VARIOUS MATHEMATICAL ASPECTS of TILING SPACES

Jean BELLISSARD [1 2]

Georgia Institute of Technology
&
Institut Universitaire de France

Collaborations:

D. SPEHNER (Essen, Germany)

M. ZARROUATI (IRSAMC, Toulouse)

D. HERRMANN (T.U. Berlin)

J. KELLENDONK (U. Cardiff, United Kingdom)

A. LEGRAND (U. Paul Sabatier, Toulouse)

J.M. GAMBAUDO (U. Dijon, France)

R. BENEDETTI (U. Pisa, Italy)

[1] Georgia Institute of Technology, School of Mathematics, Atlanta GA 30332-0160
[2] e-mail: jeanbel@math.gatech.edu

Main References

A. Connes, *Sur la théorie non commutative de l'intégration*,
Lecture Notes in Math **725**, 19-143, Springer, Berlin (1979).

J. Bellissard, *The Gap Labelling Theorems for Schrödinger's Operators*,
in *From Number Theory to Physics*, pp. 538-630, Les Houches March 89, Springer,
J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).

J.C. Lagarias, P.A.B. Pleasant, *Repetitive Delone sets and perfect quasicrystals*,
in `math.DS/9909033` (1999).

J. Bellissard, D. Herrmann, M. Zarrouati, *Hull of Aperiodic Solids and Gap Labelling Theorems*,
In *Directions in Mathematical Quasicrystals*, CRM Monograph Series,
Volume **13**, (2000), 207-259, M.B. Baake & R.V. Moody Eds., AMS Providence.

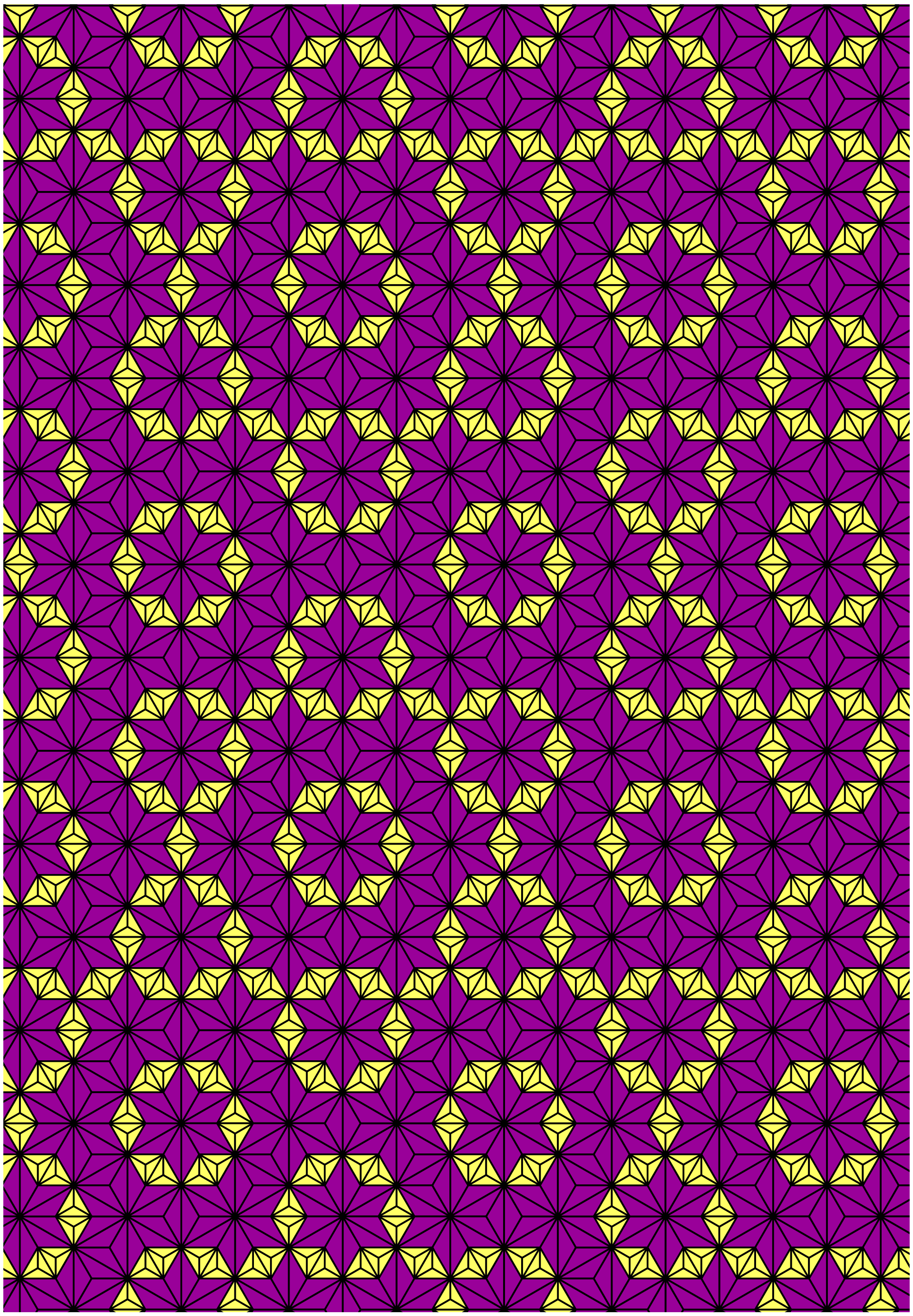
J. Bellissard, R. Benedetti, J. M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations, and Gap-Labelling*, `math.DS/0109062`, (2001).

