The INTEGER QUANTUM HALL EFFECT

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A. van Elst

Main References

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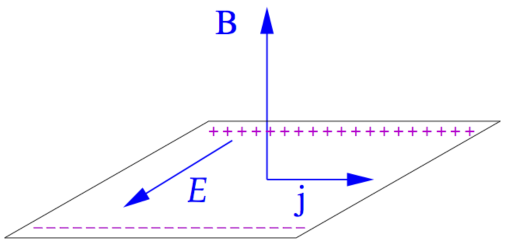
I - INTRODUCTION to the IQHE

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471

The Classical Hall Effect



B = magnetic field
j = current density
E = Hall electric field
n = charge carrier density

In the stationnary state:

$$e\,n\,\vec{\mathcal{E}} + \vec{j} \times \vec{B} = 0$$

$$\Rightarrow \quad \vec{j} = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \vec{\mathcal{E}}$$

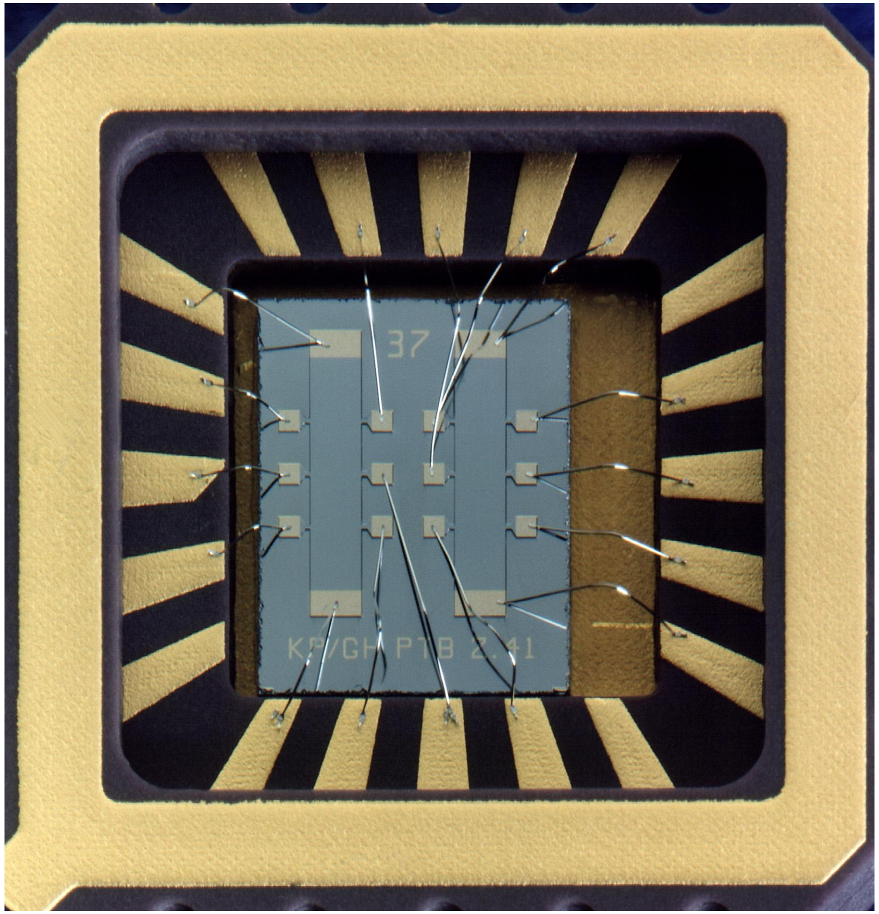
$$\sigma_H = \frac{ne}{B}$$

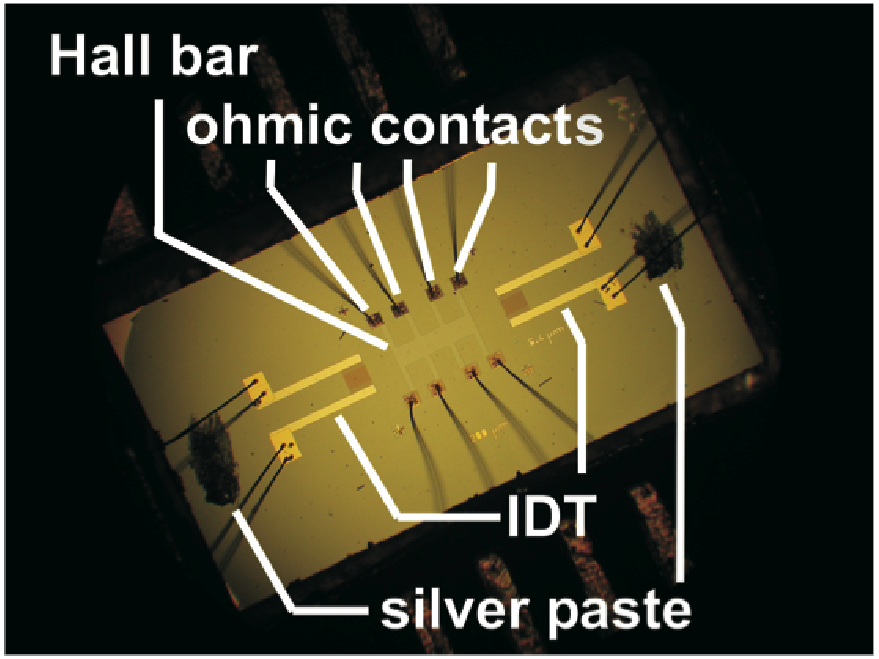
$$\text{Units : } \frac{n}{B} = \left[\frac{1}{\text{flux}}\right],\ \frac{h}{e} = [\text{flux}] \Rightarrow \nu = \frac{nh}{eB} = [1] = \textbf{(filling factor)}$$

