ELECTRONS

in

APERIODIC MEDIA

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Lecture Notes in Physics, **597**, Springer (2003), pp. 413-486.

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I - Aperiodic Materials

A List of Materials

1. Aperiodicity for Electrons
    - Crystals in a Uniform Magnetic Field
    - Semiconductors at very low temperature

2. Atomic Aperiodicity
    - Quasicrystals
    - Glasses
    - Bulk Metallic Glasses

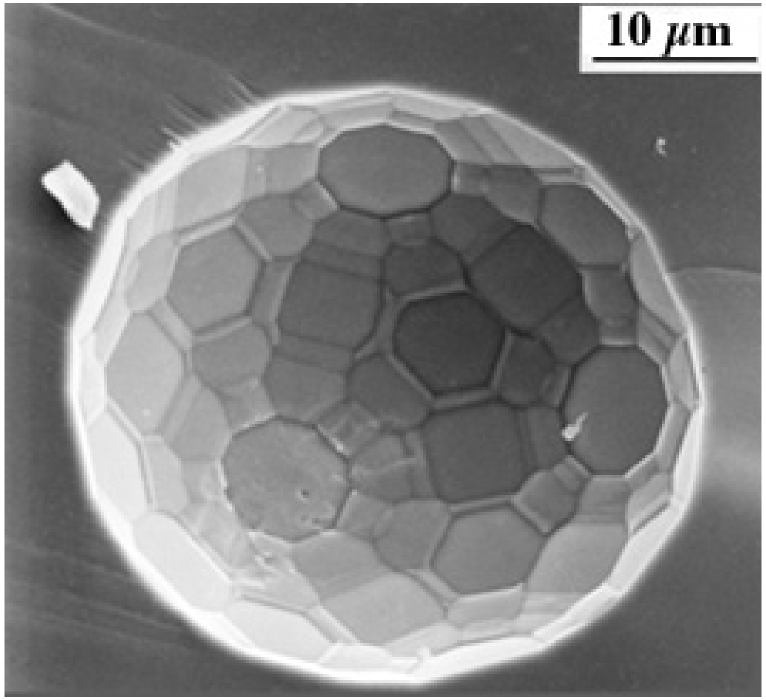
Quasicrystals

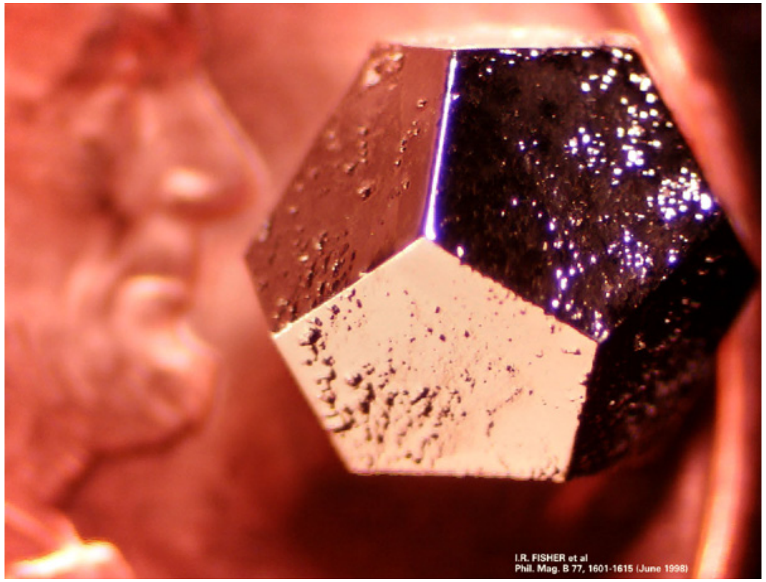
1. Stable Ternary Alloys *(icosahedral symmetry)*
   - High Quality: **AlCuFe** ($Al_{62.5}Cu_{25}Fe_{12.5}$)
   - Stable Perfect: **AlPdMn** ($Al_{70.}Pd_{22}Mn_{7.5}$)
     AlPdRe ($Al_{70.}Pd_{21}Re_{8.5}$)

2. Stable Binary Alloys
   - Periodic Approximants: $\mathbf{YbCd_6}$, $\mathbf{YbCd_{5.8}}$
   - Icosahedral Phase $\mathbf{YbCd_{5.7}}$

Quasicrystals

A hole in a sample of AlPdMn





A sample of HoMgZn compared with a US one cent coin

Bulk Metallic Glasses

1. Examples *(Ma, Stoica, Wang, Nat. Mat. '08)*

- $\mathbf{Zr}_x\mathbf{Cu}_{1-x}$ $\mathbf{Zr}_x\mathbf{Fe}_{1-x}$ $\mathbf{Zr}_x\mathbf{Ni}_{1-x}$
- $\mathbf{Cu}_{46}\mathbf{Zr}_{47-x}\mathbf{Al}_7\mathbf{Y}_x$ $\mathbf{Mg}_{60}\mathbf{Cu}_{30}\mathbf{Y}_{10}$

2. Properties *(Hufnagel web page, John Hopkins)*

- High *Glass Forming Ability* (GFA)
- High *Strength*, comparable or larger than steel
- Superior *Elastic limit*
- High *Wear* and *Corrosion* resistance
- *Brittleness* and *Fatigue* failure

