at the root. A path can be seen as an infinite sequence $\gamma = (B_n)_{n\in\mathbb{N}_0}$ such that $B_n \in \mathfrak{P}_n$ and B_{n+1} is a child of B_n, namely $B_{n+1} \subset B_n$. Such a path is said to *pass through* B_n. The set $\partial\mathcal{T}$ can be endowed with a topology that makes it completely discontinuous [8].

Theorem 2 (Existence of non degenerate Cantorization)**.** *Let X be a metrizable compact space. Let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a resolving sequence and let $\mathcal{T}$ be the corresponding Michon tree. Then*
(i) $\mathcal{T}$ is Cantorian, namely each vertex has only a finite number of children and has at least one descendent who has more than one child; in particular the boundary of this tree is a Cantor set [8]*;*
(ii) there is a continuous surjective map $\phi : \partial\mathcal{T} \to X$,
(iii) the previous Cantorization is non degenerate

Proof: (i) Since each open cover is finite, each Borel partition $\mathfrak{P}_n$ is finite as well. Hence each vertex has only a finite number of children. Let then $B_n \in \mathfrak{P}_n$ be a vertex of the tree. By construction B_n is a Borel set with nonempty interior (see Proposition 1). Let then $x \neq y$ be two interior points of B_n. Then, since the sequence is resolving, it follows that there is $N \geq n$ such that for each $m \geq N$ there are open sets U_m and V_m in $\mathcal{O}_m$ such that $x \in U_m$, $y \in V_m$ and $\overline{U}_m \cap \overline{V}_m = \emptyset$. In particular that x and y belong to different atoms of the Borel partition $\mathfrak{P}_m$. Thanks to [8], $\partial\mathcal{T}$ is a Cantor set.

(ii) Let $\gamma \in \partial\mathcal{T}$ be a path represented by the decreasing family $(B_n)_{n\in\mathbb{N}}$ of Borel sets with $B_n \in \mathfrak{P}_n$. Thus $K_n = \overline{B_n}$ is a nonempty compact subset of X and $K_{n+1} \subset K_n$. By the finite intersection property, $K(\gamma) = \bigcap_{n\in\mathbb{N}} K_n$ is not empty either. Since the sequence is resolving, this set is actually reduced to one point (use the same argument as in (i) above). Let this point be denoted by $\phi(\gamma)$. This map is onto. For if $x \in X$, then for each n there is a unique $B_n(x) \in \mathfrak{P}_n$ such that $x \in B_n(x)$. The sequence $\gamma = (B_n(x))_{n\in\mathbb{N}}$ defines an element of $\partial\mathcal{T}$ such that $\phi(\gamma) = x$. This map is also continuous. For indeed, given $x \in X$ and O an open neighborhood of x, there is $n \in \mathbb{N}$ and $U_n \in \mathcal{O}_n$ such that $x \in U_n \subset O$. Hence, there is $B_n \in \mathfrak{P}_n$ such that $x \in B_n \subset O$. Let $[B_n]$ be the set of all path $\gamma \in \partial\mathcal{T}$ passing through B_n. As shown in [8], this is a compact open subset of $\partial\mathcal{T}$ (actually such sets make up a basis for the topology of $\mathcal{T}$). In addition, by construction $\phi(\gamma) \in O$ for $\gamma \in [B_n]$, proving the claim.

(iii) It remains to prove that this Cantorization is non degenerate. Thanks to [1], Proposition 6, any compact open set $P \subset \partial\mathcal{T}$ is a finite union of disjoint vertices of $\mathcal{T}$. Namely there is an $n \in \mathbb{N}$ and a finite family $\{B_P^{(1)}, \cdots, B_P^{(l)}\}$ of disjoint atoms in $\mathfrak{P}_n$ such that this compact open set is the union of the $[B_P^{(j)}]$'s. If Q is another compact open set, disjoint of P, let $\{B_Q^{(1)}, \cdots, B_Q^{(l)}\}$ be the corresponding vertices. There is no loss of generality in assuming that the generation n is the same for both P and Q. By construction $\phi(P)$ and $\phi(Q)$ are both closed in X. Clearly $\phi(P)$ contains the union of the $B_P^{(j)}$'s while $\phi(Q)$ contains the union of the $B_Q^{(j)}$'s. Since this two sets are disjoint, by construction, the conclusion follows. $\square$