This gives the *Hall formula*

$$\sigma_H = \frac{\nu}{R_H} \qquad R_H = \frac{h}{e^2} = 25{,}812.80\ \Omega$$

The Integer Quantum Hall Effect

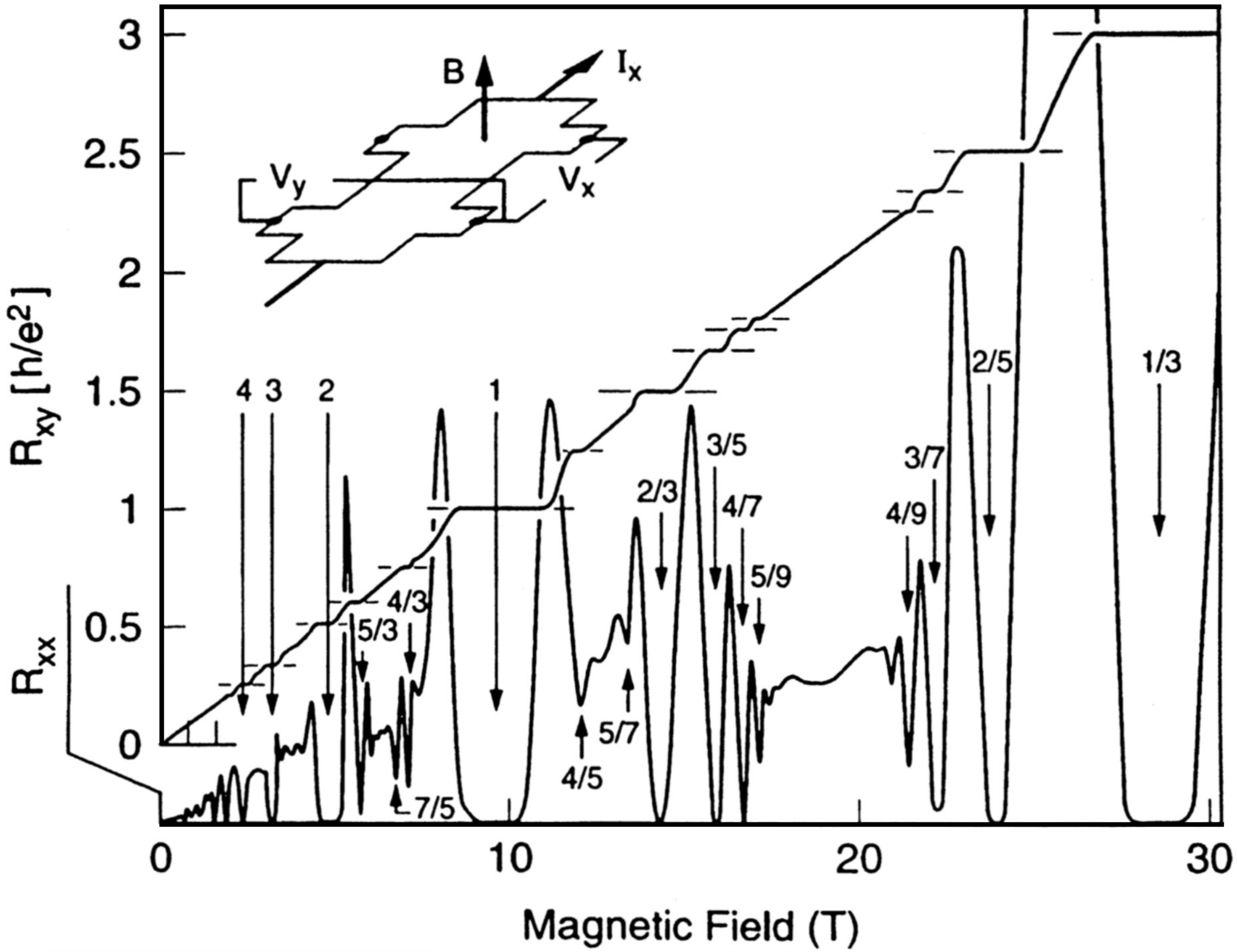




Two examples of Hall bars used in experiments

(Gossard, 2000)

The Integer Quantum Hall Effect



J. P. Eisenstein, H. L. Stormer, *Science*, (1990), **248**, 1461

The Integer Quantum Hall Effect

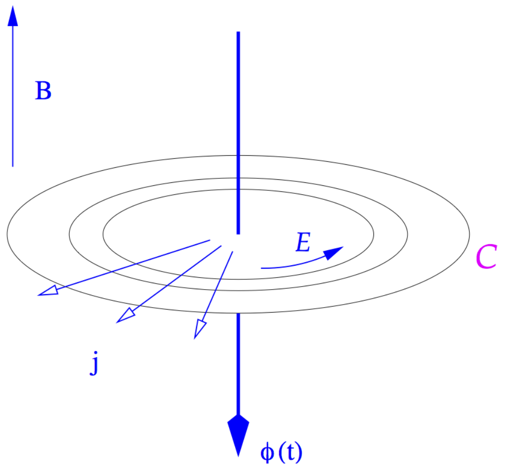
- **Conditions of Observations**
  - Low temperature ($\leq$ few Kelvins)
  - Large sample size ($\geq$ few μm)
  - High mobility & large quenched disorder
  - Two-dimensional Fermion fluid
- **Experiment show that**
  - Very flat plateaux at $\nu \sim 1, 2, 3, 4$ with $\sigma_H = \ell / R_H$, $\ell = 1, 2, 3, 4$
  - Plateaux thickness $\delta\sigma_H/\sigma_H \leq 10^{-8} - 10^{-10}$
  - Very small direct conductivity on plateaux $\Rightarrow$ *localization*
  - For $\ell \geq 2$ electron-electron *interaction* is *negligible*