J. Bellissard, *Noncommutative Geometry of Aperiodic Solids*,
in *Geometry and Topology Methods for Quantum Field Theory*, (Villa de Leyva, 2001), pp. 86-156,
World Sci. Publishing, River Edge, NJ, (2003).

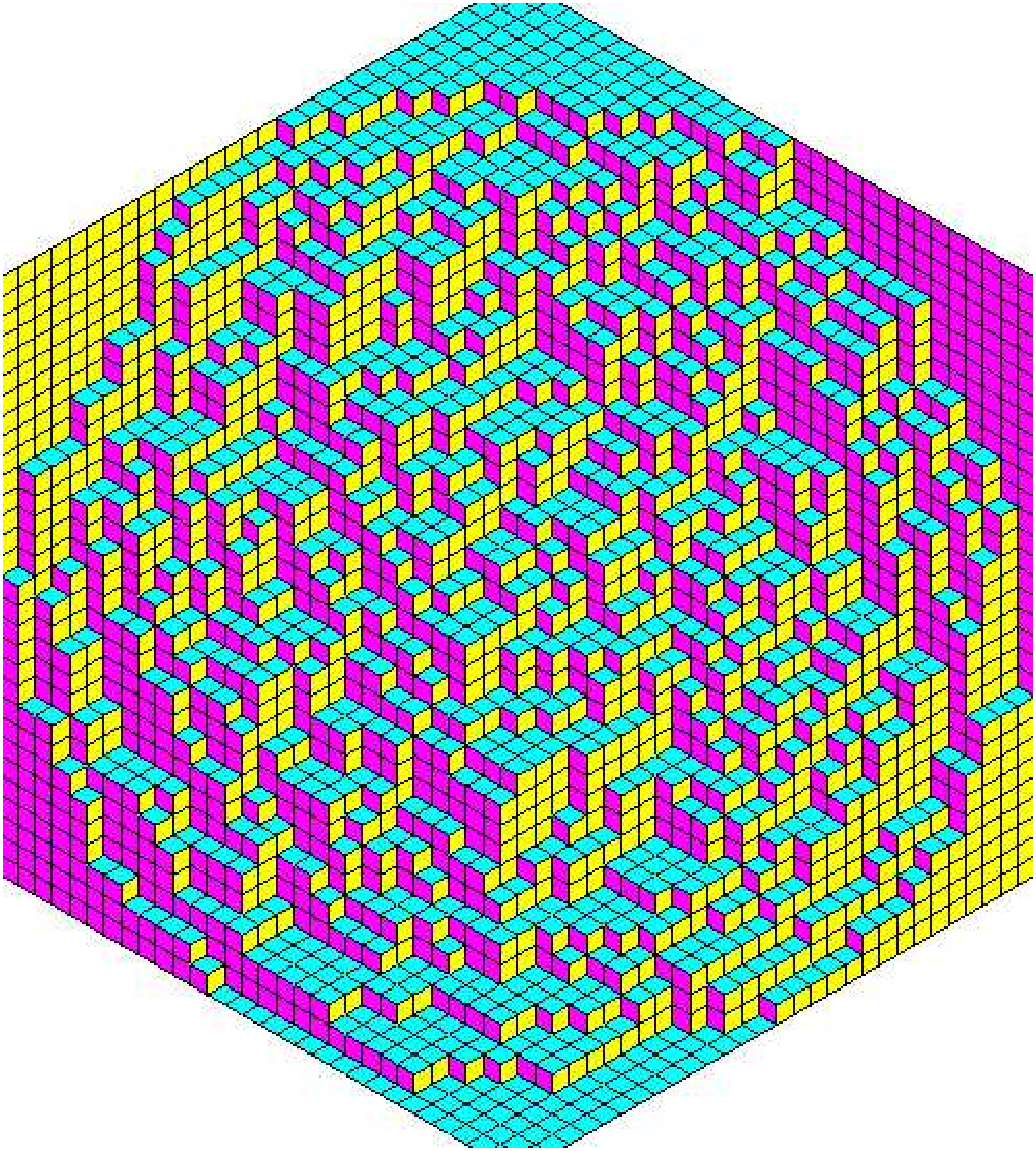
Content

1. Tilings, Tilings,...
2. Delone Sets
3. The Hull as a Dynamical System
4. Laminations
5. Branched Oriented Flat Riemannian Manifolds