1.2. **Reconnection Cohomology.** In this subsection, let X be a metrizable compact space, let C be a Cantor set and let $\phi : C \to X$ be a Cantorization. First, as a reminder

Definition 6 (see [4])**.** *A set S is called directed if there is a relation $\alpha < \beta$ between elements of S satisfying*
(i) for all element $\alpha \in S$, then $\alpha < \alpha$
(ii) if, in S, there are three elements α, β, γ such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$,
(iii) for all α, β in S there is γ such that $\gamma < \alpha$ and $\gamma < \beta$.
A subset $T \subset S$ is cofinal if for any $\alpha \in S$ there is $\beta \in T$ such that $\beta < \alpha$.

Next the notion of c-partition is defined by

Definition 7. *A c-partition $\mathcal{P}$ is a finite collection of subsets of C such that*
(i) each $P \in \mathcal{P}$ is closed and open in C
(ii) any two distinct elements of $\mathcal{P}$ are disjoint
(iii) the union of elements of $\mathcal{P}$ is C.

A c-partition $\mathcal{Q}$ is called a *refinement* of $\mathcal{P}$ whenever for each $Q \in \mathcal{Q}$ there is an element $P \in \mathcal{P}$, denoted by $P = \pi(Q)$ such that $Q \subset P$. Clearly such P is unique since two elements of $\mathcal{P}$ are disjoint. In particular, this defines a map $\pi : \mathcal{Q} \to \mathcal{P}$, called the *restriction map*. Moreover if $\mathcal{Q}(P)$ denotes the set of elements of $\mathcal{Q}$ contained in P, then it is a c-partition of P itself. Let $\mathcal{Q} \preceq \mathcal{P}$ denote the relation $\mathcal{Q}$ *is a refinement of* $\mathcal{P}$. This is an order relation on the set of c-partitions. In addition, given any pair $\mathcal{P}, \mathcal{Q}$ of c-partitions, there is a unique c-partition, denoted by $\mathcal{P} \wedge \mathcal{Q}$, such that $\mathcal{R} \preceq \mathcal{P}, \mathcal{Q}$ if and only if $\mathcal{R} \preceq \mathcal{P} \wedge \mathcal{Q}$. Actually $\mathcal{P} \wedge \mathcal{Q}$ is the c-partition the elements of which are sets of the form $P \cap Q$ for some $P \in \mathcal{P}$ and some $Q \in \mathcal{Q}$. It follows that the set $\mathfrak{P}(C) = \mathfrak{P}$ of c-partitions of C is a directed set.

By an elementary k-chain in $\mathcal{P}$, it is meant a finite family $(P_0, P_1, \cdots, P_k) \in \mathcal{P}^{k+1}$ such that $\phi(P_0) \cap \phi(P_1) \cap \cdots \cap \phi(P_k) \neq \emptyset$. Let then $C_k(\mathcal{P})$ be the free abelian group generated by elementary k-chains submitted to the following identification: if σ is a permutation of $\{0, 1, \cdots, k\}$ then

$$(P_{\sigma(0)}, P_{\sigma(1)}, \cdots, P_{\sigma(k)}) = (-)^{\sigma}\, (P_0, P_1, \cdots, P_k)\,,$$

where $(-)^{\sigma}$ denotes the *signature* of the permutation σ. Then a *boundary* operation $\partial_k = \partial : C_k(\mathcal{P}) \to C_{k-1}(\mathcal{P})$ is the group homomorphism defined on the generators by

$$\partial(P_0, P_1, \cdots, P_k) = \sum_{j=0}^{k} (-1)^j\, (P_0, P_1, \cdots, \overset{j}{\vee}, \cdots, P_k)\,,$$

where the sign $\overset{j}{\vee}$ means that the set P_j is removed. This leads to a chain complex $C_*(\mathcal{P})$ defined by

$$C_*(\mathcal{P}) = \quad \cdots \overset{\partial_{k+1}}{\to} C_k(\mathcal{P}) \overset{\partial_k}{\to} C_{k-1}(\mathcal{P}) \overset{\partial_{k-1}}{\to} \cdots \overset{\partial_1}{\to}, C_0(\mathcal{P}) \to 0 \qquad\qquad \partial_k \circ \partial_{k+1} = 0\,.$$