Bulk Metallic Glasses

Applications *(Liquidemetal Technology* www.liquidmetal.com*)*

- *Orthopedic implants* and medical Instruments
- Material for *military components*
- Sport items, *golf clubs, tennis rackets, ski, snowboard, ...*

Pieces of Titanium-Based Structural Metallic-Glass Composites

(Johnson's group, Caltech, 2008)

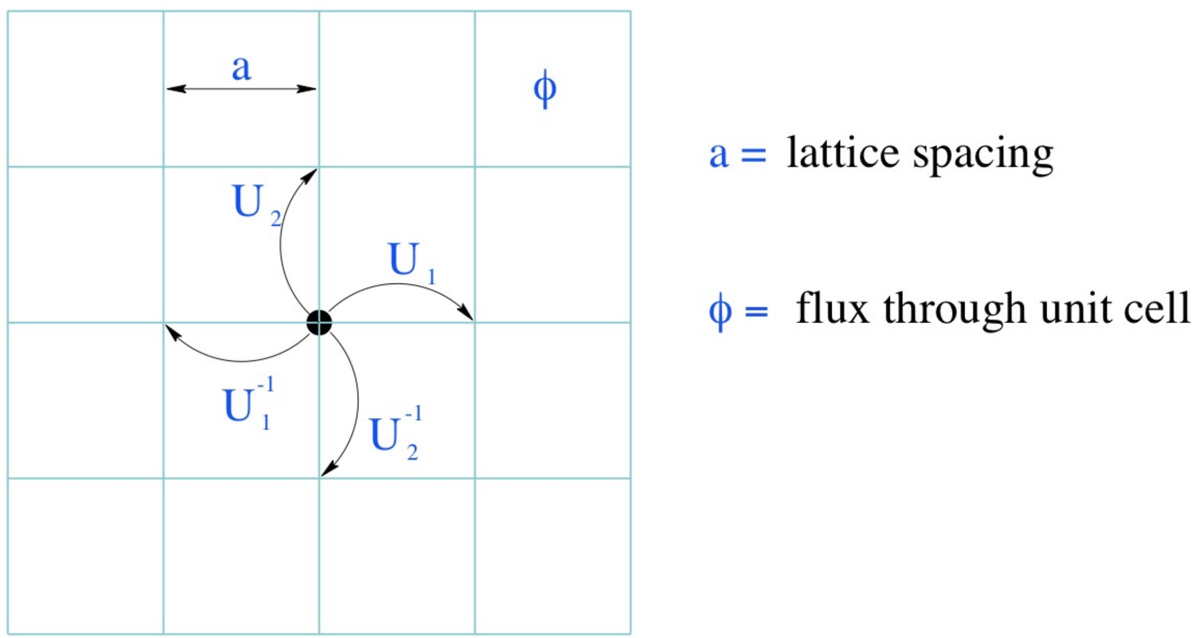
II - Harper's Model

P. G. Harper, Proc. Phys. Soc. A, **68**, 874-878, (1955)

D. R. Hofstadter, Phys. Rev. B, **14**, 2239-2249, (1976)

2*D*-Crystal Electrons in Magnetic Field

- Perfect *square lattice*, nearest neighbor hoping terms, *uniform magnetic field* B perpendicular to the plane of the lattice
- Translation operators U_1, U_2



