Topological Aspects

in the theory of

Aperiodic Solids

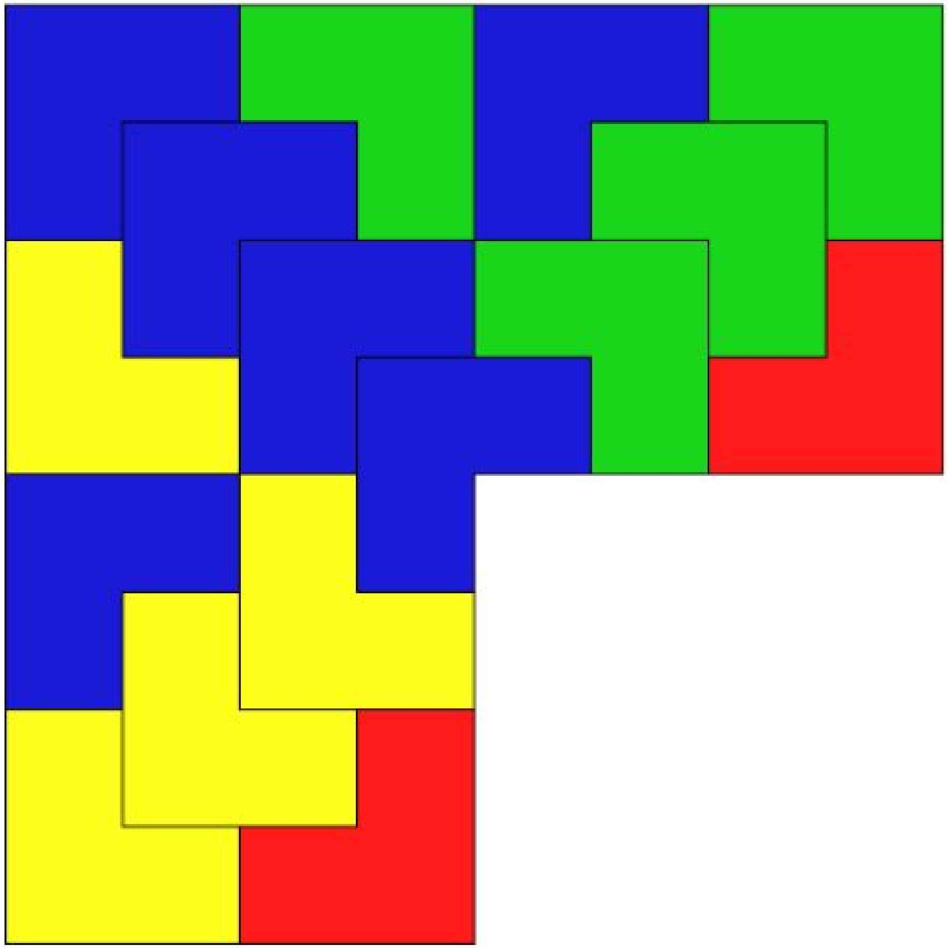
and

Tiling Spaces

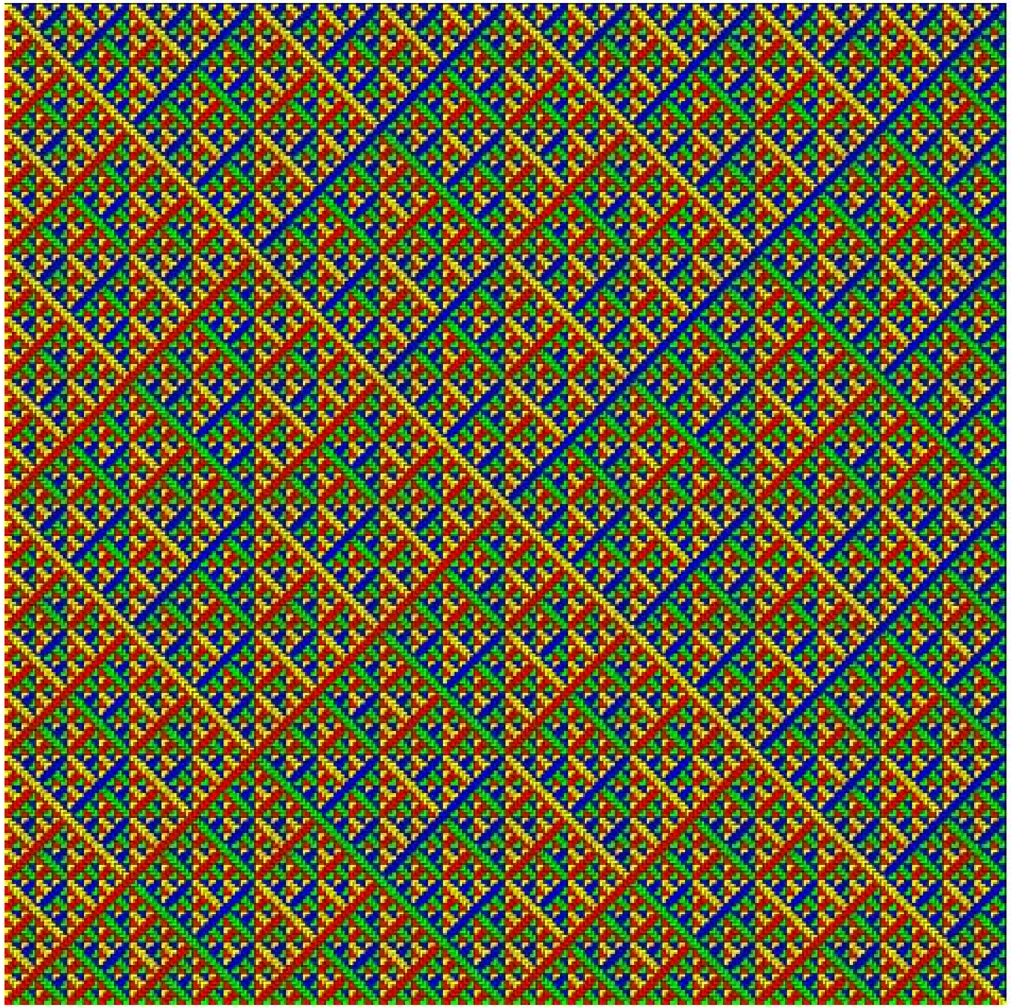
Jean BELLISSARD

Content

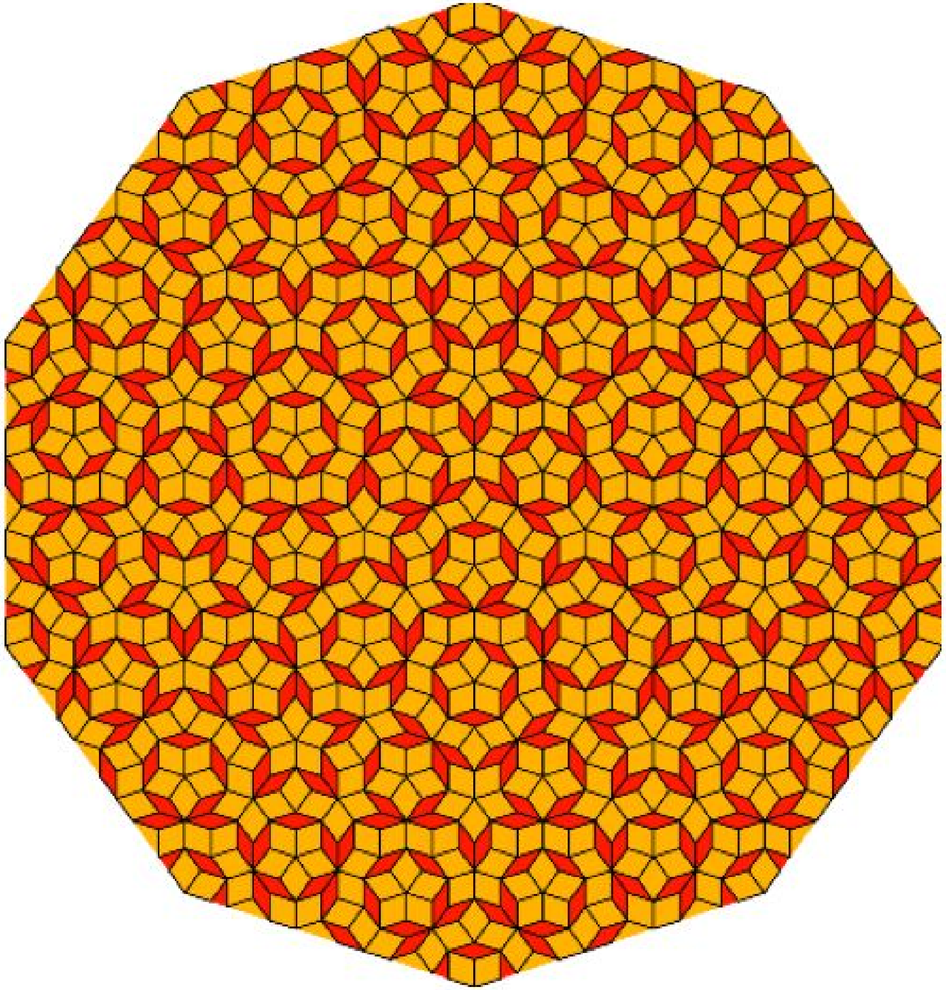
I - Tilings, Tilings,...