Given $\mathcal{Q} \preceq \mathcal{P}$, the restriction map π induces a map $\pi_k : C_k(\mathcal{Q}) \to C_k(\mathcal{P})$ through the following relation

$$\pi_k(Q_0, Q_1, \cdots, Q_k) = (P_0, P_1, \cdots, P_k)\,, \qquad\qquad P_j = \pi(Q_j)\,.$$

For indeed, if $(Q_0, Q_1, \cdots, Q_k)$ is an elementary k-chain for $\mathcal{Q}$, then $\emptyset \neq \phi(Q_0) \cap \phi(Q_1) \cap \cdots \cap \phi(Q_k) \subset \phi(P_0) \cap \phi(P_1) \cap \cdots \cap \phi(P_k)$, showing that the *r.h.s* is indeed an elementary k-chain for $\mathcal{P}$. In addition, it is immediate to show that

$$\pi_{k-1} \circ \partial_k = \partial_k \circ \pi_k \,, \qquad\qquad k \geq 1 \,.$$

This defines a map $\pi_* : C_*(\mathcal{Q}) \to C_*(\mathcal{P})$ of chain complex. In particular, if $H_k(\mathcal{P})$ denotes the homology group for $\mathcal{P}$, namely

$$H_k(\mathcal{P}) = \mathrm{Ker}\{\partial_k\}/\mathrm{Im}\{\partial_{k+1}\} \,,$$

then π_k induces a group homomorphism $\hat{\pi}_k : H_k(\mathcal{Q}) \to H_k(\mathcal{P})$.

In a similar way, let $C^k(\mathcal{P}) = \mathrm{Hom}\{C_k(\mathcal{P}), \mathbb{Z}\}$ be the dual group. Elements of $C^k(\mathcal{P})$ are called cochains. Then it can be organized into a cochain complex through using the *differential* d defined by $d_k f = f \circ \partial_{k+1}$, whenever $f \in C^k(\mathcal{P})$. Then

$$C^*(\mathcal{P}) = \quad 0 \to C^0(\mathcal{P}) \stackrel{d_0}{\to} \cdots \stackrel{d_{k-1}}{\to} C^k(\mathcal{P}) \stackrel{d_k}{\to} C^{k+1}(\mathcal{P}) \stackrel{d_{k+1}}{\to} \cdots \qquad\qquad d_{k+1} \circ d_k = 0 \,.$$

By duality, the restriction map is defined by $\pi^* : C^*(\mathcal{P}) \to C^*(\mathcal{Q})$ and again

$$\pi^{k+1} \circ d_k = d_k \circ \pi^k \,, \qquad\qquad k \geq 0 \,.$$

In much the same way, the restriction map defines a map in cohomology

$$H^k(\mathcal{P}) = \mathrm{Ker}\{d_k\}/\mathrm{Im}\{d_{k-1}\} \,, \qquad\qquad \hat{\pi}^k : H^k(\mathcal{P}) \to H^k(\mathcal{Q}) \,.$$

Since the set $\mathfrak{P}(C)$ is directed, it follows that the direct limit of the groups H^* exists. This leads to the following definition

Definition 8. *Let X be a metrizable compact space and let $\phi : C \to X$ be a Cantorization. The reconnection cohomology associated with it is defined by the abelian groups*

$$H^k(C, X, \phi) = \lim_{\to \mathfrak{P}} (H^k(\mathcal{P}), \hat{\pi}^k)$$

1.3. **Reconnection and Čech Cohomologies.** The main result of this section is provided by the following

Theorem 3. *Let X be a metrizable compact space. If $\phi : C \to X$ is a non degenerate Cantorization then the reconnection and the Čech cohomology groups are isomorphic.*

The proof of this theorem requires several technical steps that are described in this subsection. The first step is a reminder about the Čech cohomology. A more complete description can be found in [4, 5, 2].