The Integer Quantum Hall Effect

- *Why is* σ_H *quantized ?*
- *What is the role of the localization ?*

Earlier Works: Laughlin's argument

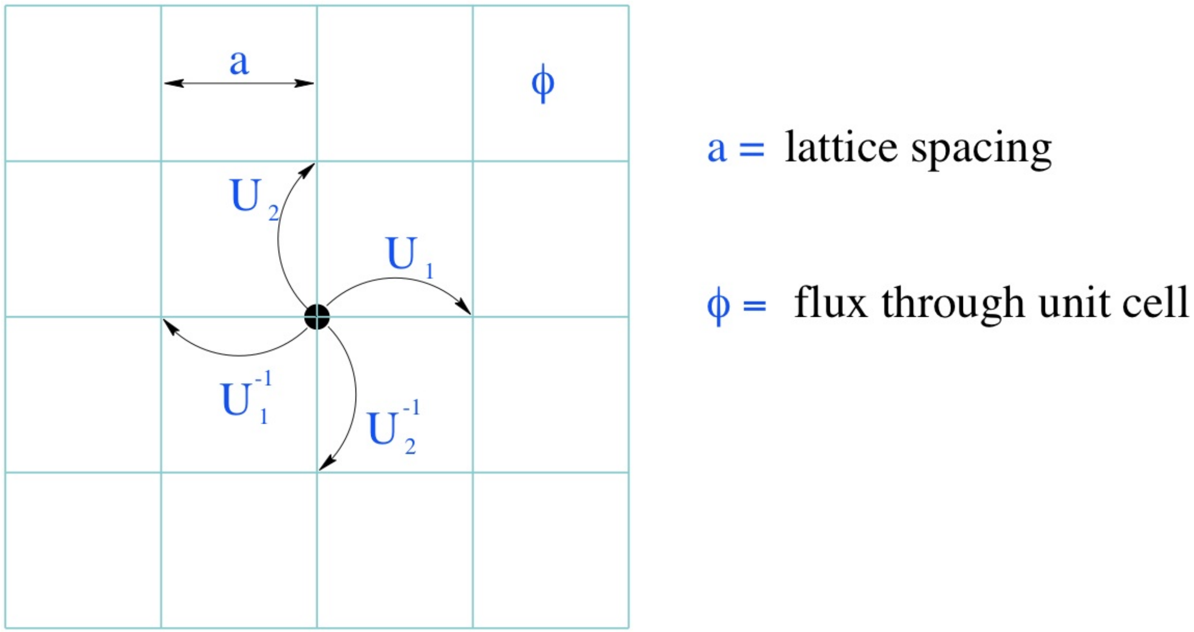
R. B. Laughlin, *Phys. Rev. B*, **23**, (1981), 5632
R. E. Prange, *Phys. Rev. B*, **23**, (1981), 4802
D. J. Thouless, *J. Phys. C*, **14**, (1981), 3475
R. Joynt, R. E. Prange, *Phys. Rev. B*, **29**, (1984), 3303



- Piercing the plane at x with a flux tube adiabatically varying from 0 to $\phi_0 = h/e$ forces one charge per filled Landau level to transfer from $x \to \infty$
- This adiabatic change induces a unitary transformation u on the Landau Hamiltonian (gauge transformation)
- This gives the quantization of the Hall conductance
- Localized states do not participate to this transport

Earlier Works: TKN_2

- Use the Harper model on a *square lattice*, nearest neighbor hopping terms, *uniform magnetic field* B perpendicular to the lattice
- Translation operators U_1, U_2



Earlier Works: TKN_2

- Commutation rules *(Rotation Algebra)*

$$U_1\, U_2 = e^{2\imath\pi\alpha}\, U_2\, U_1 \qquad \alpha = \frac{\phi}{\phi_0} \qquad \phi = Ba^2 \quad \phi_0 = \frac{h}{e}$$

- Kinetic Energy *(Hamiltonian)*

$$H = t\left(U_1 + U_2 + U_1^{-1} + U_2^{-1}\right)$$

- Landau gauge $\psi(m,n) = e^{2\imath\pi mk}\varphi(n)$.
  Hence $H\psi = E\psi$ means

$$\varphi(n+1) + \varphi(n-1) + 2\cos 2\pi(n\alpha - k)\varphi(n) = \frac{E}{t}\,\varphi(n)$$