2*D*-Crystal Electrons in Magnetic Field

- Commutation rules *(Rotation Algebra)*

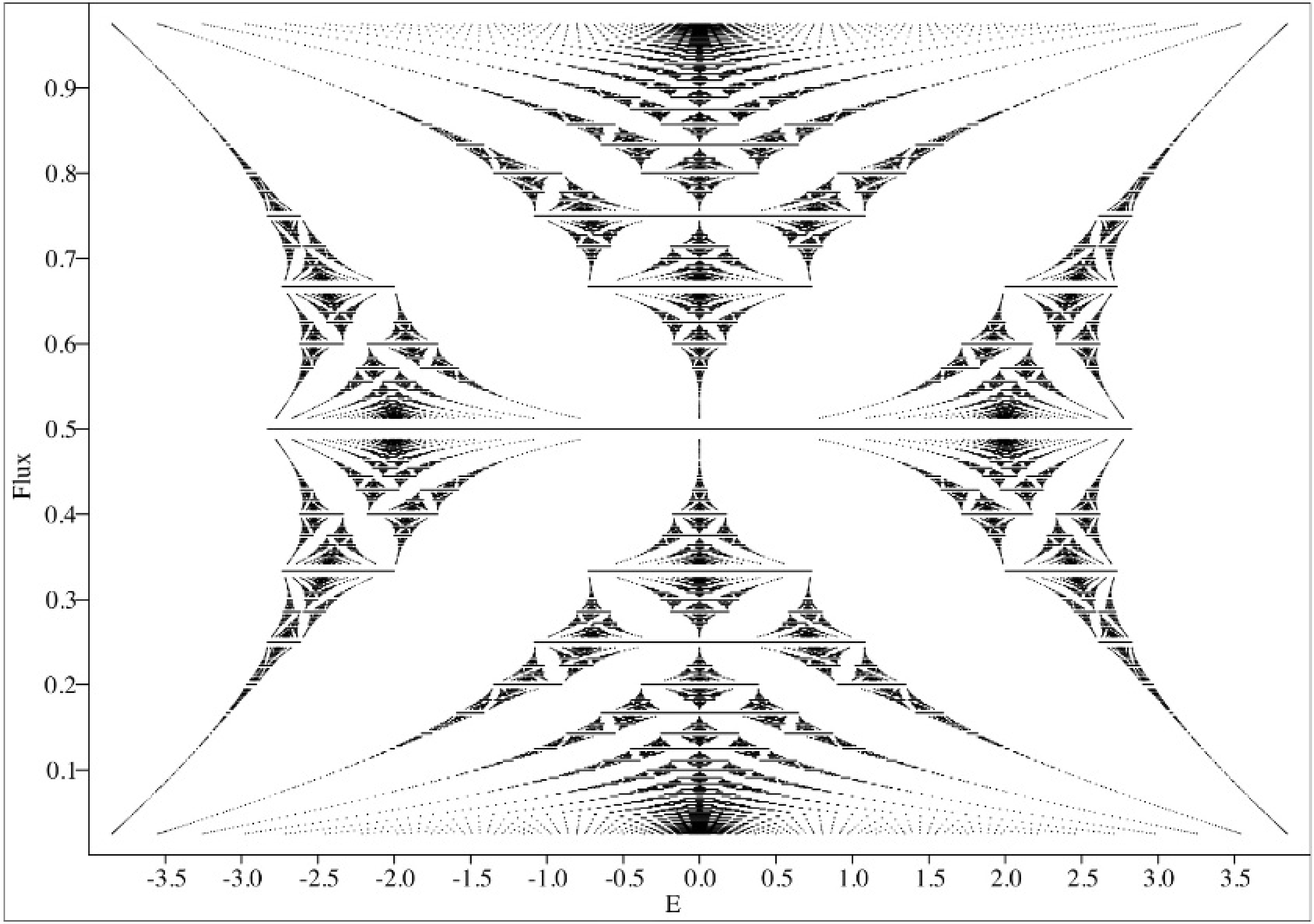
$$U_1\,U_2 = e^{2\imath\pi\alpha}\,U_2\,U_1 \qquad \alpha = \frac{\phi}{\phi_0} \qquad \phi = Ba^2 \quad \phi_0 = \frac{h}{e}$$

- Kinetic Energy *(Hamiltonian)*

$$H = t\left(U_1 + U_2 + U_1^{-1} + U_2^{-1}\right)$$

- Landau gauge $\psi(m,n) = e^{2\imath\pi mk}\varphi(n)$.
  Hence $H\psi = E\psi$ means

$$\varphi(n+1) + \varphi(n-1) + 2\cos 2\pi(n\alpha - k)\varphi(n) = \frac{E}{t}\,\varphi(n)$$


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Flux
-3.5
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
E

$2D$-Crystal Electrons in Magnetic Field

For $\alpha = p/q$, the following properties hold

- The spectrum has q nonoverlapping bands, *touching only at* $E = 0$

  (Bellissard-Simon '82,..., Avila-Jitomirskaya '09)

- The spectral gaps are bounded below by e^{-Cq} for some $C > 0$

  (Helffer-Sjöstrand '86-89, Choi-Elliot-Yui. 90)

$2D$-Crystal Electrons in Magnetic Field

For $\alpha \notin \mathbb{Q}$,

- The spectrum is a Cantor set

  (Bellissard-Simon '82,..., Avila-Jitomirskaya '09)

- The spectrum has zero Lebesgue measure

  (Avron-van Mouche-Simon, '90, ..., Avila-Jitomirskaya '09)

- The gap edges are Lipshitz continuous as long as they do not close, otherwise they are Hölder with exponent $1/2$

  (Bellissard '94, Avron-van Mouche-Simon, '90, Haagerup et al.)

- The derivative of gap edges *w.r.t.* α is discontinuous at each rational

  (Wilkinson '84, Rammal '86, Bellissard-Rammal '90)

Rotation Algebra

- The C^*-algebra $\mathcal{A}_\alpha$ generated by two unitaries U_1, U_2 such that $U_1 U_2 = e^{2\imath\pi\alpha} U_2 U_1$ is called the *rotation algebra* *(Rieffel '81)*

- $\mathcal{A}_\alpha$ has a *trace* defined by

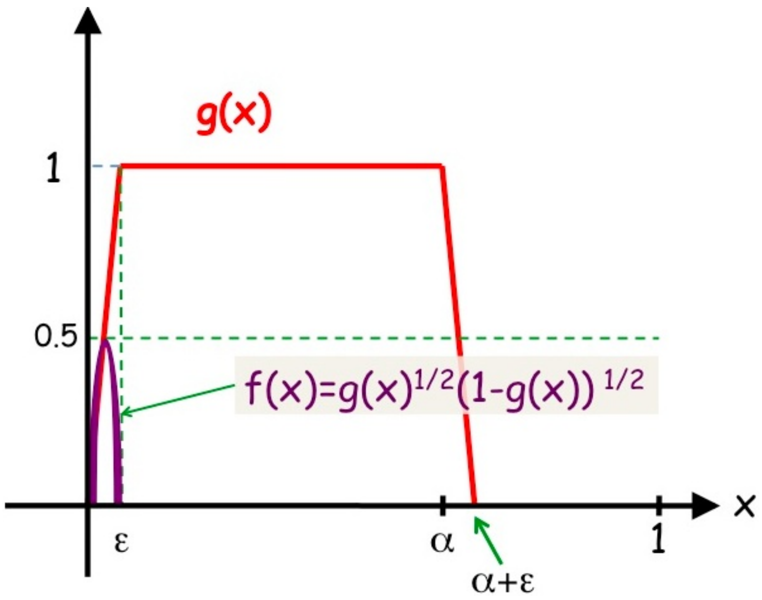
$$\mathcal{T}(U_1^m U_2^n) = \delta_{m,0}\,\delta_{n,0}$$