An open cover $\mathcal{U}$ is a family of open subsets of X the union of which equals X. A subcover is a family $\mathcal{V} \subset \mathcal{U}$ of opens sets in $\mathcal{U}$ covering X. Since X is compact, all open cover contains a finite subcover. A *refinement* of $\mathcal{U}$ is a cover $\mathcal{V}$ such that each open set $V \in \mathcal{V}$ is contained in at least one open set $U \in \mathcal{U}$. The relation $\mathcal{V}$ *is a refinement of* $\mathcal{U}$ will be denoted by $\mathcal{V} \preceq \mathcal{U}$. With this relation, the set of finite covers is a directed set. For indeed, clearly $\mathcal{U} \preceq \mathcal{U}$, if $\mathcal{W} \preceq \mathcal{V}$ and $\mathcal{V} \preceq \mathcal{U}$ implies $\mathcal{W} \preceq \mathcal{U}$. Moreover, given two open covers $\mathcal{U}$ and $\mathcal{V}$, the open cover $\mathcal{U} \wedge \mathcal{V}$ is defined by $\{U \cap V \,;\, U \in \mathcal{U} \,,\, V \in \mathcal{V}\}$. In particular, since both $\mathcal{U}$ and $\mathcal{V}$ are finite, so is $\mathcal{U} \wedge \mathcal{V}$. It satisfies $\mathcal{U} \wedge \mathcal{V} \preceq \mathcal{U}$, $\mathcal{U} \wedge \mathcal{V} \preceq \mathcal{V}$.

Let $\mathfrak{U}$ be the set of finite subcovers of X. Given such a cover $\mathcal{U} \in \mathfrak{U}$, an elementary k-chain in $\mathcal{U}$ is a family $(U_0, U_1, \cdots, U_k)$ of open sets in $\mathcal{U}$ such that $U_0 \cap U_1 \cap \cdots \cap U_k \neq \emptyset$. Then let $C_k(\mathcal{U})$ be the free group generated by elementary k-chains submitted to the condition $(U_{\sigma(0)}, U_{\sigma(1)}, \cdots, U_{\sigma(k)}) = (-)^{\sigma}(U_0, U_1, \cdots, U_k)$. Elements of $C_k(\mathcal{U})$ are called k-chains. Like in the case of c-partitions, a *boundary* operator $\partial = \partial_k : C_k(\mathcal{U}) \to C_{k-1}(\mathcal{U})$ is defined by

$$\partial_k(U_0, U_1, \cdots, U_k) = \sum_{j=0}^{k} (-1)^k \, (U_0, U_1, \cdots, \overset{j}{\vee}, \cdots, U_k)\,.$$

This leads again to a chain complex

$$C_*(\mathcal{U}) = \quad \cdots \overset{\partial_{k+1}}{\to} C_k(\mathcal{U}) \overset{\partial_k}{\to} C_{k-1}(\mathcal{U}) \overset{\partial_{k-1}}{\to} \cdots \overset{\partial_1}{\to}, C_0(\mathcal{U}) \to 0 \qquad\qquad \partial_k \circ \partial_{k+1} = 0\,.$$

The very same argument as in Section 1.2 leads to (i) if $\mathcal{V} \preceq \mathcal{U}$ and if $\pi : \mathcal{V} \to \mathcal{U}$ is a restriction map, then it induces a group homomorphism $\pi_k : C_k(\mathcal{V}) \to C_k(\mathcal{U})$ (ii) $\pi_{k-1} \circ \partial_k = \partial_k \circ \pi_k$, (iii) this leads to a morphism of chain complex $\pi_* : C_*(\mathcal{V}) \to C_*(\mathcal{U})$.

Similarly, by duality, the group of k-cochains $C^k(\mathcal{U}) = \mathrm{Hom}\{C_k(\mathcal{U}), \mathbb{Z}\}$ and the differential $d_k : C_k(\mathcal{U}) \to C_{k+1}(\mathcal{U})$ defined by $d_k f = f \circ \partial_k$ leads to the cochain complex

$$C^*(\mathcal{U}) = \quad 0 \to C^0(\mathcal{U}) \overset{d_0}{\to} \cdots \overset{d_{k-1}}{\to} C^k(\mathcal{U}) \overset{d_k}{\to} C^{k+1}(\mathcal{U}) \overset{d_{k+1}}{\to} \cdots \qquad\qquad d_{k+1} \circ d_k = 0\,.$$