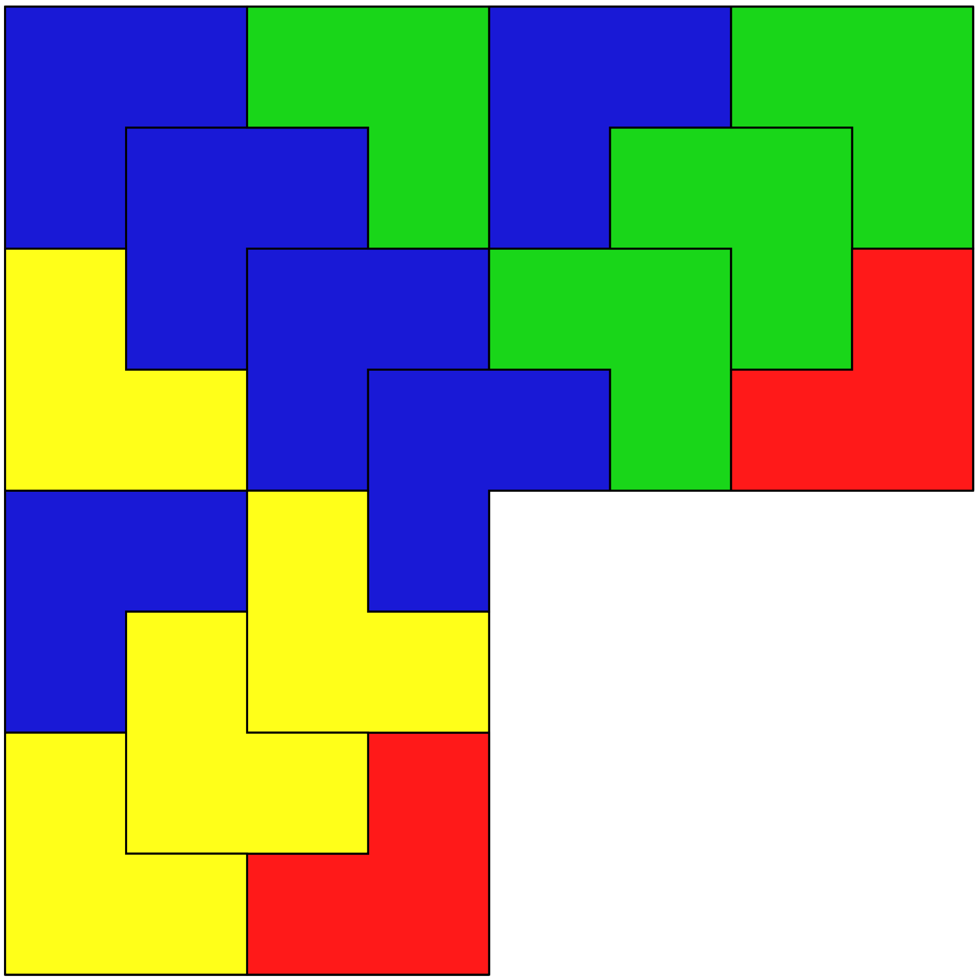
I - Tilings, Tilings,...