Earlier Works: TKN_2

- Choose $\alpha = p/q$ to make H q-periodic. Use *Bloch theory* with quasimomentum $\vec{k} = (k_1, k_2) \in \mathbb{B} \approx \mathbb{T}^2$
- At H is a $q \times q$-matrix valued smooth function of $\vec{k}$
- At $\vec{k}$ fixed, any eigenstate $\Psi_{\vec{k}}$ of $H_{\vec{k}}$, defines a *line bundle* over $\mathbb{B}$
- Its non triviality is controlled by the *Chern number*

$$\mathbf{Ch}(\Psi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \Im m \, \langle \frac{\partial \Psi}{\partial k_1} | \frac{\partial \Psi}{\partial k_2} \rangle \, dk_1 dk_2$$

- $\mathbf{Ch}(\Psi) \in \mathbb{Z}$ and is *homotopy invariant* under deformation of H

Earlier Works: TKN_2

- If $P : \vec{k} \in \mathbb{B} \mapsto P(\vec{k})$ is a *projection* valued smooth map then *(example: $P = |\Psi\rangle\langle\Psi|$)*

$$\mathbf{Ch}(P) = \frac{1}{2\imath\pi} \int_0^{2\pi} \int_0^{2\pi} \mathrm{Tr}\left(P(\vec{k}) \left[\frac{\partial P}{\partial k_1}, \frac{\partial P}{\partial k_2} \right] \right) dk_1 dk_2 \; \in \; \mathbb{Z}$$

- If P, Q are two orthogonal projections, $PQ = QP = 0$, then

$$\mathbf{Ch}(P \oplus Q) = \mathbf{Ch}(P) + \mathbf{Ch}(Q)$$

Earlier Works: TKN_2

- If the *Fermi level* E_F belongs to an energy gap, let P_F be the *Fermi projection* *(namely the eigenprojection onto states with energy $E \leq E_F$)*

- Then the following *Chinese-Japanese* relation holds

$$\sigma_H = \frac{e^2}{h} \mathbf{Ch}(P_F)$$

(Chern-Kubo formula)

- This formula *explains the quantization* of the Hall conductance *for rational magnetic fields !*

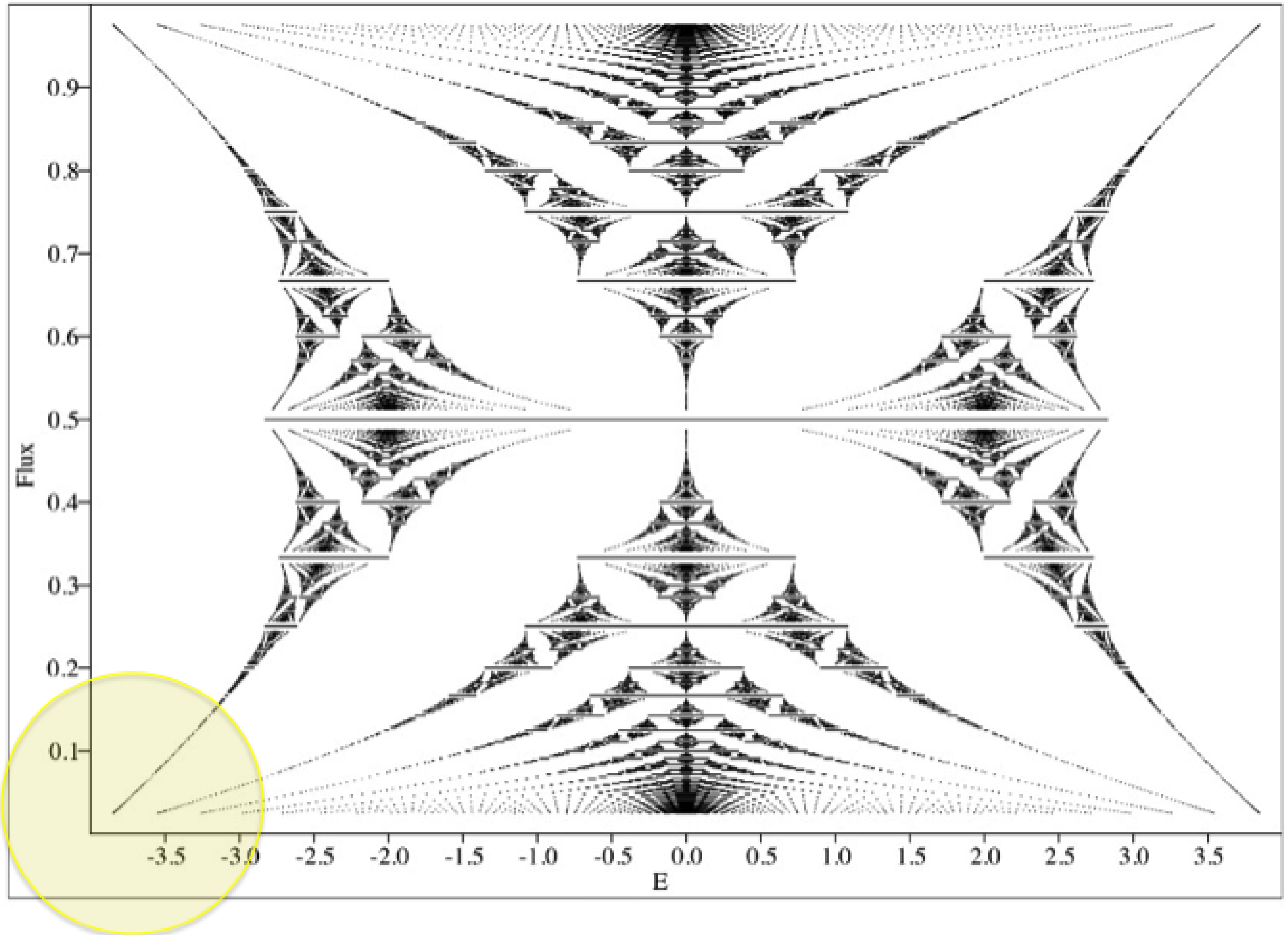
It does NOT explain the appearance of plateaux !