- $\mathcal{A}_\alpha$ admits two $*$*-derivations* ∂_1, ∂_2 defined by *(Connes '82)*

$$\partial_i U_j = 2\imath\pi\,\delta_{i,j} U_j$$

Rotation Algebra

- Rieffel's projection $P_R = -f(U_2)U_1 + g(U_2) - U_1^{-1}f(U_2)$



$$\mathcal{T}(P_R) = \alpha$$

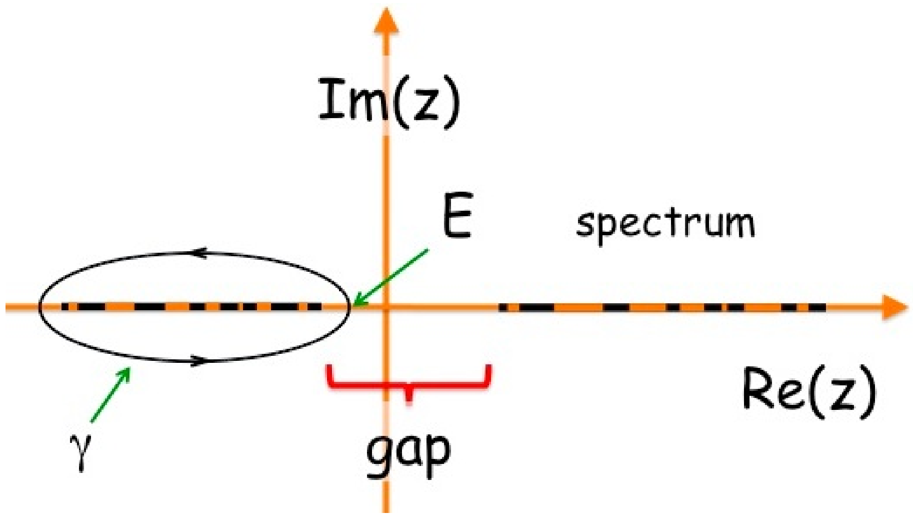
$$\frac{1}{2\imath\pi}\mathcal{T}(P_R[\partial_1 P_R, \partial_2 P_R]) = 1$$

- If $P \in \mathcal{A}_\alpha$ is a *projection*, then *(Rieffel '81, Pimsner-Voiculescu '80, Connes '82)*

$$\mathcal{T}(P) = n\alpha - [n\alpha] \qquad n = \mathrm{Ch}(P) = \frac{1}{2\imath\pi}\mathcal{T}(P[\partial_1 P, \partial_2 P]) \in \mathbb{Z}$$

Gap Labels

- If $H = U_1 + U_1^{-1} + U_2 + U_2^{-1}$, and if E belongs to a gap of the spectrum of H, set



$$P_E = \frac{1}{2\imath\pi} \oint_\gamma \frac{dz}{zI - H}$$

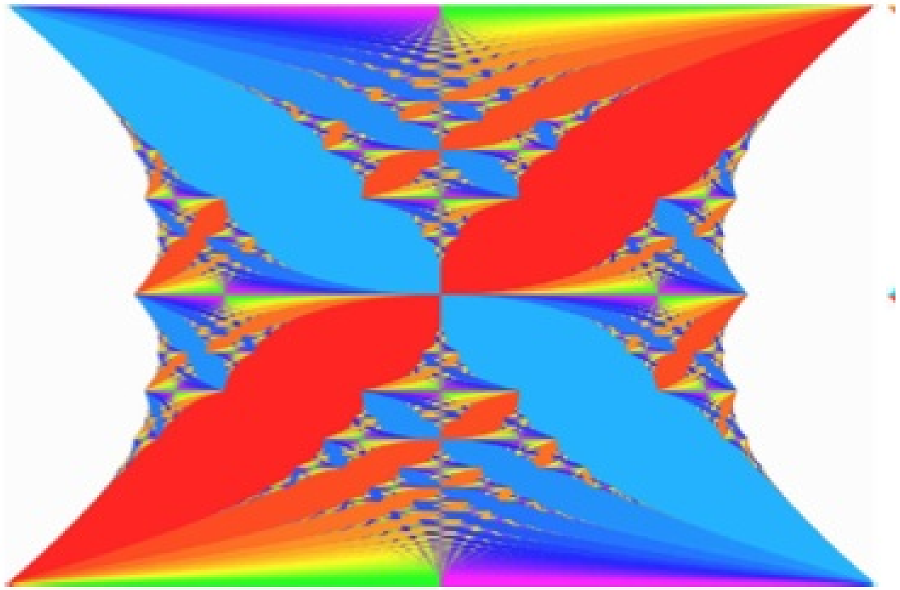
- Then $P_E \in \mathcal{A}_\alpha$!! Hence $\mathcal{T}(P_E) = n\alpha - [n\alpha]$ for some $n \in \mathbb{Z}$!!