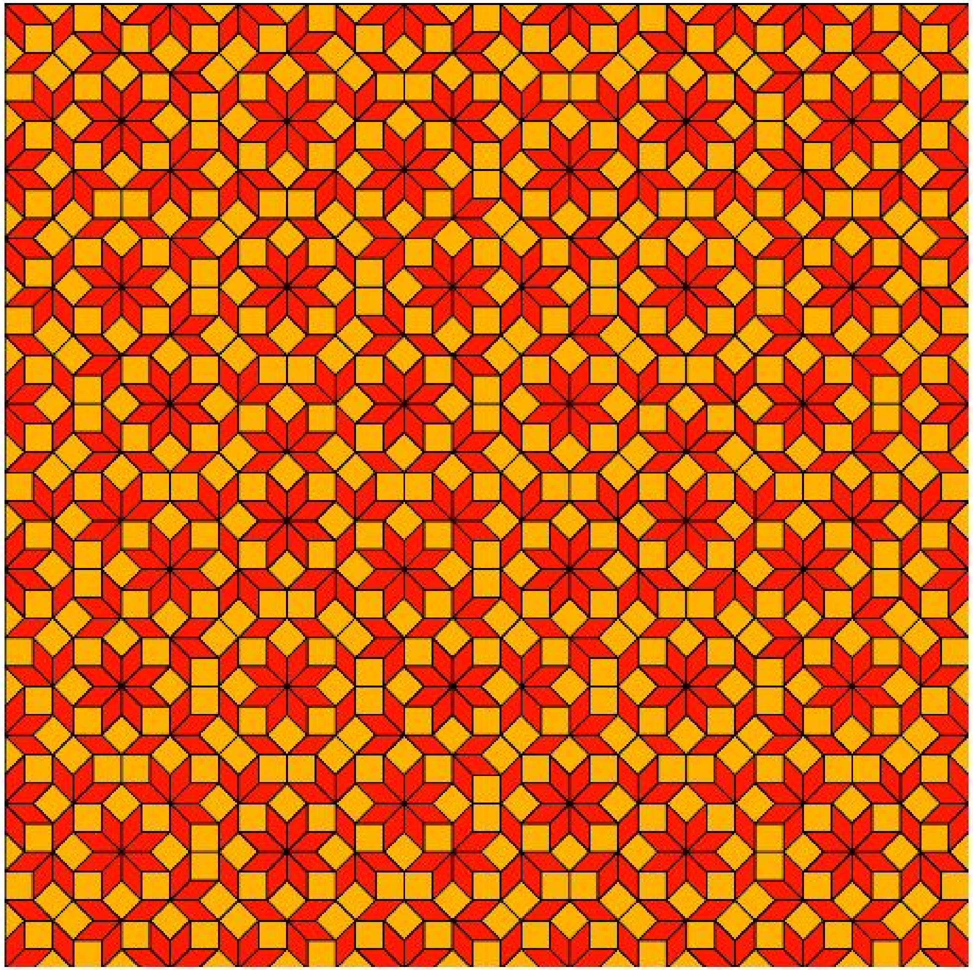
- Building the chair tiling -



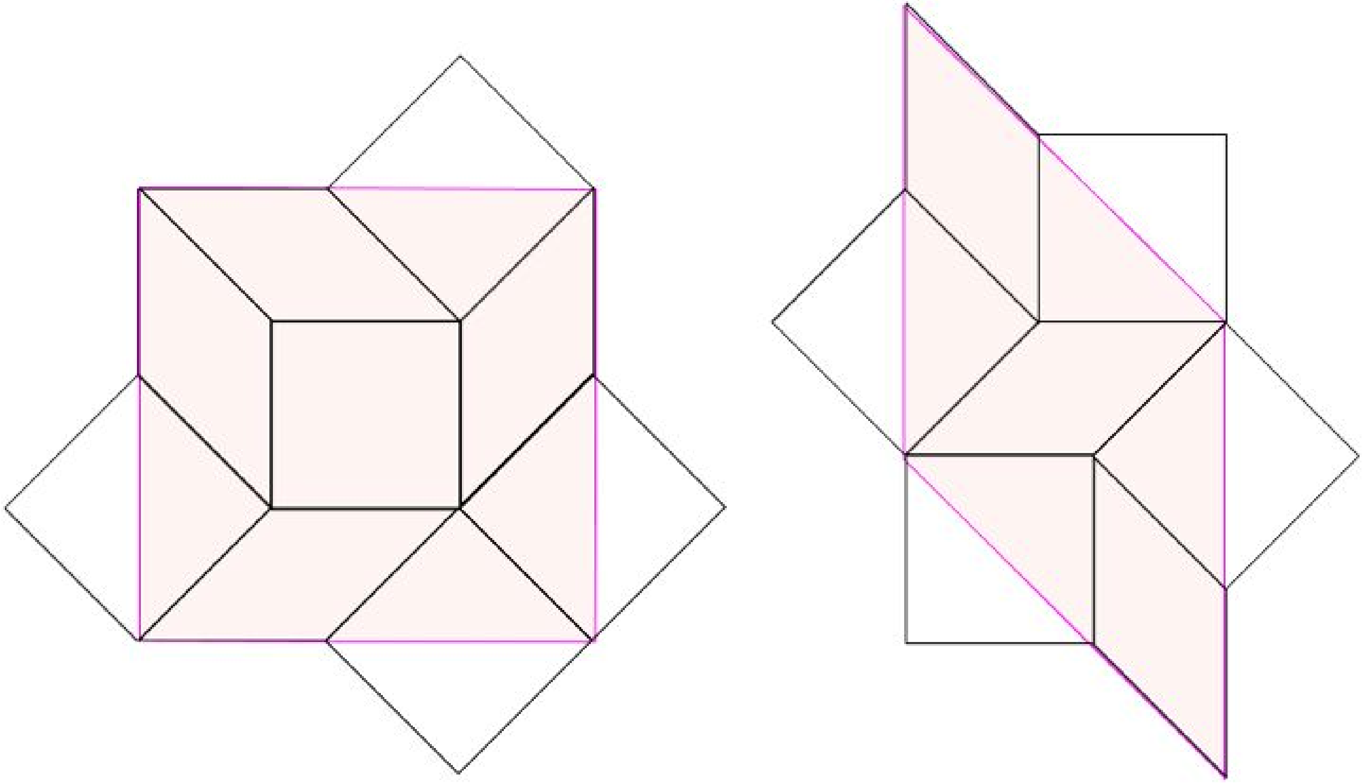
- The chair tiling -



- The Penrose tiling -



- The octagonal tiling -

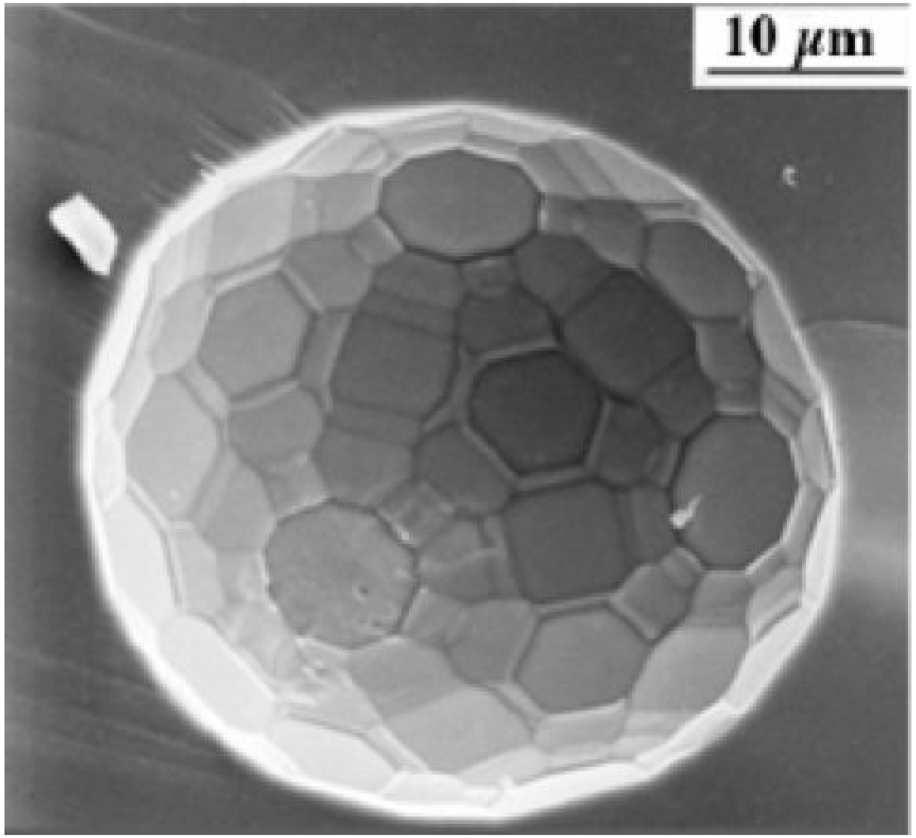


- Octagonal tiling: inflation rules -

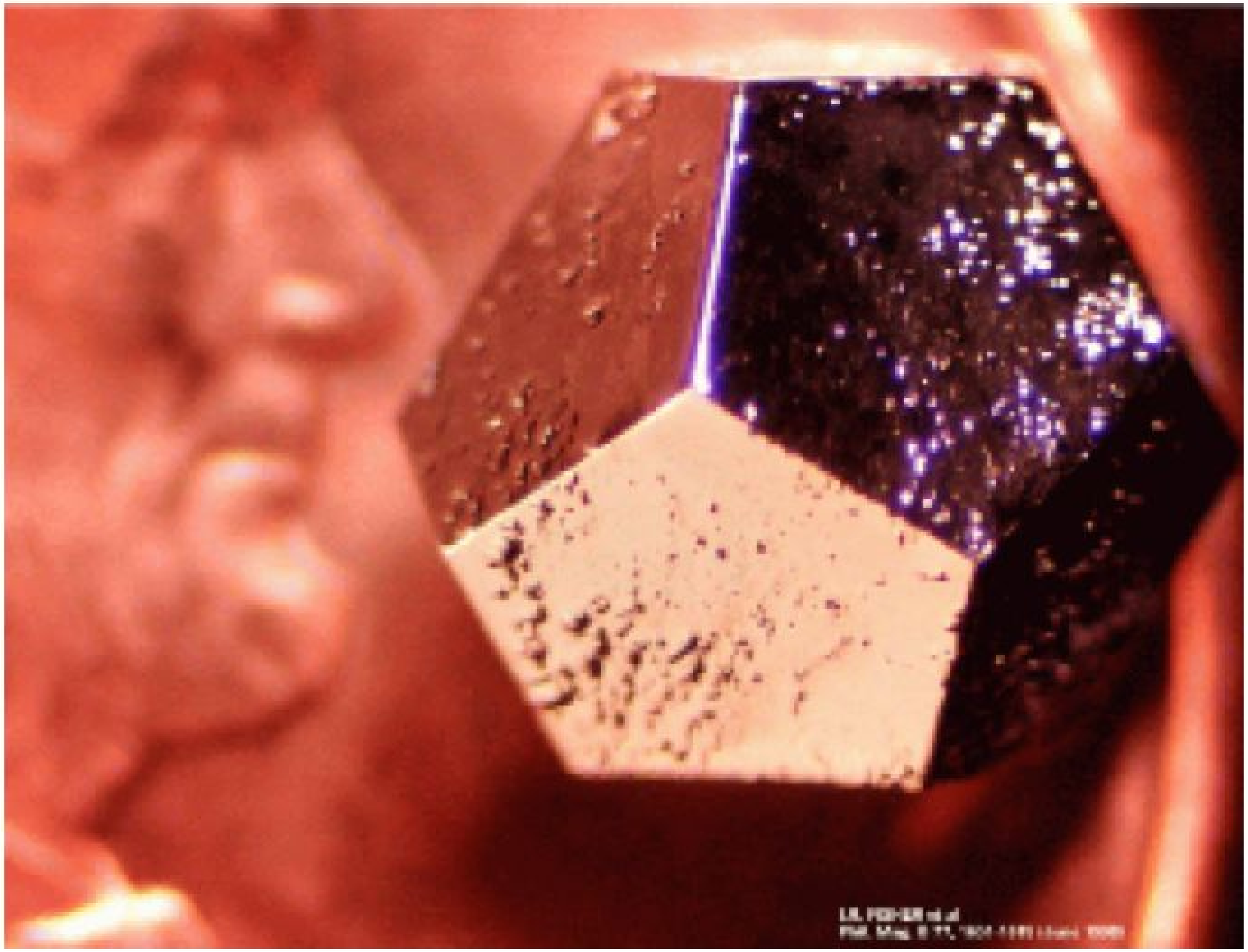
Quasicrystals

No translation symmetry, but icosahedral symmetry. Ex.:

1. $\mathbf{Al_{62.5}Cu_{25}Fe_{12.5}}$;
2. $\mathbf{Al_{70}Pd_{22}Mn_{8}}$;
3. $\mathbf{Al_{70}Pd_{22}Re_{8}}$;



- The icosahedral quasicrystal *AlPdMn* -



- The icosahedral quasicrystal *HoMgZn*-

II - The Hull as a Dynamical System

Point Sets

A subset $\mathcal{L} \subset \mathbb{R}^d$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.
4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.
5. *Finite Local Complexity (FLC)*: $\mathcal{L} - \mathcal{L}$ is discrete and closed.
6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