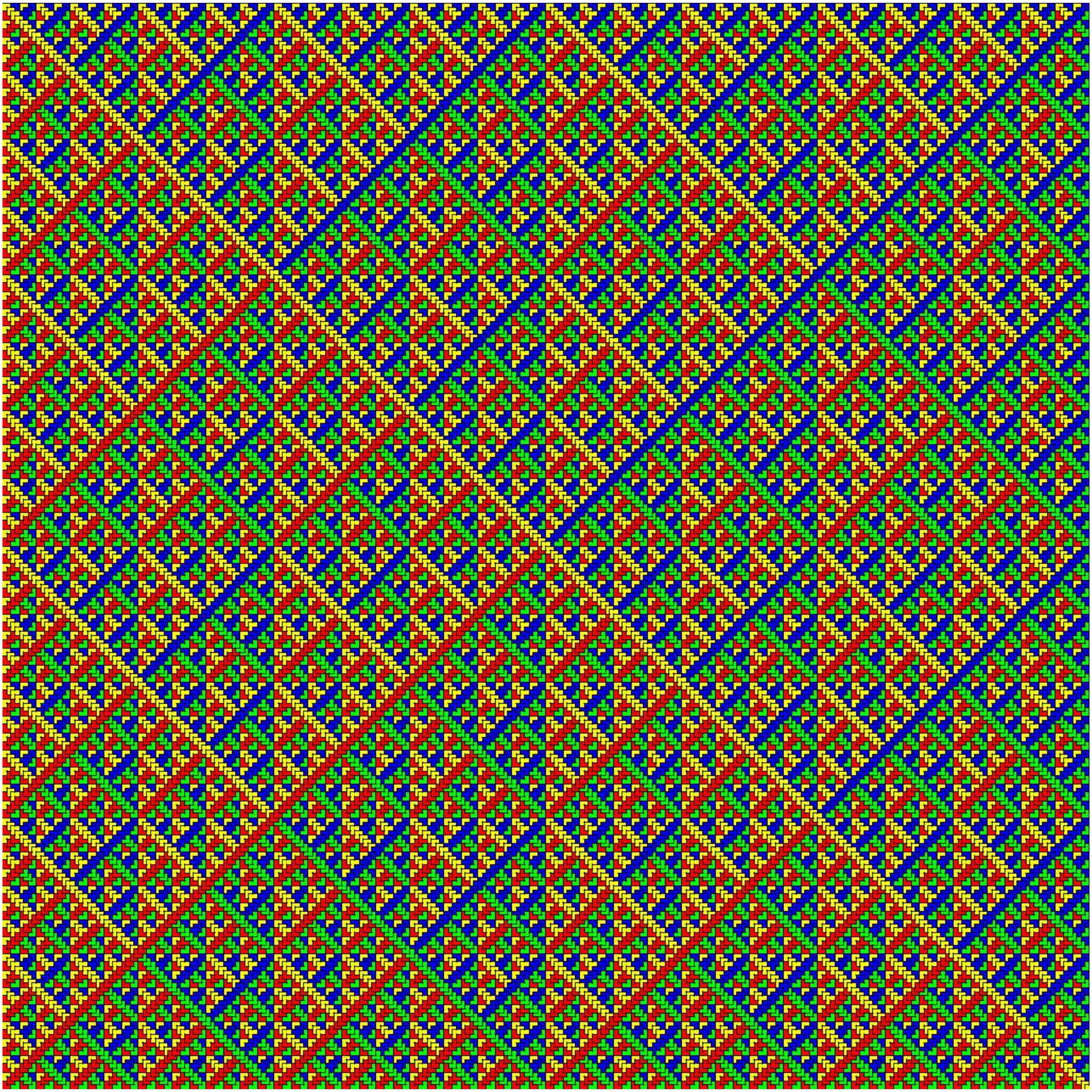
- A triangle tiling -



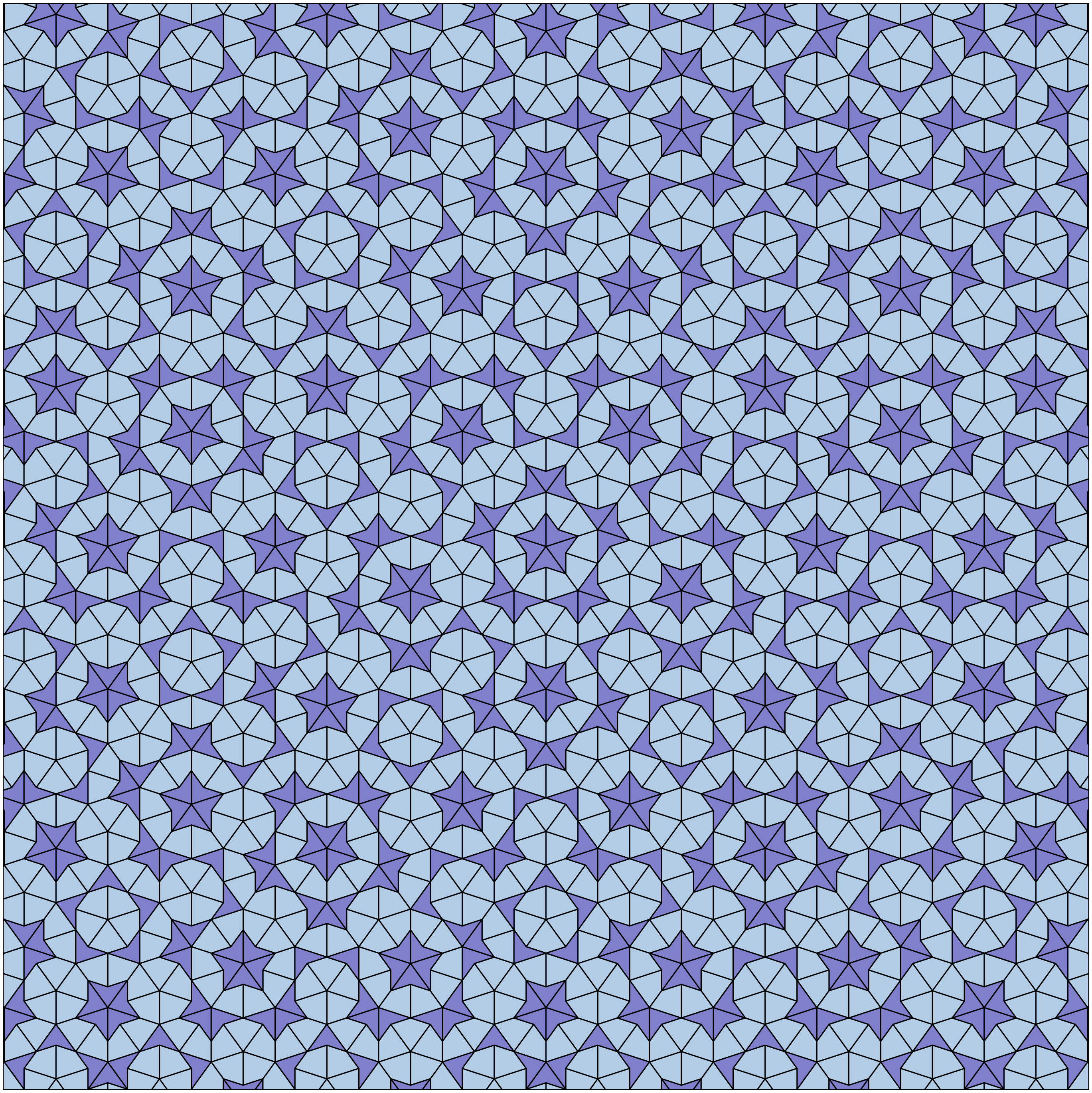