II - Disorder and Magnetic Field

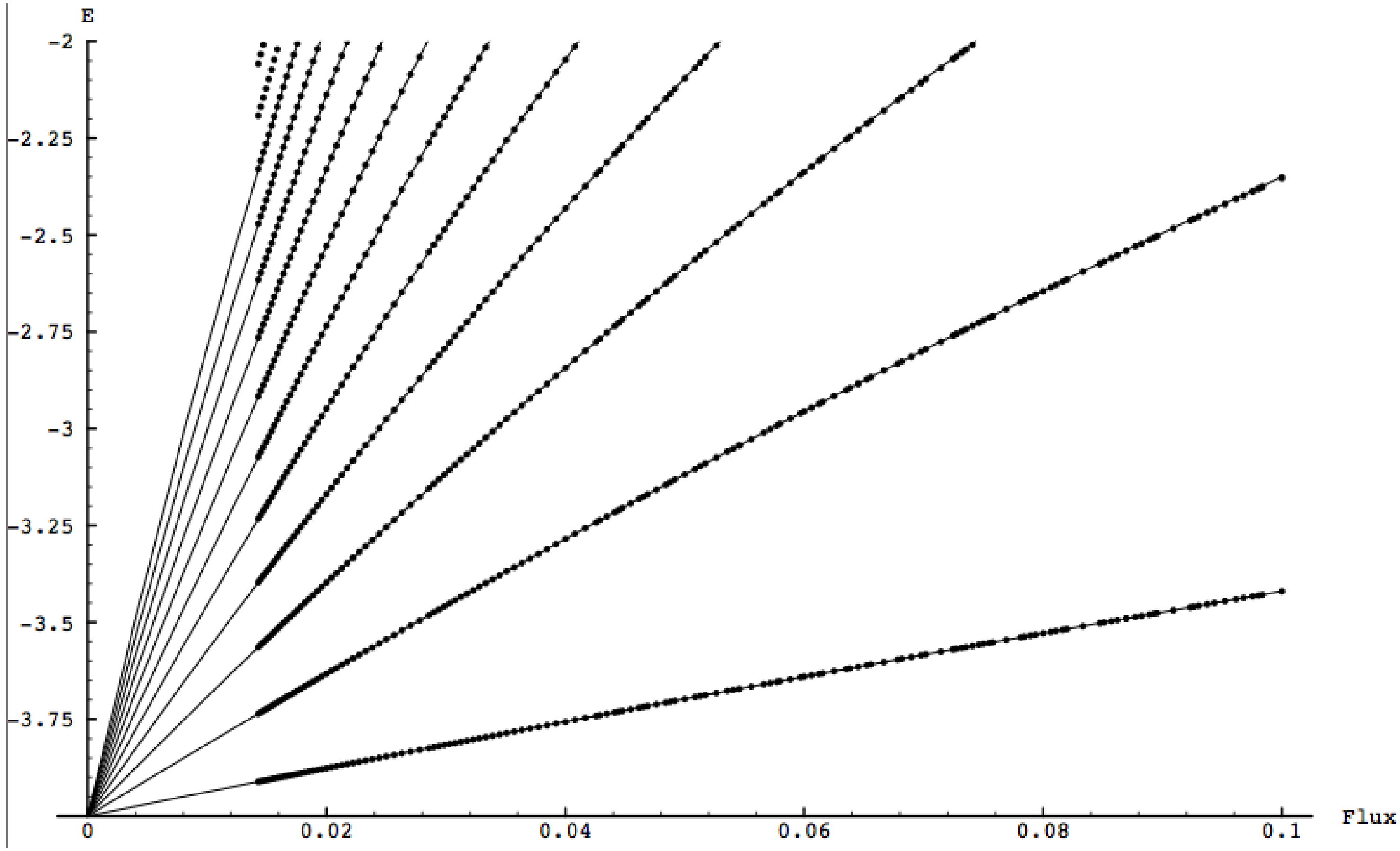
Noncommutativity of the Brillouin Zone

- If $\alpha = \phi/\phi_0 \notin \mathbb{Q}$, the Bloch theory *fails* !
- Adding a *random potential* adds up to the *failure* of Bloch theory !
- **Disordered potential:** $V_\omega(x) = W\,\omega_x\,,\ x \in \mathbb{Z}^2$ with
  - W is the disorder strength
  - $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ and the ω_x's are *i.i.d.*'s with *uniform distribution* on $[-1/2, +1/2]$
  - $\omega \in \Omega = \prod_{x\in\mathbb{Z}^2}[-1/2, +1/2]$ is compact *(Tychonov Theorem)* and $\mathbb{Z}^2$ acts by *shift*.
- The groupoid is now $\Omega \rtimes \mathbb{Z}^2$
  The observable algebra is again $\mathcal{A} = C(\Omega) \rtimes_B \mathbb{Z}^d$.