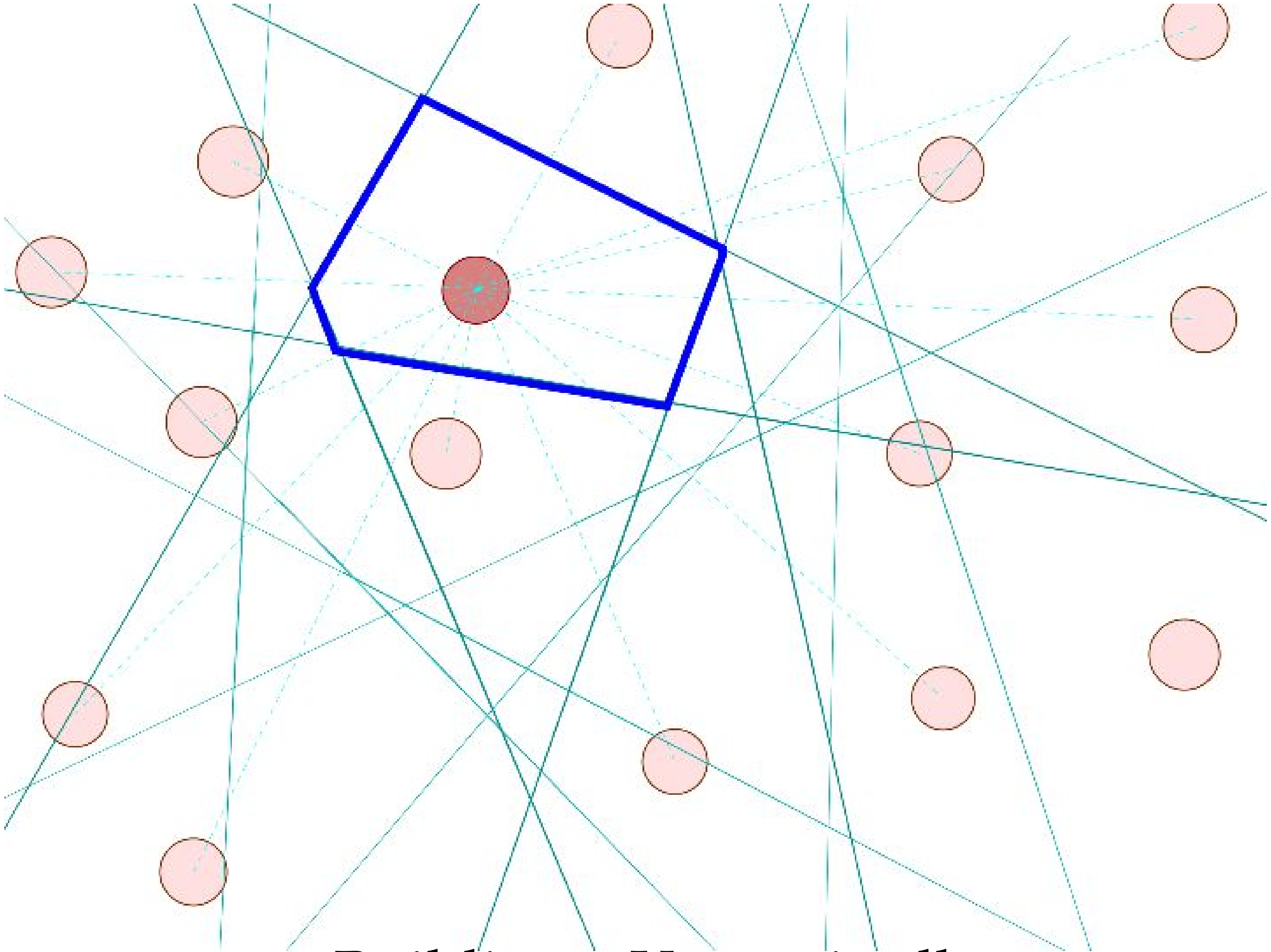
Point Sets and Tilings

Given a tiling with finitely many tiles *(modulo translations)*, a Delone set is obtained by defining a point in the interior of each *(translation equivalence class of)* tile.

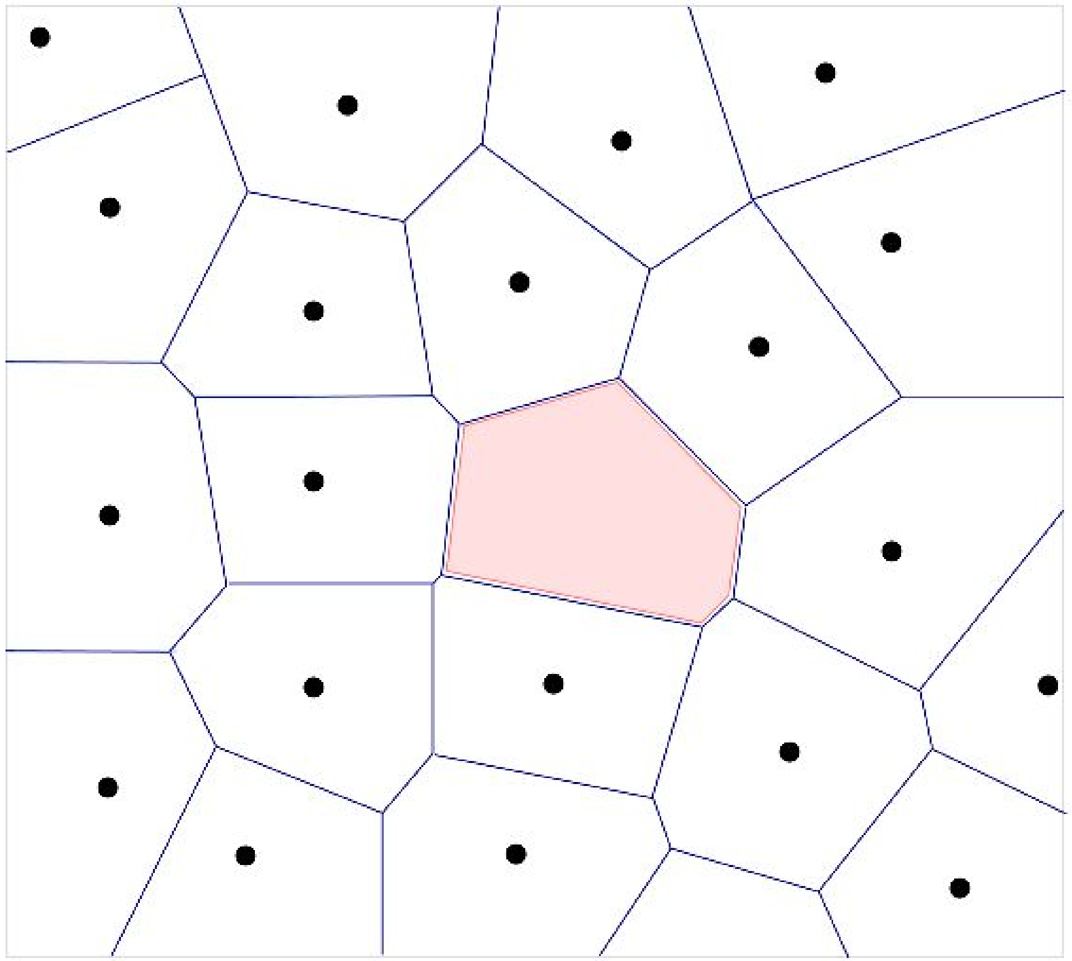
Conversely, given a Delone set, a tiling is built through the *Voronoi cells*

$$V(x) = \{a \in \mathbb{R}^d \,;\, |a - x| < |a - y| \,, \forall y \mathcal{L} \setminus \{x\}\}$$

1. $V(x)$ is an *open convex polyhedron* containing $B(x; r)$ and contained into $\overline{B(x; R)}$.
2. Two Voronoi cells touch face-to-face.
3. If $\mathcal{L}$ is *FLC*, then the Voronoi tiling has finitely many tiles modulo translations.



- Building a Voronoi cell-



- A Delone set and its Voronoi Tiling-

The Hull

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $C_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology. For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu \ := \ \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x - y) \qquad \in \mathfrak{M}(\mathbb{R}^d) \, .$$

Definition 1 *Given $\mathcal{L}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{L}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $\nu^{\mathcal{L}}$.*

Proposition 1 *The Hull is a compact space.*

Properties of the Hull

If $\mathcal{L} \subset \mathbb{R}^d$ is r-uniformly discrete with Hull Ω then using compactness

1. each point $\omega \in \Omega$ *is an r-uniformly discrete* point measure with support $\mathcal{L}_\omega$.
2. if $\mathcal{L}$ is (r, R)*-Delone*, so are all $\mathcal{L}_\omega$'s.
3. if, in addition, $\mathcal{L}$ is *FLC*, so are all the $\mathcal{L}_\omega$'s.
   Moreover then $\mathcal{L} - \mathcal{L} = \mathcal{L}_\omega - \mathcal{L}_\omega \ \forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{L}_\omega$.*

Theorem 1 *If $\mathcal{L}$ is FLC, then its transversal is completely discontinuous.*

Local Isomorphism Classes and Tiling Space

A *patch* is a finite subset of $\mathcal{L}$ of the form

$$p = (\mathcal{L} - x) \cap \overline{B(0, r_1)} \qquad x \in \mathcal{L}\,,\; r_1 \geq 0$$

Given $\mathcal{L}$ a repetitive, FLC, Delone set let $\mathcal{W}$ be its set of finite patches: it is called the *the $\mathcal{L}$-dictionary*.