- Dominos on a triangular lattice -



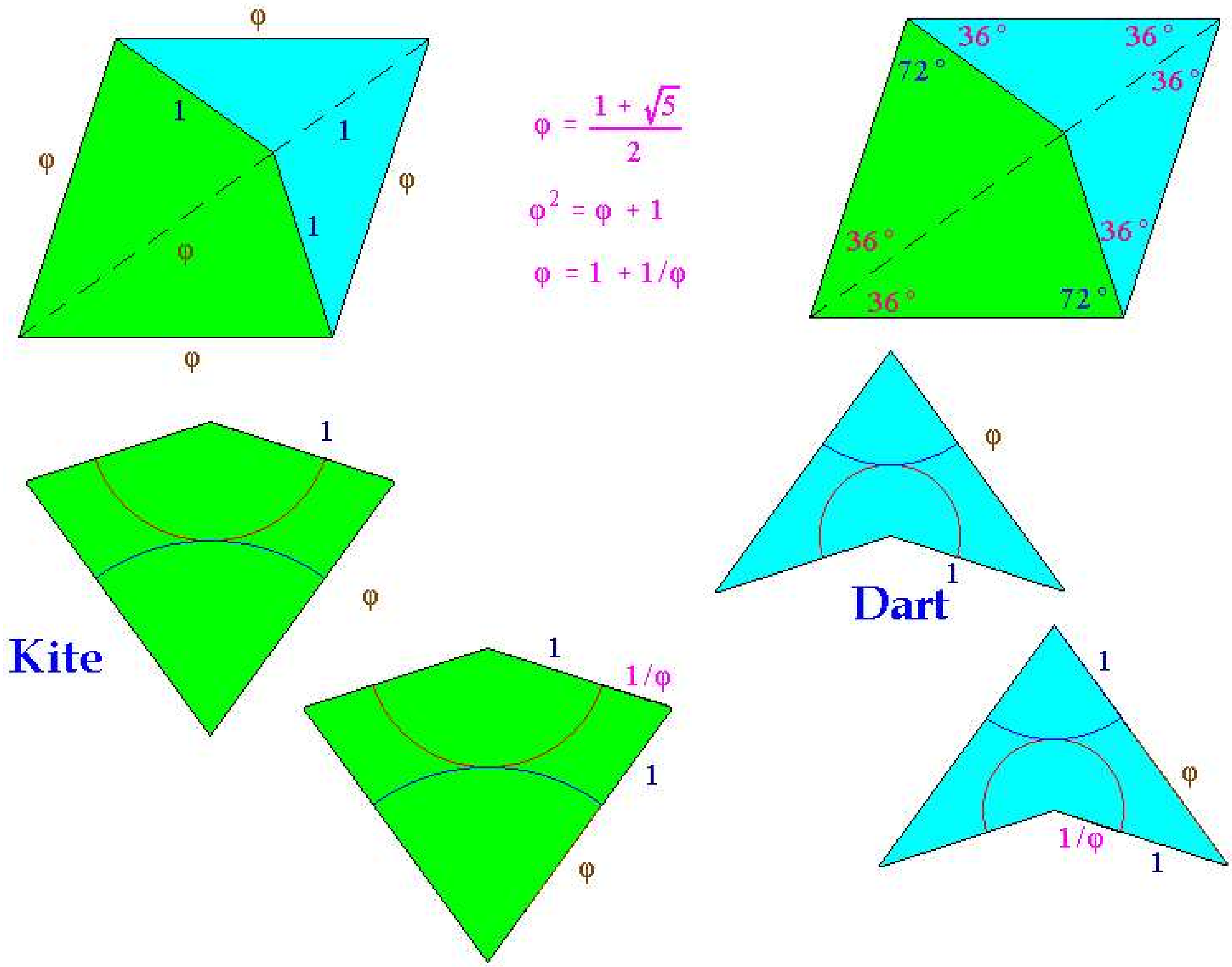
- Building the chair tiling -



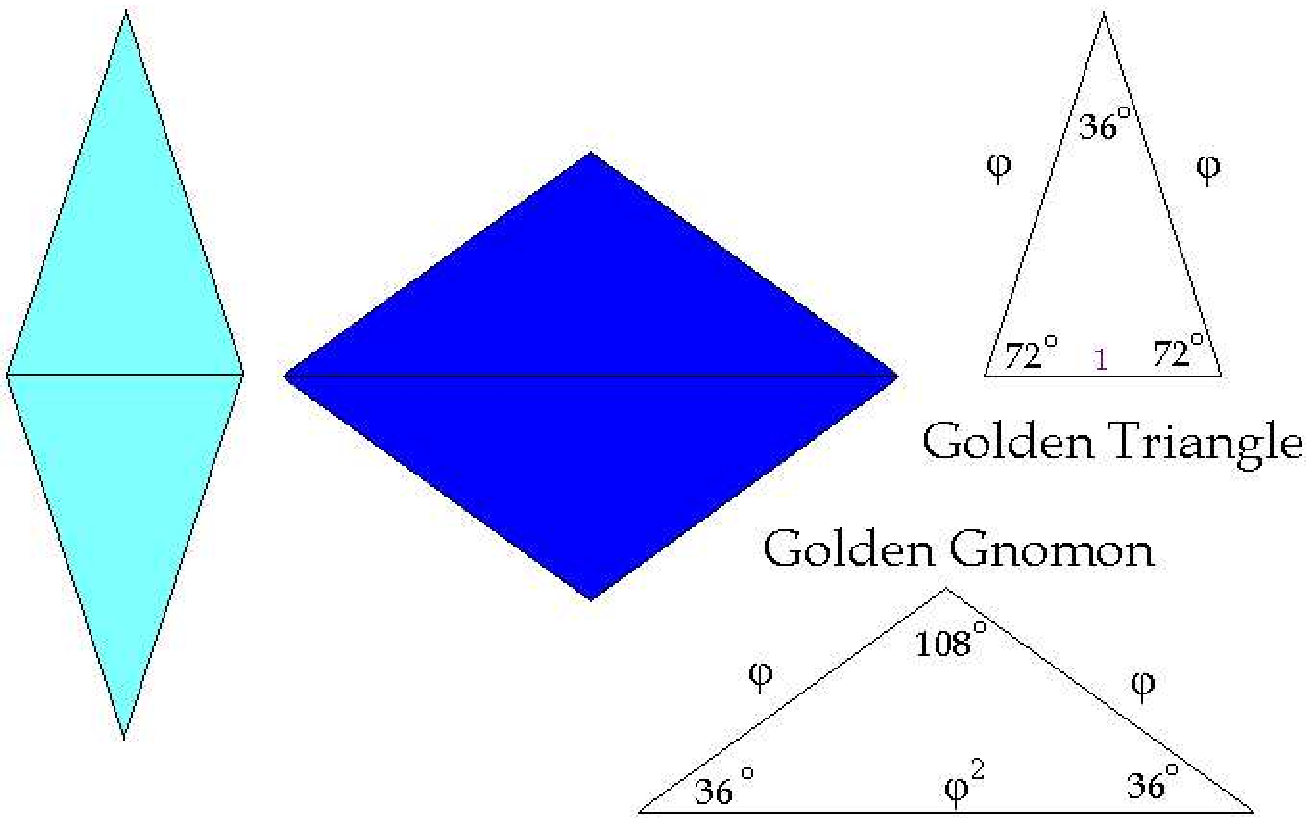