Similarly, given $\mathcal{V} \preceq \mathcal{U}$, any restriction map π leads to group homomorphisms $\pi^k : C^k(\mathcal{U}) \to C^k(\mathcal{V})$, then to a morphism $\pi^* : C^*(\mathcal{U}) \to C^*(\mathcal{V}))$ of cochain complex. In particular

$$d_k \circ \pi^k = \pi^{k+1} \circ d_k\,. \tag{1}$$

In turn, this implies that the cohomology groups $H^k(\mathcal{U}) = \mathrm{Ker}\{d_k\}/\mathrm{Im}\{d_{k-1}\}$ and the group homomorphisms $\hat{\pi}^k : H^k(\mathcal{U}) \to H^k(\mathcal{V})$ are well defined. The following result is essential

Proposition 2 (see [4])**.** *The homomorphism $\hat{\pi}^k : H^k(\mathcal{U}) \to H^k(\mathcal{V})$ is independent of the choice of the restriction map.*

Since the set $\mathfrak{U}$ of (finite) open covers is directed by the relation of refinement, the following definition makes sense

Definition 9. *The Čech cohomology group $\check{H}^k(X)$ is defined as the direct limit*

$$\check{H}^k(X) = \lim_{\to \mathfrak{U}} (H^k(\mathcal{U}), \pi^k)$$

Since X is metrizable, let d be a metric on X defining its topology. Then, if $A \subset X$ is a subset and if $\delta > 0$, then A^{δ} will denote the following neighborhood of A

$$A^{\delta} = \{x \in X\,;\, \mathrm{dist}(x, A) < \delta \cdot \mathrm{diam}(A)\}$$

In particular if $0 < \delta_1 < \delta_2$ this gives $A^{\delta_1} \subset \overline{A^{\delta_1}} \subset A^{\delta_2}$. Consequently, thanks to the finite intersection property, $\bigcap_{\delta} A^{\delta} = \bigcap_{\delta} \overline{A^{\delta}} = \overline{A}$.

Starting from a c-partition $\mathcal{P}$ of C, let $\phi(\mathcal{P})^{\delta}$ denote the open cover $\{\phi(P)^{\delta}\,;\, P \in \mathcal{P}\}$. This is a cover because $\bigcup_{P \in \mathcal{P}} \phi(P) = X$ and because $\phi(P) \subset \phi(P)^{\delta}$. The next result will be crucial for the proof

Lemma 2. *Given any c-partition $\mathcal{P}$ of the Cantor set, there is $\delta_0 > 0$, depending of both $\mathcal{P}$ and of ϕ, such that for $0 < \delta < \delta_0$ the set of elementary chains for both $\mathcal{P}$ and $\phi(\mathcal{P})^\delta$ are in bijection.*

Proof: (i) If $C = (P_0, P_1, \cdots, P_k)$ is an elementary k-chain for $\mathcal{P}$, then clearly $\phi_k(C) = (\phi(P_0)^\delta, \phi(P_1)^\delta, \cdots, \phi(P_k)^\delta)$ is also an elementary k-chain for $\phi(\mathcal{P})^\delta$. If C, C' are two distinct such k-chains in $\mathcal{P}$, then there is at least one index j such that $P_j \neq P'_j$. Since both P_j, P'_j belong to $\mathcal{P}$ it follows that they do not intersect. Since ϕ is non degenerate, $\phi(P'_j) \setminus \phi(P_j) \neq \emptyset$. Therefore there is $y \in \phi(P'_j) \setminus \phi(P_j)$, so that $\mathrm{dist}(y, \phi(P_j)) > 0$. Hence there is $\delta_1 > 0$ such that $\phi(P'_j) \setminus \phi(P_j)^\delta \neq \emptyset$ for $0 < \delta < \delta_1$. In particular $\phi(P'_j)^\delta \neq \phi(P_j)^\delta$. Since $\mathcal{P}$ is finite, the set of elementary chain is also finite and therefore, it is possible to choose δ_1 uniformly with respect to the elementary chain. Therefore $\phi_k(C) \neq \phi_k(C')$. Hence, if $\delta > 0$ is small enough, ϕ_k is one-to-one on the set of elementary chains.