Landau Levels



Landau Levels

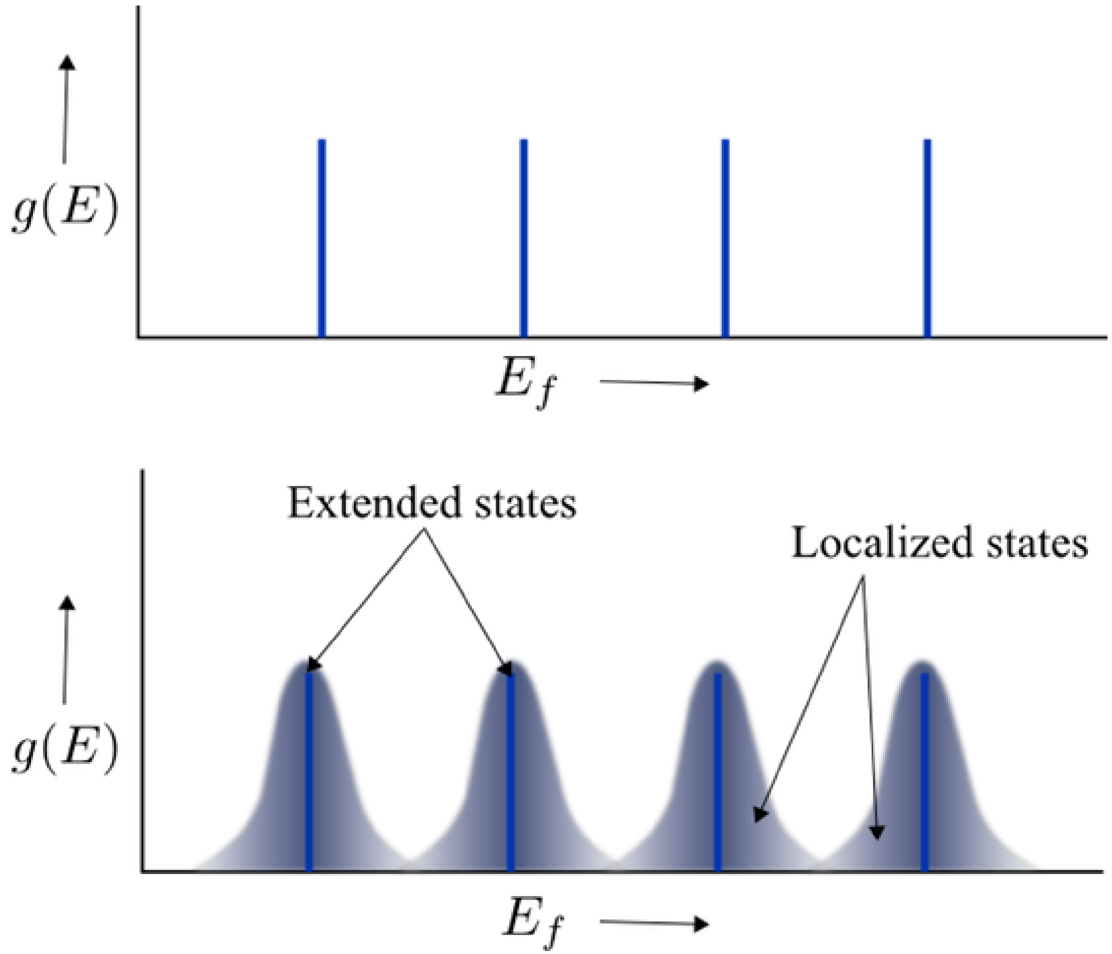


Landau levels

$E_\ell = \ell\alpha$

$\ell = 1, 2, \cdots$

Density of States



$$\int_{\mathbb{R}} f(E)\, g(E)\, dE = \mathcal{T}_{\mathbb{P}}(f(H))$$

(for $f \in C_c(\mathbb{R})$)

NO gap !

Noncommutativity of the Brillouin Zone

- The *Chern-Kubo formula* becomes

$$\sigma_H = -2\imath\pi \frac{e^2}{h} \mathcal{T}_{\mathbb{P}} (P_F \; [\partial_1 P_F \, , \, \partial_2 P_F]) = \frac{e^2}{h} \, \mathbf{Ch}(P_F)$$

- **Questions:**
  - How does one prove that $\mathbf{Ch}(P_F) \in \mathbb{Z}$?
  - How does one define $\mathbf{Ch}(P_F)$ if the Fermi level *does NOT belong* to a gap !

III - The Four Traces Way

J. Bellissard, H. Schulz-Baldes, A. van Elst, *J. Math. Phys.*, **35**, (1994), 5373-5471

Trace and Trace per Unit Volume

- For a *trace class* operator T acting on a separable Hilbert space

$$\mathrm{Tr}(T) = \sum_{n=1}^{\infty} \langle e_n | T e_n \rangle \qquad (e_n)_{n\in\mathbb{N}} \ \textbf{orthonormal basis}$$

- If Γ is a locally compact *groupoid*, with unit space Ξ equipped with an invariant *probability* measure $\mathbb{P}$

$$\mathcal{T}_{\mathbb{P}}(A) = \int_{\Xi} A(\xi, 0)\, d\mathbb{P}(\xi) \qquad A \in C_c(\Gamma)$$

Graded Trace

- **Spinors:** here $\Gamma^{\xi} \subset \mathbb{R}^2$! If $\mathcal{H}_{\xi} = L^2(\Gamma^{\xi})$ set $\widehat{\mathcal{H}}_{\xi} = \mathcal{H}_{\xi} \otimes \mathbb{C}^2$
- **Grading:**

$$G = \mathbf{1}_{\mathcal{H}_{\xi}} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad\qquad G^* = G = G^{-1}$$