- The chair tiling -



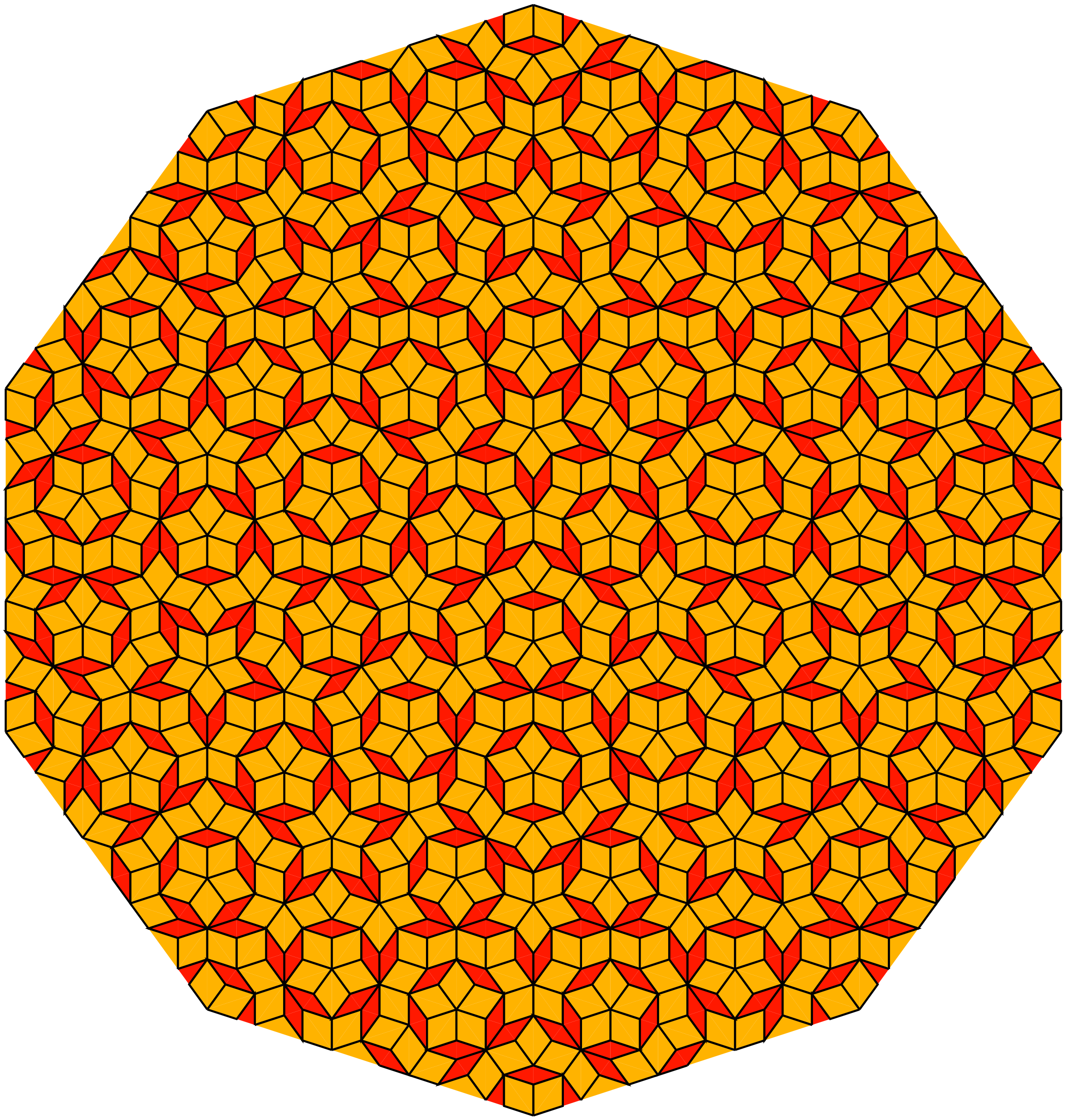
- The Penrose tiling -



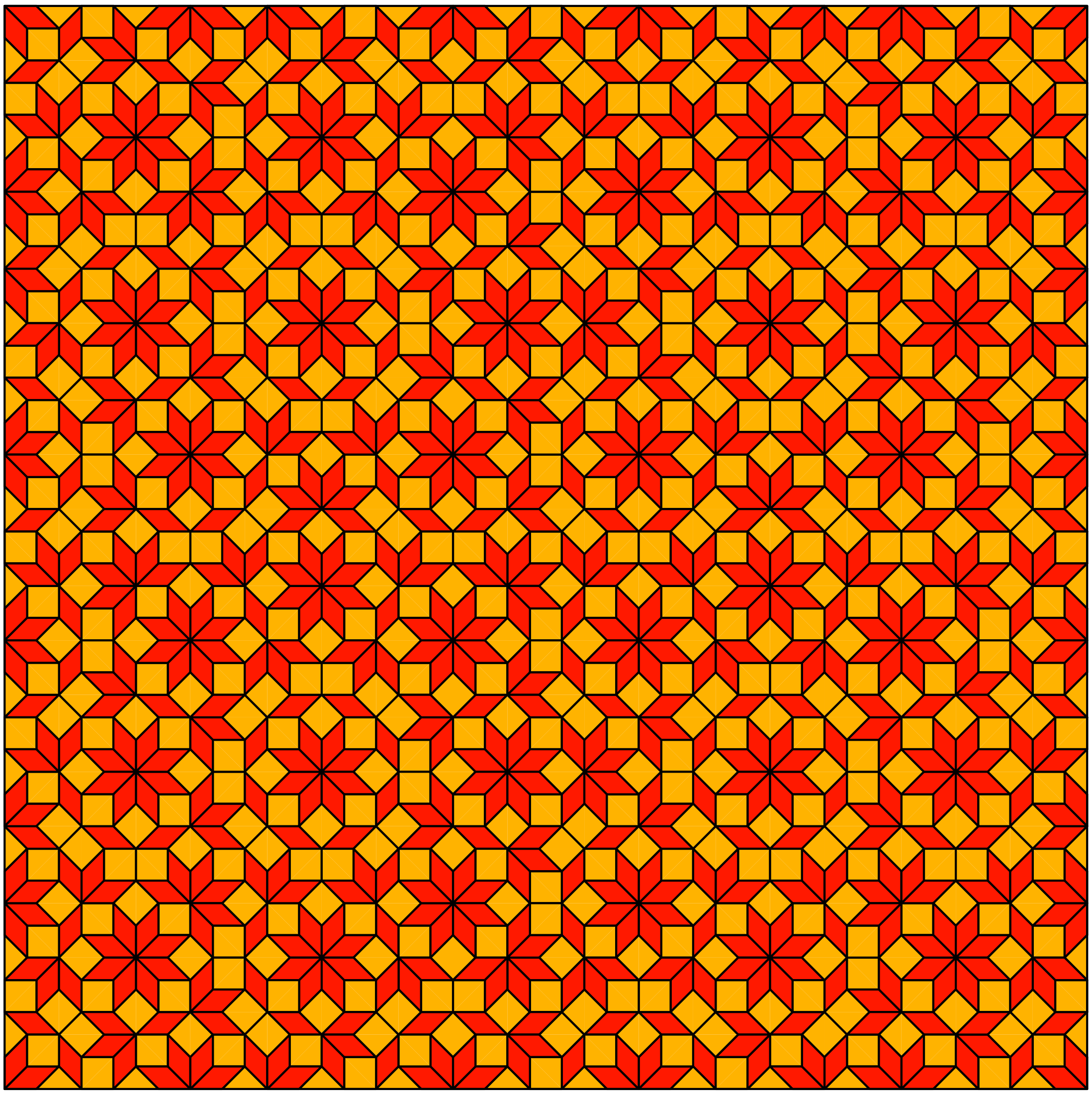