(ii) Let $(P_0, P_1, \ldots, P_k)$ be a family of k-distinct elements in $\mathcal{P}$ such that for any $\delta > 0$ the intersection $\phi(P_0)^\delta \cap \phi(P_1)^\delta \cap \cdots \cap \phi(P_k)^\delta$ is nonempty. The intersection over $\delta > 0$ coincides with the closed set $\phi(P_0) \cap \phi(P_1) \cap \cdots \cap \phi(P_k)$ and, thanks to the finite intersection property, is nonempty. Conversely, if the family $(\phi(P_0), \phi(P_1), \ldots, \phi(P_k))$ has empty intersection, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$, $\phi(P_0)^\delta \cap \phi(P_1)^\delta \cap \cdots \cap \phi(P_k)^\delta = \emptyset$. Again, because $\mathcal{P}$ is finite, it is possible to choose δ_2 uniformly for all finite family in $\mathcal{P}$. Consequently, if $\delta_0 = \min\{\delta_1, \delta_2\}$ the statement follows. □

By using the same type of machinery it can be shown that the maps ϕ_k defines an isomorphism of chain complex, then by duality a similar result holds for the cochain complex, so that

Corollary 1. *If $\mathcal{P}$ is a c-partition and if δ is small enough, then $H^k(\mathcal{P})$ and $H^k(\phi(\mathcal{P})^\delta)$ are isomorphic.*

Lemma 3. *Given any finite open cover $\mathcal{U}$ in X, there is a c-partition $\mathcal{P}$ and some $\delta > 0$, such that $\phi(\mathcal{P})^\delta$ is a refinement of $\mathcal{U}$. In other words, the family of open covers of the form $\phi(\mathcal{P})^\delta$ is cofinal in the set $\mathfrak{U}$ of finite open covers.*

Proof: (i) let $\phi^{-1}(\mathcal{U}) = \{\phi^{-1}(U)\,;\, U \in \mathcal{U}\}$. Since ϕ is continuous, each $\phi^{-1}(U)$ is open in C. Moreover if $c \in C$ there is $U \in \mathcal{U}$ such that $\phi(c) \in U$, or, equivalently, $c \in \phi^{-1}(U)$. It follows that $\phi^{-1}(\mathcal{U})$ is an open cover of C.

(ii) Let now $\mathcal{O}$ be an open cover in C. Then there is a c-partition refining $\mathcal{O}$. For indeed given any $O \in \mathcal{O}$ and any $c \in O$, there is a clopen set P_c containing c and contained in O. The family $(P_c)_{c \in C}$ is an open cover. Since C is compact, it has a finite subcover $\mathcal{O}_1$ made of clopen sets. In particular this subcover is a refinement of $\mathcal{O}$. Let now $\sigma(\mathcal{O}_1)$ be the σ-algebra generated by $\mathcal{O}_1$: it is finite by construction. Since each generator is a clopen set, it follows that all elements of this σ-algebra are clopen as well. The subset of minimal elements will be called $\mathcal{P}$ and it is clear that it is a c-partition refining $\mathcal{O}_1$ thus $\mathcal{O}$.

(iii) If $\mathcal{U}$ is a finite open cover of X, the previous construction shows the existence of a c-partition $\mathcal{P}$ refining $\phi^{-1}(\mathcal{U})$. In particular for each $P \in \mathcal{P}$ there is $U \in \mathcal{U}$ such that $\phi(P) \subset U$. Therefore there is $\delta_P > 0$ such that $\phi(P)^\delta \subset U$ for $0 < \delta < \delta_P$. Taking the minimum $\delta_{\mathcal{P}} = \min\{\delta_P\,;\, P \in \mathcal{P}\}$ it follows that $\phi(\mathcal{P})^\delta \preceq \mathcal{U}$ for $0 < \delta < \delta_{\mathcal{P}}$. □

Proof of Theorem 3 : The Corollary 1, shows that for each c-partition $\mathcal{P}$, there is $\delta_{\mathcal{P}} > 0$ such that for $0 < \delta < \delta_{\mathcal{P}}$ the two cohomology groups $H^k(\mathcal{P})$ and $H^k(\phi(\mathcal{P})^\delta)$ are isomorphic. Moreover it is clear that if $\mathcal{Q} \preceq \mathcal{P}$, then $\phi(\mathcal{Q})^\delta \preceq \phi(\mathcal{P})^\delta$ as well. Therefore, taking the direct limit it gives

$H^k(C, X, \phi) \simeq \lim_{\to \mathfrak{P}} H^k(\phi(\mathcal{P})^\delta)$. Thanks to Lemma 3, this direct limit coincides with the Čech cohomology group $\check{H}^k(X)$. $\square$

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