(Bellissard '81, '86)

Gap Labels

- The spectral projection of the Harper model between any two gaps can be labelled by an integer, using the previous results, *(Claro-Wannier '78)*

- This integer corresponds to the quantization of the Hall conductivity in such systems *(Thouless-Kohmoto-den Nijs-Nightingale '82)*



Each color corresponds to the integer gap label, for the eigenprojection between the l.h.s *and the gap.*
(Avron-Osadchy-Seiler '03)

III - C^*-algebras: an apology

Fourier Transform

JOSEPH FOURIER, *Mémoire sur la propagation de la chaleur dans les corps solides, Nouveau Bull. Sci. Soc. Philomatique Paris*, I (6), (1808), 112-116.

- *"All"* complex valued functions defined on the interval $[0,1]$ can be written as a Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}_n \, e^{2\imath\pi \, n \, x} \qquad\qquad \hat{f}_n \in \mathbb{C}$$

- *"All"* complex valued functions defined on $\mathbb{R}$ can be written as a Fourier integral

$$f(x) = \int_{\mathbb{R}} \hat{f}(p) \, e^{\imath \, px} \, dp$$

Fourier Transform

- The expansion extends to a parallelepiped $P = \prod_{i=1}^{d}[a_i, b_i]$ or to $\mathbb{R}^d$

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}_n \, e^{2\imath\pi \sum_{i=1}^{d} n_i (x_i - a_i)/(b_i - a_i)} \qquad f(x) = \int_{\mathbb{R}^d} \hat{f}(p) \, e^{\imath \, p \cdot x} \, d^d p$$

- It was used by Fourier to *"solve"* the *heat equation* in a parallelepipedic box and to compute the heat transfer

$$\frac{\partial f}{\partial t} = \Delta f \qquad \Leftrightarrow \qquad \frac{\partial \hat{f}_n}{\partial t} = -\left(\sum_i \frac{4\pi^2 \, n_i^2}{(b_i - a_i)^2} \right) \hat{f}_n$$

An amazingly efficient tool

- The entry *Fourier transform* on Google produces

$$8,100,000 \textit{ results !!}$$

- Used to analyze *partial differential equations*
- Used to solve PDE's by numerical computations through *(fast Fourier transform)*, in
  - Celestial Mechanics (satellites),
  - weather forecast
  - Optic, gratings,
  - Quantum Physics: nuclei, atoms, molecules, band calculations,...

Reconstruction Formula & Hilbert Spaces

(Second half of 19th century, PARSEVAL, orthonormal polynomials, HILBERT)

- The Fourier coefficients $\hat{f}_n$ can be computed from f

$$\hat{f}_n = \int_0^1 e^{-2\imath n x} f(x)\, dx$$

- The sequence $\left(\hat{f}_n\right)_{n\in\mathbb{Z}}$ as an analog of the *coordinates of a vector* in the complex analog of Euclidean spaces, a *Hilbert space*

$$\sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 = \int_0^1 |f(x)|^2\, dx = \|f\|^2 \qquad \textbf{(Parseval)}$$

Pontryagin Duality

L.S. PONTRYAGIN, *The theory of topological commutative groups*, *Ann. of Math.*, **35**, (1934), 361-388

- If G is an *abelian group*, a *character* is a group homomorphism $\chi : G \to \mathbb{S}^1$
- The set G^* of character is an abelian group for the *pointwise* multiplication $\chi\eta : g \in G \mapsto \chi(g)\eta(g) \in \mathbb{S}^1$ and $G \subset G^{**}$
- If G is a *topological group* the weak*-topology make G^* topological
- If G is locally compact
  - so is G^*
  - $G = G^{**}$
  - G and G^* have a *Haar measure* $dg, d\chi$

Pontryagin Duality

If $F : G \to \mathbb{C}$ is *continuous*, there is a suitable normalization of the dual Haar measure $d\chi$ such that, its *Fourier transform* satisfies

$$\widehat{F}(\chi) = \int_G \chi(g)\, F(g)\, dg \qquad \Leftrightarrow \qquad F(g) = \int_{G^*} \overline{\chi(g)}\, \widehat{F}(\chi)\, d\chi$$

$$\int_G |F(g)|^2\, dg = \int_{G^*} |\widehat{F}(\chi)|^2\, d\chi$$ **(Parseval)**

Fourier transform $: L^2(G) \to L^2(G^*)$ **is unitary.**

Fourier transform is (abelian) group theory !

Convolution & Product

- Dual aspect of the group law: *convolution*
  If $F, F' : G \to \mathbb{C}$ are continuous with comact support

$$F * F'(g) = \int_G F(h)\, F'(h^{-1}g)\, dh$$

- The Fourier transform it into the *pointwise product*

$$\widehat{F * F'}(\chi) = \widehat{F}(\chi)\, \widehat{F'}(\chi)$$

Fourier transform is (abelian) algebra !