A Delone set (or a Tiling) $\mathcal{L}'$ is *locally isomorphic* to $\mathcal{L}$ if it has the same dictionary. The *Tiling Space* of $\mathcal{L}$ is the set of *Local Isomorphism Classes* of $\mathcal{L}$.

Theorem 2 *The Tiling Space of $\mathcal{L}$ coincides with its Hull.*

Minimality

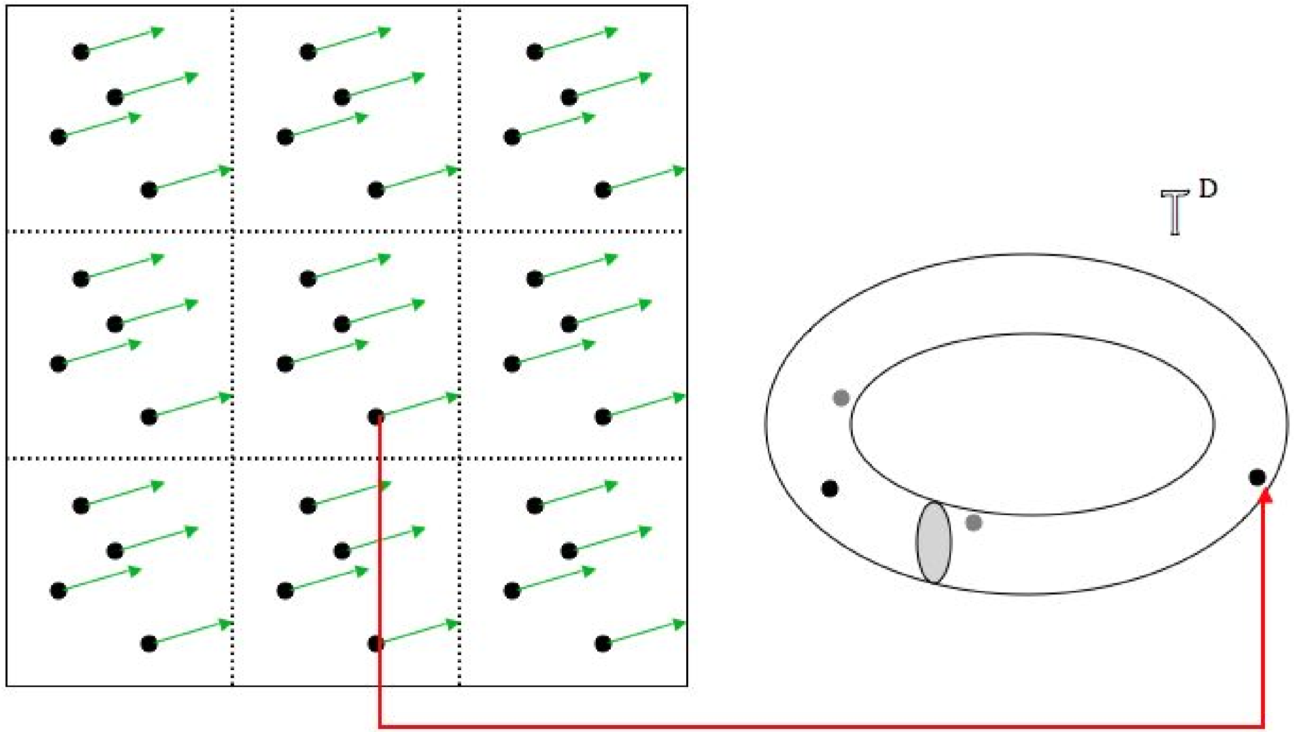
$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

Theorem 3 $\mathbb{R}^d$ *acts minimaly on* Ω *if and only if* $\mathcal{L}$ *is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for *Si* atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

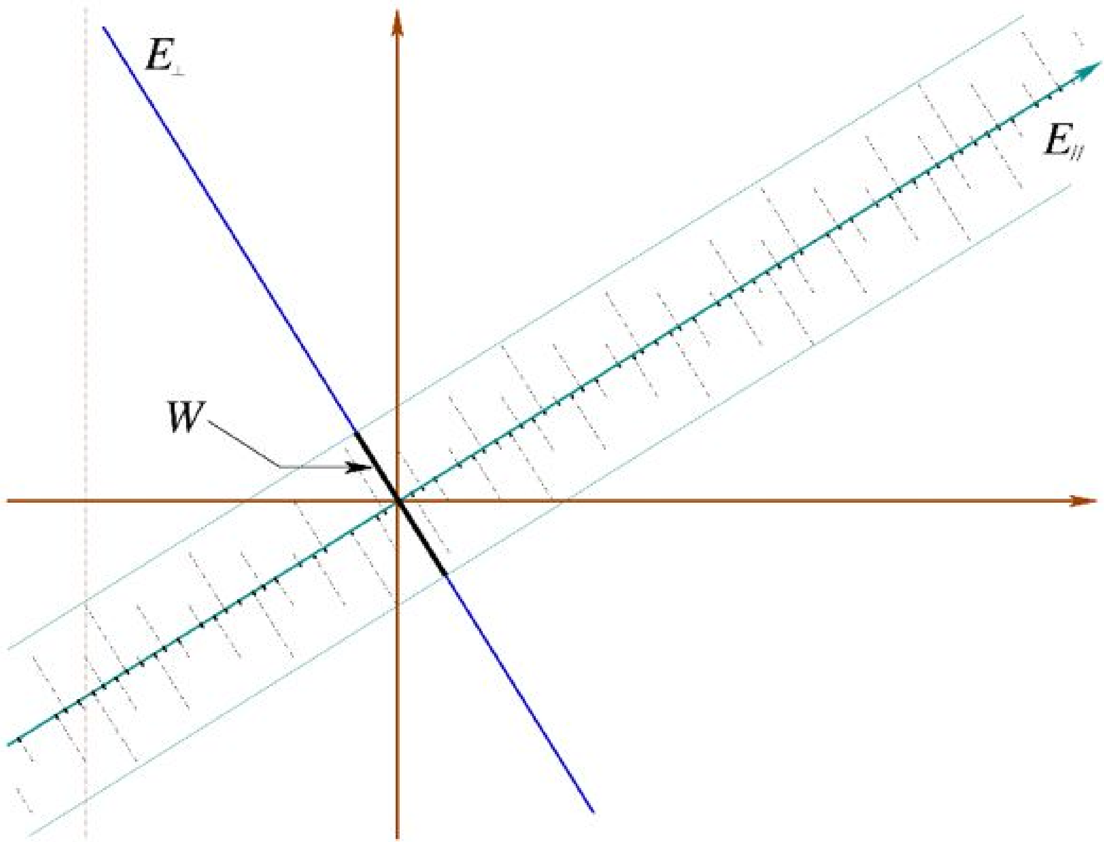
$$\mathbb{R}^d \simeq \mathcal{E}_{\|} \xleftarrow{\pi_{\|}} \mathbb{R}^n \xrightarrow{\pi_{\perp}} \mathcal{E}_{\perp} \simeq \mathbb{R}^{n-d}$$

$$\mathcal{L} \xleftarrow{\pi_{\|}} \tilde{\mathcal{L}} \xrightarrow{\pi_{\perp}} W \quad ,$$