- An operator $T \in \mathcal{B}(\widehat{\mathcal{H}}_{\xi})$ has *degree* $\deg(T)$ whenever

$$GT - (-1)^{\deg(T)}\, TG = 0$$

- Any operator $T \in \mathcal{B}(\widehat{\mathcal{H}}_{\xi})$ can be uniquely decomposed into $T = T_0 + T_1$ with $\deg(T_i) = i$

Graded Trace

- **Graded Commutator:**

$$[T\,,\,T']_S = TT' - (-1)^{\deg(T)\,\deg(T')}\;T'T$$

- **Dirac Operator:** if $X = X_1 + \imath X_2$ is the *position* operator

$$D = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \qquad\qquad F = \frac{D}{|D|} \;\Rightarrow\; F = F^* = F^{-1}\,,\ \deg(F) = 1$$

- **Graded Trace:**

$$\mathrm{Tr}_S(T) = \frac{1}{2}\,\mathrm{Tr}\left(GF\;[F,T]_S\right)$$

Graded Trace

- **Differential:**

$$dT = [F\,,\,T]_S$$

- **Leibniz rule:**

$$d(TT') = dT\,T' + (-1)^{\deg(T)}\,T\,dT'$$

- Tr_S is *linear* and satisfies, for dT, dT' *trace class* operator

$$\mathrm{Tr}_S(TT') = (-1)^{\deg(T)\,\deg(T')}\,\mathrm{Tr}_S(T'T)$$ **(graded trace)**

Graded Trace

- **Representation of** $C^*(\Gamma)$

$$\widehat{\pi}_\xi(A) = \begin{bmatrix} \pi_\xi(A) & 0 \\ 0 & \pi_\xi(A) \end{bmatrix} \qquad A \in \mathcal{A}_0\,, \ \deg(\widehat{\pi}_\xi(A)) = 0$$

- **Laughlin argument:** It is worth noticing that $u = X/|X|$ is a unitary operator on $\mathcal{H}_\xi$ representing the *gauge transformation* corresponding to an *adiabatic change* of a pointwise flux at the origin, from 0 to ϕ_0.

The Dixmier Trace

J. Dixmier, *C. R. Acad. Sci. Paris Sér. A-B*, **262**, (1966), A1107-A1108

- If $\mathcal{H}$ is a Hilbert space $L^p(\mathcal{H})$ denotes the *Schatten ideal* of compact operators with $\mathrm{Tr}(|T|^p) < \infty$
- If T is compact, let $\mu_1 \geq \cdots \geq \mu_n \geq 0$ be its singular values (eigenvalues of $|T|$) labelled in nonincreasing order. Then

$$\|T\|_{p+} = \left(\limsup_{N \in \mathbb{N}} \frac{1}{\ln(N+1)} \sum_{n=1}^{N} \mu^p \right)^{1/p}$$

- **Mačaev ideal:** $L^{p+}(\mathcal{H})$ is the set of T compact with $\|T\|_{p+} < \infty$

The Dixmier Trace

Theorem *Let* $L^{p-}(\mathcal{H}) = \{T \text{ compact} \,;\, ||T||_{p+} = 0\}$. *Then*

1. $L^{p\pm}(\mathcal{H})$ *are two-sided ideals in* $\mathcal{B}(\mathcal{H})$

2. *If* $0 \leq p < p' < \infty$

$$L^{p}(\mathcal{H}) \subset L^{p-}(\mathcal{H}) \subset L^{p+}(\mathcal{H}) \subset L^{p'}(\mathcal{H})$$

3. $|| \cdot ||_{p+}$ *is a seminorm making* $L^{p+}(\mathcal{H})/L^{p-}(\mathcal{H})$ *a Banach space*

The Dixmier Trace

- **Abstract nonsense:** using the theory of amenable groups, Dixmier proves the existence of a *linear form* $\Upsilon : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ such that
  - $\Upsilon(a_1, a_2, a_3 \cdots) = \Upsilon(a_2, a_3, a_4, \cdots)$
  - $\Upsilon(a_1, a_2, a_3 \cdots) = \Upsilon(a_1, a_1, a_2, a_2, \cdots)$
  - Is $a \in \ell^\infty(\mathbb{N})$ converges, then $\Upsilon(a) = \lim_{n\to\infty}(a_n)$
- **Dixmier Trace:** given such Υ, then

$$\mathrm{Tr}_\Upsilon(T) = \Upsilon\left(\frac{1}{\ln(N+1)}\sum_{n=1}^{N}\mu\right) \qquad T \in L^{1+}(\mathcal{H}),\, T \geq 0$$

The Dixmier Trace

- Then Dixmier proves that Tr_Υ extends as a *positive linear map* on $L^{p+}(\mathcal{H})$ vanishing on $L^{p-}(\mathcal{H})$ and such that

$$\mathrm{Tr}_\Upsilon(UTU^{-1}) = \mathrm{Tr}_\Upsilon(T) \qquad \mathrm{Tr}_\Upsilon(ST) = \mathrm{Tr}_\Upsilon(TS)$$

if U is unitary and $S, T \in L^{p+}(\mathcal{H})$

IV - Connes Formulae

A. Connes, *Noncommutative Geometry*, Acad. Press, (1994)

First Connes Formula

- If $A \in C^*(\Gamma)$ then for $\mathbb{P}$-almost all ξ's and all Υ

$$\mathcal{T}_{\mathbb{P}}\left(|\vec{\nabla} A|^2\right) \stackrel{def}{=} \mathcal{T}_{\mathbb{P}}(|\partial_1 A|^2 + |\partial_2 A|^2) = \frac{1}{\pi}\,\mathrm{Tr}_{\Upsilon}\left(|d\pi_{\xi}(A)|^2\right)$$