C^*-Algebras

A very long story: from Gelfand *et al.* 1940, to the mid sixties
G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, 1979.

A *C^*-algebra* $\mathcal{A}$ is a Banach space with a bilinear associative product $(a, b) \mapsto ab$ and an antilinear involution $a \mapsto a^*$ such that

$$||ab|| \leq ||a|| \, ||b|| \qquad\qquad ||a^* a|| = ||a||^2$$

The *r.h.s* implies that the norm is of purely algebraic origin because

1. The norm of $||a^* a||$ coincides with the *spectral radius* of $a^* a$
   (namely the minimum of $|z|$ for $z \in \mathbb{C}$ such that $(z - a^* a)$ is invertible in $\mathcal{A}$)
2. If $\rho : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism between C^*-algebras, then it is *isometric*.

C*-Algebras

A *character* is a $*$-homomorphism $\chi : \mathcal{A} \to \mathbb{C}$. Then the *kernel* of a character is a *closed $*$-ideal*.

If $\mathcal{A}$ is simple, there are no characters other than $0, 1$.

The set $X = X(\mathcal{A})$ of characters is equipped with the *weak*-topology*: then X is *locally compact* (*compact* if $\mathcal{A}$ has a unit).

Gelfand Theorem *A C*-algebra $\mathcal{A}$ is an abelian if and only if there is a *-isomorphism $\mathcal{G} : \mathcal{A} \to C_0(X)$*

C^*-Algebras

Let G be a *locally compact abelian group*. The convolution and the adjoint ($F, F' \in C_c(G)$ are continuous with compact support)

$$F * F'(g) = \int_G F(h)F'(h^{-1}g)\, dh \qquad F^*(g) = \overline{F(g^{-1})}$$

$C_c(G)$ becomes a $*$-*algebra*.
It has a unique C^*-*norm* with completion $C^*(G)$.

Theorem *If G is a locally compact abelian group $C^*(G)$ is commutative and its space of character is homeomorphic to G^*.*
The Gelfand transform $\mathcal{G}$ coincides with the Fourier transform

Fourier transform is (abelian) C^*-algebras !

C^*-Algebras as Noncommutative Spaces

- A C^*-algebra can be seen as the space of continuous functions *(vanishing at infinity)* on an hypothetical locally compact *"non-commutative"* space X.

- Hence expressing algebraically operations like integrals, derivatives, vector fields, connections, *etc.*, leads to *"noncommutative geometry"*.

- Similarly a C^*-algebra can be seen as a *Fourier transform without symmetries* !

VI - The Hull

Fundamental Properties

- **Pointlike Nuclei:** The *atomic nuclei* in a solid are located on a discrete subset of $\mathbb{R}^3$. These nuclei can be considered as pointlike.

- **Exclusion Principle:** Due to the electron-electron repulsion, produced by the Pauli's *exclusion principle*, there is a *minimum distance* between nuclei. Hence the nuclei positions make up a uniformly discrete subset of $\mathbb{R}^3$.

- **Condensed Media:** Both in liquid and solids, big holes are unstable, hence unlikely.

- **Homogeneity:** All solids considered are *homogeneous*, namely their large scale physical properties are invariant by translation

Uniformly Discrete Sets

- A discrete subset $\mathcal{L} \subset \mathbb{R}^d$ is called *uniformly discrete* whenever there is $r > 0$ such that $\#\{B(x;r) \cap \mathcal{L}\} = 0, 1$ for any $x \in \mathbb{R}^d$
- Associated with $\mathcal{L}$ is the *Radon measure*

$$\nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta_y$$

- $\nu^{\mathcal{L}}$ is characterized by two properties
  - If B is any bounded *Borel* subset of $\mathbb{R}^d$ then $\nu^{\mathcal{L}}(B) \in \mathbb{N}$
  - For all $x \in \mathbb{R}^d$ then $\nu^{\mathcal{L}}(B(x;r)) \in \{0, 1\}$

- Let UD_r be the set of such measures on $\mathbb{R}^d$.

UD-sets

- The space $\mathfrak{M}(\mathbb{R}^d)$ of *Radon measures* on $\mathbb{R}^d$ is the dual to the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support. It will be endowed with the *weak*-topology*. $\mathbb{R}^d$ acts on it and this action is weak*-continuous. Then T will denote this action.
- **Theorem:**
  - *A Radon measure μ belongs to* UD_r *if and only if it has the form* $\nu^{\mathcal{L}}$ *with* $\mathcal{L}$ *being r-uniformly discrete*
  - UD_r *is invariant by the translation group* $\mathbb{R}^d$
- **Theorem:** *For any $r > 0$, the space* UD_r *is compact*

The Hull

- **Hull of μ:** if $\mu \in \mathrm{UD}_r$ its Hull is the closure of its translation orbit.

$$\mathrm{Hull}(\mu) = \overline{\{\tau^a \mu \,;\, a \in \mathbb{R}^d\}}$$

- It follows immediately that the *Hull* is *compact* and that $\mathbb{R}^d$ acts on it by *homeomorphisms*. Hence

$$(\mathrm{Hull}(\mu), \mathbb{R}^d, \tau) \text{ is a } \textit{topological dynamical system}$$

- The support of μ is denoted by $\mathcal{L}_\mu$

The Canonical Transversal

- If $\mu \in UD_r$ its *canonical transversal* is the subset $\mathrm{Trans}(\mu)$ defined by those elements $\xi \in \mathrm{Hull}(\mu)$ with $\xi(\{0\}) = 1$

- If $\xi \in \mathrm{Trans}(\mu)$ and if $a \in \mathbb{R}^d$ is small enough and nonzero $0 < |a| < r$ then $\tau^a \xi \notin \mathrm{Trans}(\mu)$

- $\mathrm{Trans}(\mu)$ is also compact.