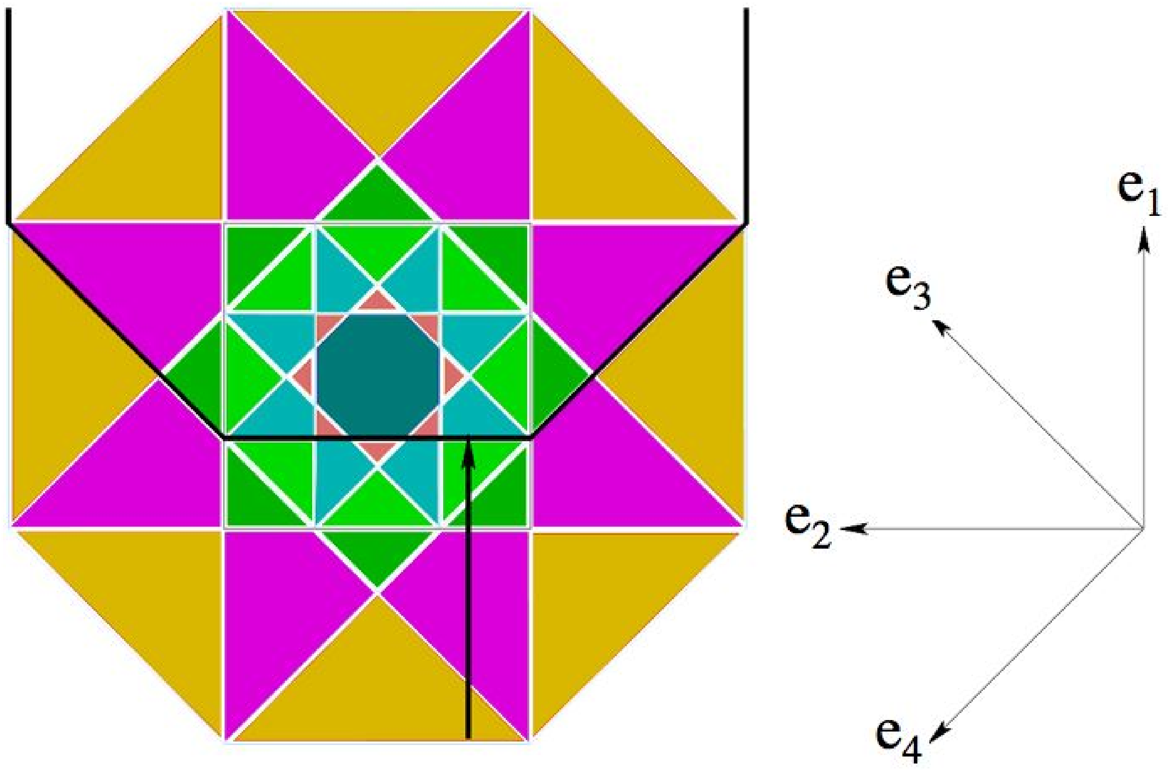
Here

1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{L}$ is the *quasilattice* in $\mathcal{E}_{\|}$ defined as

$$\mathcal{L} \;=\; \{\pi_{\|}(m) \in \mathcal{E}_{\|}\,;\, m \in \tilde{\mathcal{L}},\, \pi_{\perp}(m) \in W\}$$



– The cut–and–project construction –



- The transversal of the Octagonal Tiling is completely disconnected -

III - The Gap Labeling Theorem

J. Bellissard, R. Benedetti, J.-. Gambaudo, Commun. Math. Phys., **261**, (2006), 1-41.
J. Kaminker, I. Putnam, Michigan Math. J., **51**, (2003), 537-546.
M. Benameur, H. Oyono-Oyono, C. R. Math. Acad. Sci. Paris, **334**, (2002), 667-670.

Schrödinger's Operator

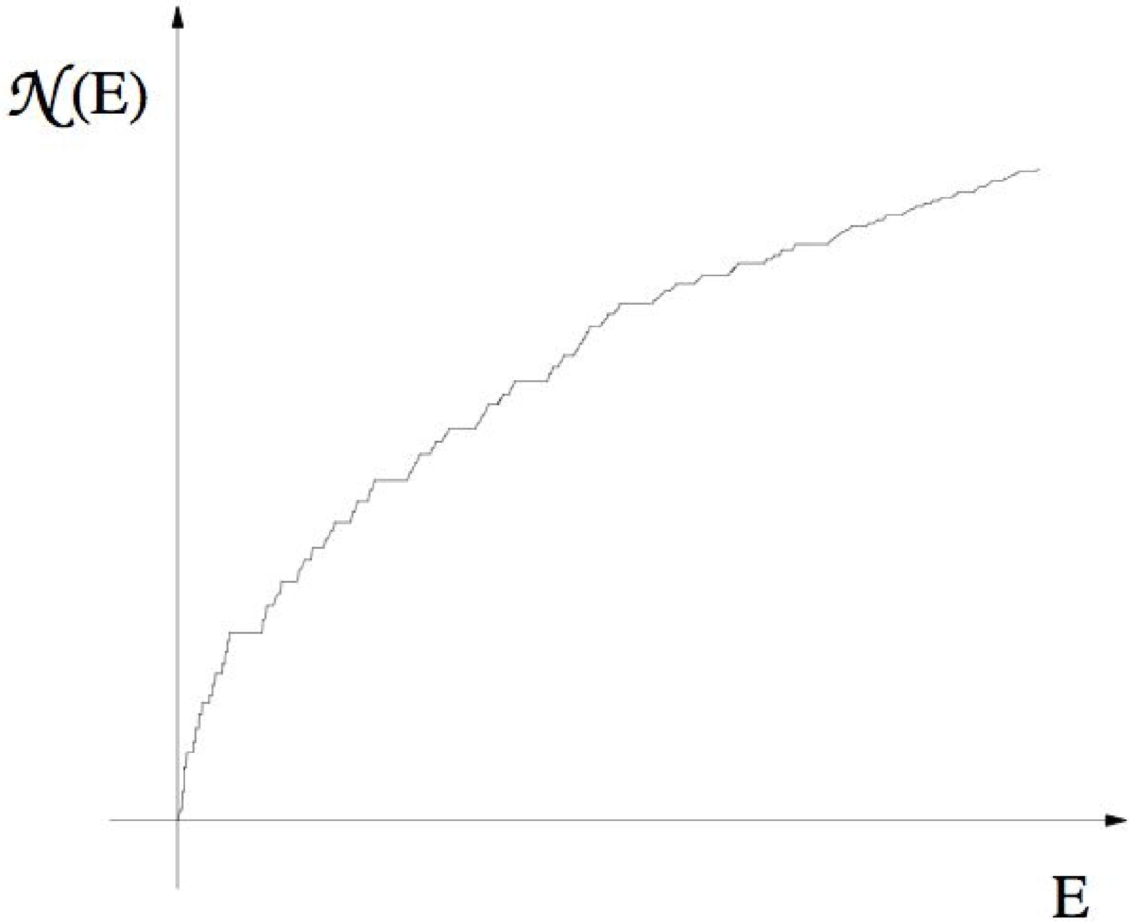
Ignoring electrons-electrons interactions, the one-electron Hamiltonian is given by

$$H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y \in \mathcal{L}_\omega} v(\cdot - y)$$

Its *integrated density of states (IDS)* is defined by

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega \restriction_\Lambda \leq E\}$$

For any $\mathbb{R}^d$*-invariant probability measure* $\mathbb{P}$ on Ω the limit exists a.e. and is independent of ω. It defines a nondecreasing function of E constant on the spectral gaps of H_ω. It is asymptotic at large E's to the IDS of the free Hamiltonian.



- An example of IDS -

Gap Labels

Theorem 4 *The value of the IDS on gaps is a linear combination of the occurrence probabilities of finite patches with integer coefficients.*

The proof goes through the group of K-theory of the hull. The result is model independent.