- Kites and Darts -



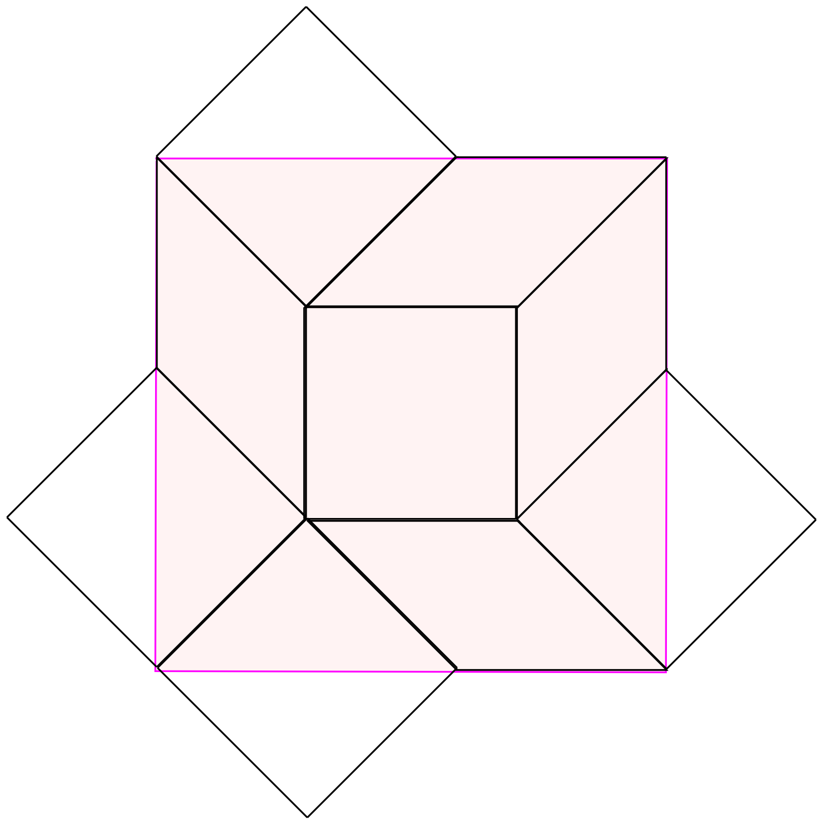
- Rhombi in Penrose's tiling -