- $\xi \in \mathrm{Trans}(\mu)$ then its *fiber* is the set of points $\mathcal{L}_\xi \subset \mathbb{R}^d$ such that $a \in \mathcal{L}_\xi \;\Rightarrow\; \tau^{-a}\xi \in \mathrm{Trans}(\mu)$. Hence $\mathcal{L}_\xi$ is nothing but the *support* of ξ

Delone Sets

- A measure $\mu \in \mathrm{UD}_r$ is *Delone* if there is $R \geq r$ so that

$$\mu(\overline{\mathrm{B}}(x;R)) \geq 1 \qquad\qquad \forall x \in \mathbb{R}^d$$

  The space of R-Delone sets is closed, thus $\mathrm{Hull}(\mu) \subset \mathrm{Del}_{r,R}$

- A Delone set $\mathcal{L}$ has *finite local complexity (FLC)*, if the set $\mathcal{L} - \mathcal{L}$ is discrete and closed *(Lagarias '99)* where $\mathcal{L} - \mathcal{L} = \{y - z\,;\, y,z \in \mathcal{L}\}$
  Then its transversal is *completely disconnected*.

- $\mathcal{L}$ is a *Meyer set* whenever both $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone. *Quasicrystal* are described by Meyer sets.

V - The Noncommutative Brillouin Zone

Set Up

- Ω is a *compact metrizable* space equipped with an *action* of the translation group $\mathbb{R}^d$ by homeomorphism
  (for example $\Omega = \mathrm{Hull}(\mu)$*)*

- $\Xi \subset \Omega$ is a *closed* subspace *(for example* $\Xi = \Omega$, $\mathrm{Trans}(\mu)$*)*

- $\Gamma^{\xi} = \{x \in \mathbb{R}^d\,;\, \tau^{-x}\xi \in \Xi\}$. Let dy denotes either the Lebesgue or the counting measure on Γ^{ξ}.

- **Magnetic field:** $(x, y) \in \mathbb{R}^d \mapsto \mathcal{B}(x, y) \in \mathbb{R}$ is *bilinear, antisymmetric*.

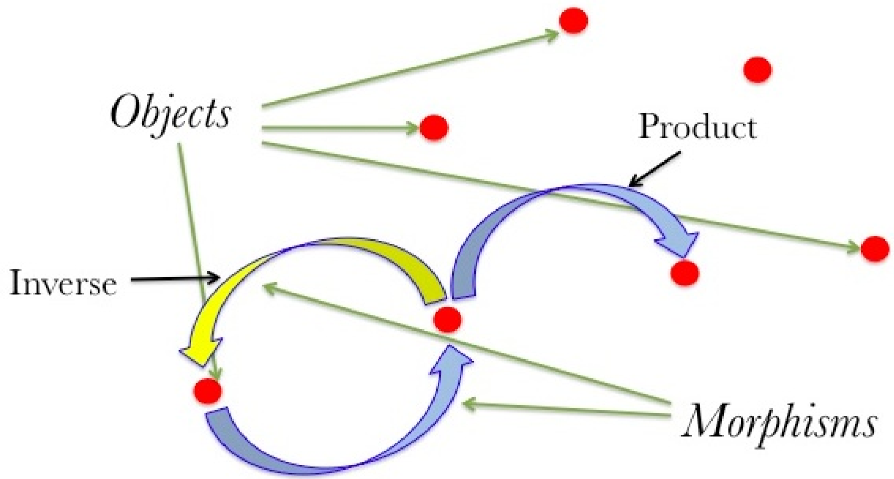
Groupoids

The lack of periodicity makes *homogeneity* (translation invariance at large scale) more complicate to express.

Instead of a *group* action, it is rather a *groupoid* action.

Groupoids

A groupoid can be seen as a *category*, with objects and morphisms being *sets*.



A groupoid can be seen as a group with *several units* (objects) and a *partially defined product*. It can be endowed with a topology, a probability on the set of units, ...

Groupoids

$\Gamma \subset \Xi \times \mathbb{R}^d$ is the *groupoid* induced Ξ:

- Elements of Γ are pairs $\gamma = (\xi, x) \in \Xi \times \mathbb{R}^d$ such that $\mathrm{T}^{-x}\xi \in \Xi$
- $\Gamma \subset \Xi \times \mathbb{R}^d$ is closed.
- **Source, Range:** $r(\gamma) = \xi$, $s(\gamma) = \mathrm{T}^{-x}\xi$ are continuous maps $s, r : \Gamma \to \Xi$
- **Product:** if $r(\gamma') = s(\gamma)$ and $\gamma = (\xi, x)$, $\gamma' = (\xi', x') = (\mathrm{T}^{-x}\xi, x')$

$$\gamma \circ \gamma' = (\xi, x + x')$$

- **Inverse:** $\gamma^{-1} = (\mathrm{T}^{-x}\xi, -x)$

Groupoids

- In the *periodic case* $\mathcal{L} = \mathbb{Z}^d$ then $\Omega = \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$
- If $\Xi = \{0\} \subset \mathbb{T}^d$ then $\Gamma \simeq \mathbb{Z}^d$ is the *period group.*