The abstract result goes back to 1982 (J.B). In 1*D, proved in 1993 (JB). Recent proof in any dimension for aperiodic, repetitive, aperiodic tilings by* KAMINKER-PUTNAM, BENAMEUR & OYONO-OYONO, JB-BENDETTI-GAMBAUDO *in 2001.*

IV - Branched Oriented Flat Riemannian Manifolds

The Fibonacci Tiling

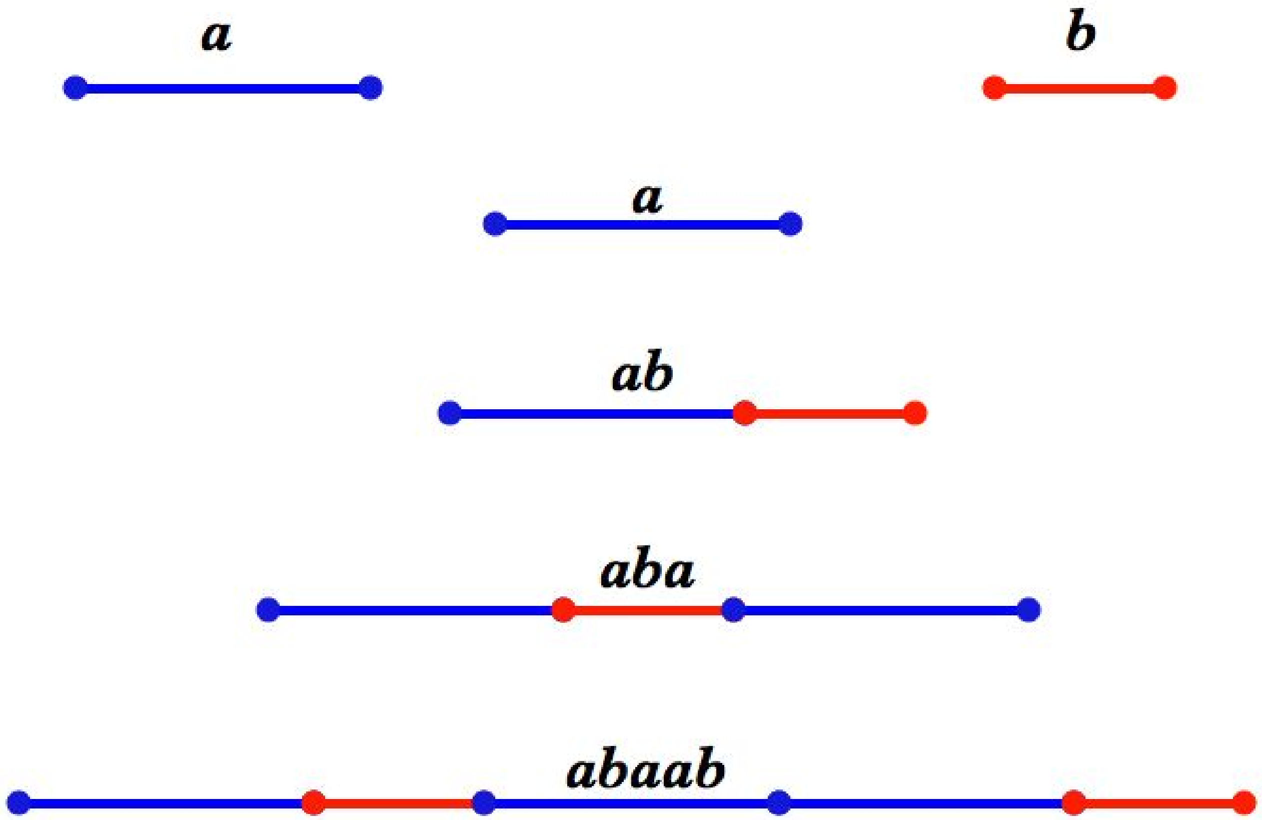
Starting from two tiles in $\mathbb{R}$ as intervals a of lenght 1 and interval b of length $\sigma = (\sqrt{5}-1)/2$, the tiling is built by using the substitution

$$a \mapsto ab \qquad\qquad b \mapsto a$$

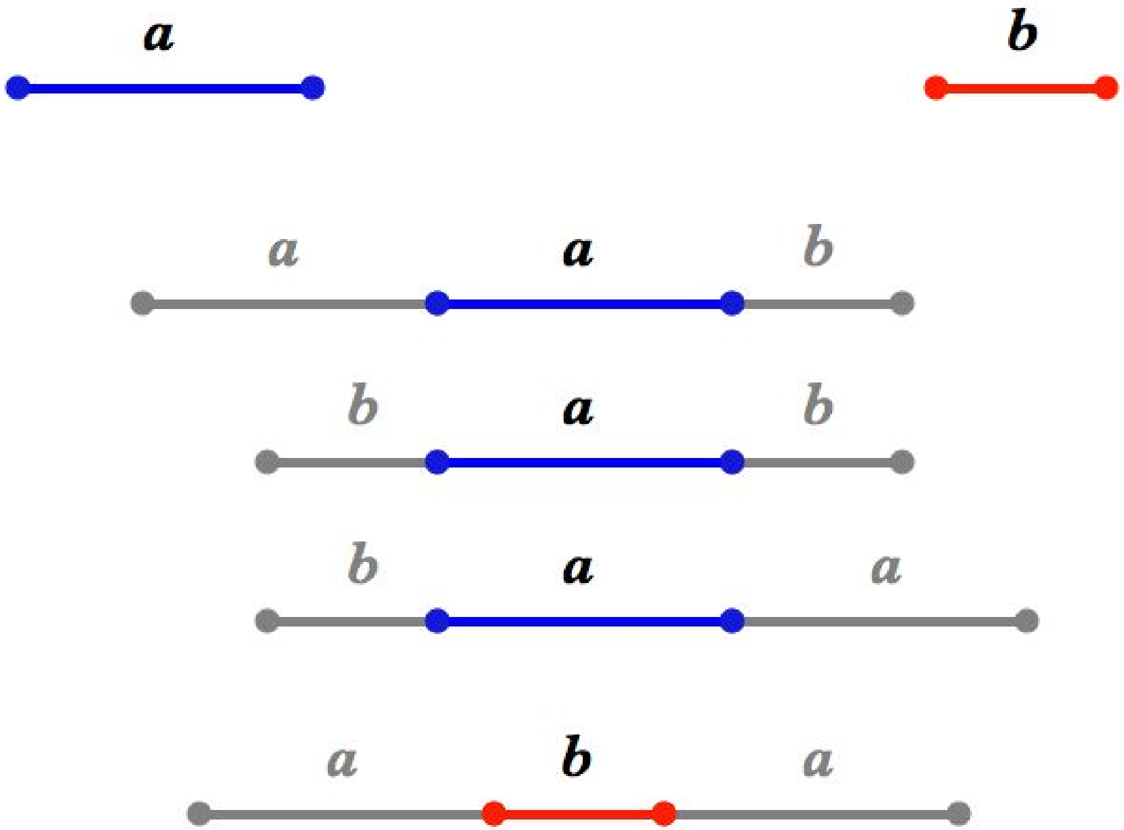
After several iterations this gives

$$abaab{\cdot}aba{\cdot}abaab{\cdot}abaab{\cdot}aba{\cdot}abaab{\cdot}aba{\cdot}abaab{\cdot}abaab{\cdot}aba{\cdot}abaab{\cdot}abaab{\cdot}aba$$