- If $\mathcal{S}$ denotes the *Sobolev space* generated by $A \in \mathcal{A}_0$ such that $\mathcal{T}_{\mathbb{P}}(|A|^2 + |\vec{\nabla} A|^2) < \infty$ then

$$A \in \mathcal{S} \;\Rightarrow\; d\pi_{\xi}(A) \in L^{2+}(\mathcal{H})$$

Second Connes Formula

- **A cyclic 2-cocycle:** for $A_0, A_1, A_2 \in \mathcal{S}$

$$\mathcal{T}_2(A_0, A_1, A_2) = 2\imath\pi\, \mathcal{T}_{\mathbb{P}}\left(A_0\left(\partial_1 A_1 \partial_2 A_2 - \partial_2 A_1 \partial_1 A_2\right)\right)$$

- **Cyclicity:**

$$\mathcal{T}_2(A_0, A_1, A_2) = \mathcal{T}_2(A_2, A_0, A_1)$$

- $\mathcal{T}_2$ **is Hochschild-closed:**

$$\begin{aligned}(b\mathcal{T}_2)(A_0, A_1, A_2, A_3) &= \mathcal{T}_2(A_0 A_1, A_2, A_3) - \mathcal{T}_2(A_0, A_1 A_2, A_3) \\ &+ \mathcal{T}_2(A_0, A_1, A_2 A_3) - \mathcal{T}_2(A_3 A_0, A_1, A_2) \\ &= 0\end{aligned}$$

Second Connes Formula

- for $A_0, A_1, A_2 \in \mathcal{S}$

$$\mathcal{T}_2(A_0, A_1, A_2) = \int_{\Xi} \mathrm{Tr}_{\mathcal{S}}(\widehat{\pi}_{\xi}(A_0)\, d\widehat{\pi}_{\xi}(A_1)\, d\widehat{\pi}_{\xi}(A_2))\, d\mathbb{P}(\xi)$$

- $$\mathrm{Tr}_{\mathcal{S}}(\widehat{\pi}_{\xi}(A_0)\, d\widehat{\pi}_{\xi}(A_1)\, d\widehat{\pi}_{\xi}(A_2)) = \frac{1}{2}\,\mathrm{Tr}(G\, d\widehat{\pi}_{\xi}(A_0)\, d\widehat{\pi}_{\xi}(A_1)\, d\widehat{\pi}_{\xi}(A_2))$$

- $A_i \in \mathcal{S} \;\Rightarrow\; d\widehat{\pi}_{\xi}(A) \in L^{2+}(\mathcal{H}) \subset L^3(\mathcal{H})$ so that the *r.h.s* is well defined

Fredholm Index

- **Integrality:** *(Connes)* If P is a projection on $\mathcal{H}_\xi$ then set $\widehat{P} = P \otimes \mathbf{1}_2$. If $d\widehat{P} \in L^3(\mathcal{H})$ then PuP is *Fredholm* and

$$\mathrm{Tr}_S(\widehat{P}\, d\widehat{P}\, d\widehat{P}) = \mathrm{Ind}(PuP) \in \mathbb{Z}$$

- **Fedosov formula:** $d\widehat{P} \in L^3(\mathcal{H}) \Leftrightarrow (PuP - P) \in L^3(\mathcal{H})$ and

$$\mathrm{Ind}(PuP) = \mathrm{Tr}((PuP - P)^{2n+1}) \qquad \forall n \geq 1$$

- Hence $\mathrm{Ind}(PuP)$ measure the *change of dimension* of P under the Laughlin gauge transformation, namely the *number of charges* sent to infinity *(Avron, Seiler, Simon)*

Quantization of the Chern Number

- If $P_F \in \mathcal{S}$, then $d\widehat{\pi}_\xi(P_F) \in L^{2+}(\mathcal{H}_\xi) \subset L^3(\mathcal{H}_\xi)$ *(1st Connes formula)*
- Then *(2nd Connes formula)*

$$\mathbf{Ch}(P_F) = \int_\Xi \mathrm{Tr}_{\mathcal{S}}(\widehat{\pi}_\xi(P_F)\, d\widehat{\pi}_\xi(P_F)\, \widehat{\pi}_\xi(P_F))\, d\mathbb{P}(\xi) = \int_\Xi n(\xi)\, d\mathbb{P}(\xi)$$

- Since it is a Fredholm index $n(\xi) \in \mathbb{Z}$. By covariance it is *translation invariant*. Since $P_F \in \mathcal{S}$ it follows that $n(\xi)$ is *measurable* in ξ. Since $\mathbb{P}$ is *ergodic*, then $n(\xi)$ is almost surely constant. Hence

$$P_F \in \mathcal{S} \Rightarrow \mathbf{Ch}(P_F) \in \mathbb{Z}$$

which measures the *number of charges* sent to infinity.

Localization

The condition $P_F \in \mathcal{S}$ is implied by the condition that the Fermi level E_F lies in a region of *localized states*

Thanks for listening !