Γ *is the remnant of the translation group !*

Groupoids

- **Hilbert Space:** $\mathcal{H}_\xi = L^2(\Gamma^\xi)$

- **Γ-Action:** if $\gamma \in \Gamma : \eta = \mathrm{T}^{-x}\xi \to \xi$ then

$$U(\gamma) : \mathcal{H}_\eta \to \mathcal{H}_\xi$$

$$U(\gamma)\psi(y) = e^{-\imath \mathcal{B}(x,y)}\, \psi(y-x) \qquad \psi \in \mathcal{H}_\eta \;\; y \in \Gamma^\xi$$

The Algebra $C^*(\Gamma)$

Endow $\mathcal{A}_0 = C_c(\Gamma)$ with *(here $A, B \in \mathcal{A}_0$)*

- **Product:** *(convolution over the groupoid)*

$$A \cdot B(\xi, x) = \int_{\Gamma^\xi} A(\xi, y)\, B(\mathrm{T}^{-y}\xi, x - y)\, e^{\imath \mathcal{B}(x,y)}\, dy$$

- **Adjoint:** *(Like $F^*(\gamma) = \overline{F(\gamma^{-1})}$)*

$$A^*(\xi, x) = \overline{A(\mathrm{T}^{-x}\xi, -x)}$$

- **Representation:** *(Left regular)* on $\mathcal{H}_\xi = L^2(\Gamma^\xi)$

$$\pi_\xi(A)\, \psi(x) = \int_{\Gamma^\xi} A(\mathrm{T}^{-x}\xi, y - x)\, e^{-\imath \mathcal{B}(x,y)}\, \psi(y)\, dy \qquad \psi \in \mathcal{H}_\xi$$

The Algebra $C^*(\Gamma)$

- **Covariance:** if $\gamma : \eta \to \xi$

$$U(\gamma)\pi_\eta(A)U(\gamma)^{-1} = \pi_\xi(A)$$

- **C*-norm:**

$$\|A\| = \sup_{\xi \in \Xi} \|\pi_\xi(A)\|$$

- $C^*(\Gamma, \mathcal{B})$ is the *completion* of $\mathcal{A}_0$ under this norm

The Algebra $C^*(\Gamma)$

- **Periodic Case:** $\mathcal{L}$ is a discrete subgroup of $\mathbb{R}^d$, such that $\mathbb{B} = \mathbb{R}^d/\mathcal{L}$ is compact. $\mathbb{B}$ is called the *Brillouin zone*.

$$C^*(\Gamma) \simeq \mathcal{C}(\mathbb{B}) \otimes \mathcal{K}$$

- In particular any $A \in C^*(\Gamma)$ can be seen as a matrix valued function $A_{ij}(k),\ k \in \mathbb{B}$

Calculus

- Let $\mathbb{P}$ be a Γ-*invariant probability* on Ξ and set

$$\mathcal{T}_{\mathbb{P}}(A) = \int_{\Xi} A(\xi, 0)\ \mathbb{P}(d\xi) \qquad\qquad A \in C_c(\Gamma)$$

- $\mathcal{T}_{\mathbb{P}} : \mathcal{A}_0 \to \mathbb{C}$ is a trace:
  - $\mathcal{T}_{\mathbb{P}} : \mathcal{A}_0 \to \mathbb{C}$ is linear
  - It is positive $\mathcal{T}_{\mathbb{P}}(A^*A) \geq 0$
  - It is tracial $\mathcal{T}_{\mathbb{P}}(AB) = \mathcal{T}_{\mathbb{P}}(BA)$

Calculus

- $\mathcal{T}_{\mathbb{P}}$ is the *trace per unit volume*

$$\mathcal{T}_{\mathbb{P}}(A) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda \cap \Gamma^{\xi}|} \mathrm{Tr}\left(\pi_{\xi}(A)\chi_{\Lambda}\right) \qquad \forall \xi \ \mathbb{P} - a.\ s.$$

- In the periodic case

$$\mathcal{T}_{\mathbb{P}}(A) = \int_{\mathbb{B}} \mathrm{Tr}\left(\hat{A}(k)\right)\, dk$$

Trace per unit volume = integral over Brillouin's zone

Calculus

- **Dual $\mathbb{R}^d$-action:**

$$\eta_k(A)(\xi, x) = e^{\imath k\cdot x} A(\xi, x) \qquad \partial_i A = \frac{\partial}{\partial k_i}\eta_k(A) \restriction_{k=0}$$

- Then if $X = (X_1, \cdots, X_d)$ is the *position* operator defined by $(X_i\psi)(x) = x_i\,\psi(x)$ for $\psi \in \mathcal{H}_\xi$, then

$$\pi_\xi(\partial_i A) = -\imath[X_i, \pi_\xi(A)]$$

∂_i is is differentiation w.r.t quasi-momentum

Thanks for listening !