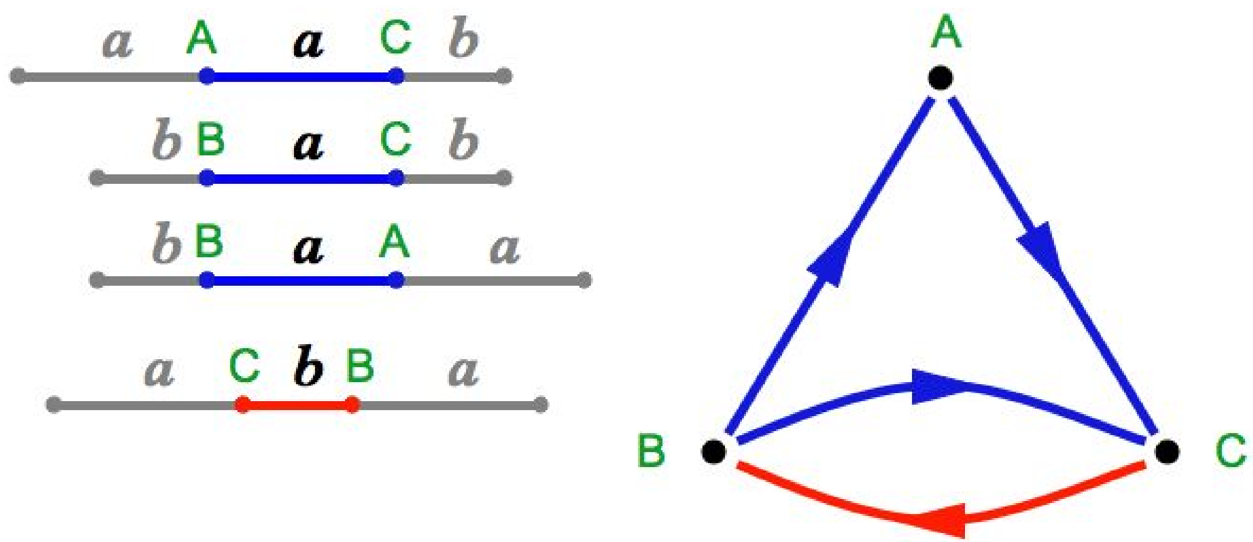
$$abaababa \cdot abaab \cdot abaababa \cdot abaababa \cdot abaab \cdot abaababa \cdot abaab \cdot abaababa$$



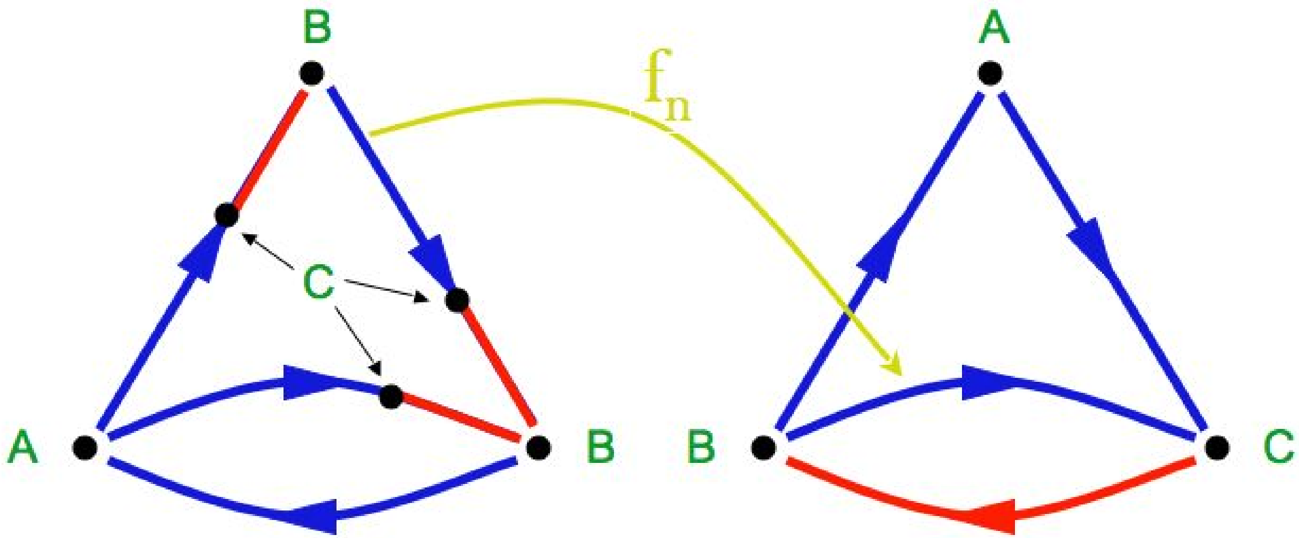
- Constructing the Fibonacci tiling -



- Collared tiles in the Fibonacci tiling -



- The Anderson-Putnam complex for the Fibonacci tiling -



$$X_{n+1} \xrightarrow{\quad f_n \quad} X_n$$

- The substitution map -

The Hull as an Inverse Limit

In general the following theorem holds

Theorem 5 *Given an aperiodic, repetitive, FLC Delone set* $\mathcal{L} \subset \mathbb{R}^d$*, there is a countable family* $(X_n)_{n\in\mathbb{N}}$ *of CW-complexes and maps* $f_{n+1} : X_{n+1} \mapsto X_n$ *that are local homeomorphisms such that the Hull of* $\mathcal{L}$ *can be seen as the inverse limit*

$$\Omega = \lim_{\leftarrow}(X_n, f_n)$$

In addition, the structure of the X_n*'s allows to see them as flat oriented branched Riemannian manifolds so that* $Df_n = \mathbf{1}$*. The action of* $\mathbb{R}^d$ *can be recovered form the local action by constant vector fields.*

V - Cohomology and *K*-Theory

Čech Cohomology of the Hull

Let $\mathcal{U}$ be an *open covering* of the Hull. If $U \in \mathcal{U}$, $\mathcal{F}(U)$ is the space of integer valued locally constant function on U.

For $n \in \mathbb{N}$, the n-chains are the element of $C^n(\mathcal{U})$, namely the *free abelian group* generated by the elements of $\mathcal{F}(U_0 \cap \cdots \cap U_n)$ when the U_i varies in $\mathcal{U}$. A differential is defined by

$$d : C^n(\mathcal{U}) \mapsto C^{n+1}(\mathcal{U})$$

$$df\Big(\bigcap_{i=0}^{n+1} U_i\Big) = \sum_{j=0}^{n} (-1)^j f\Big(\bigcap_{i:i\neq j} U_i\Big)$$

This defines a *complex* with cohomology $\check{H}^n(\mathcal{U}, \mathbb{Z})$. The Čech cohomology group of the Hull Ω is defined as

$$\check{H}^n(\Omega, \mathbb{Z}) = \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathbb{Z})$$

with ordering given by *refinement* on the set of open covers. Thanks to properties of the cohomology, if f_n^* is the map induced by f_n on the cohomology

$$\check{H}^n(\Omega, \mathbb{Z}) = \varinjlim_{n} \left(\check{H}^n(X_n, \mathbb{Z}), f_n^*\right)$$

Examples

J. E. ANDERSON, I. PUTNAM, Ergodic Theory Dynam. Systems, **18**, (1998), 509-537.
L. SADUN, Topology of Tiling Spaces. AMS (2008)

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- Thue-Morse: substitution $a \mapsto ab\,,\ b \mapsto ba$
  $H^0 = \mathbb{Z}\,,\ H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}$

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- Penrose 2D:
  $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^5$, $H^2 = \mathbb{Z}^8$

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  $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}$
- Penrose 2D:
  $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^5$, $H^2 = \mathbb{Z}^8$
- Chair tiling:
  $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$, $H^2 = \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$

Other Cohomologies

- Longitudinal Cohomology (Connes, Moore-Schochet)
- Pattern-equivariant cohomology (Kellendonk-Putnam, Sadun)
- PV-cohomology (Savinien-Bellissard)

In maximal degree the Čech *Homology* does exists. It contains a natural *positive cone* isomorphic to the set of *positive* $\mathbb{R}^d$*-invariant measures* on the Hull (Bellissard-Benedetti-Gambaudo).

Cohomology and *K*-theory

The main topological property of the Hull (or tiling psace) is summarized in the following

Theorem 6 *(i) The various cohomologies, Čech, longitudinal, pattern-equivariant and PV, are isomorphic.*
(ii) There is a spectral sequence converging to the K-group of the Hull with page 2 given by the cohomology of the Hull.
(iii) In dimension $d \leq 3$ *the K-group coincides with the cohomology.*

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.

2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.

3. This dynamical system can be seen as a *lamination* or, equivalently, as the *inverse limit* of *Branched Oriented Flat Riemannian Manifolds*.

4. The Čech cohomology is equivalent to the longitudinal one, obtained by inverse limit, to the pattern-equivariant one or to the Pimsner cohomology are equivalent *Cohomology* of the Hull. The *K-group* of the Hull can be computed through a spectral sequence with the cohomology in page 2.

5. In maximum degree, the *Homology* gives the family of *invariant measures* and the *Gap Labelling Theorem*.