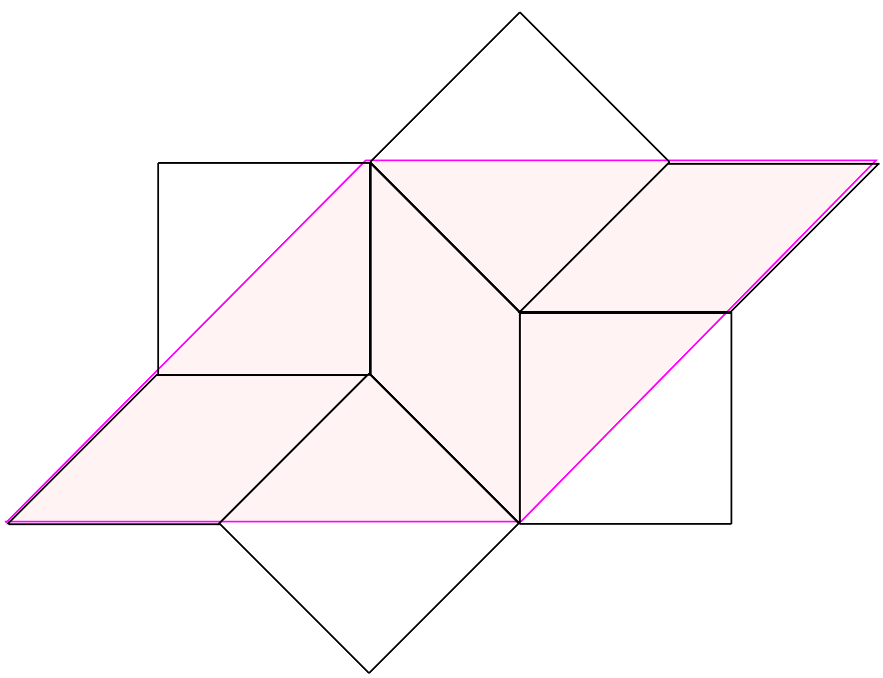


- The Penrose tiling -

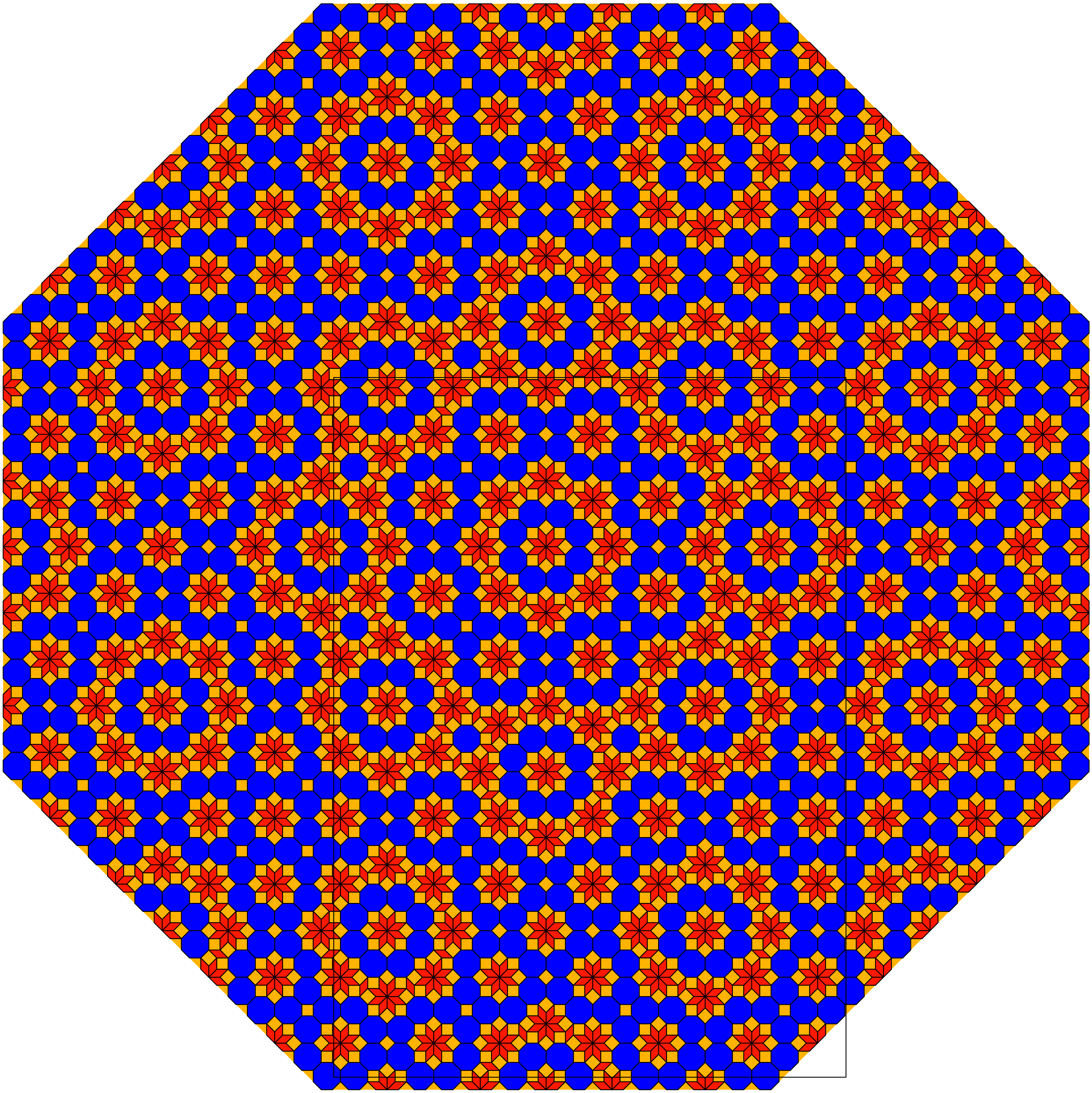


- The octagonal tiling -

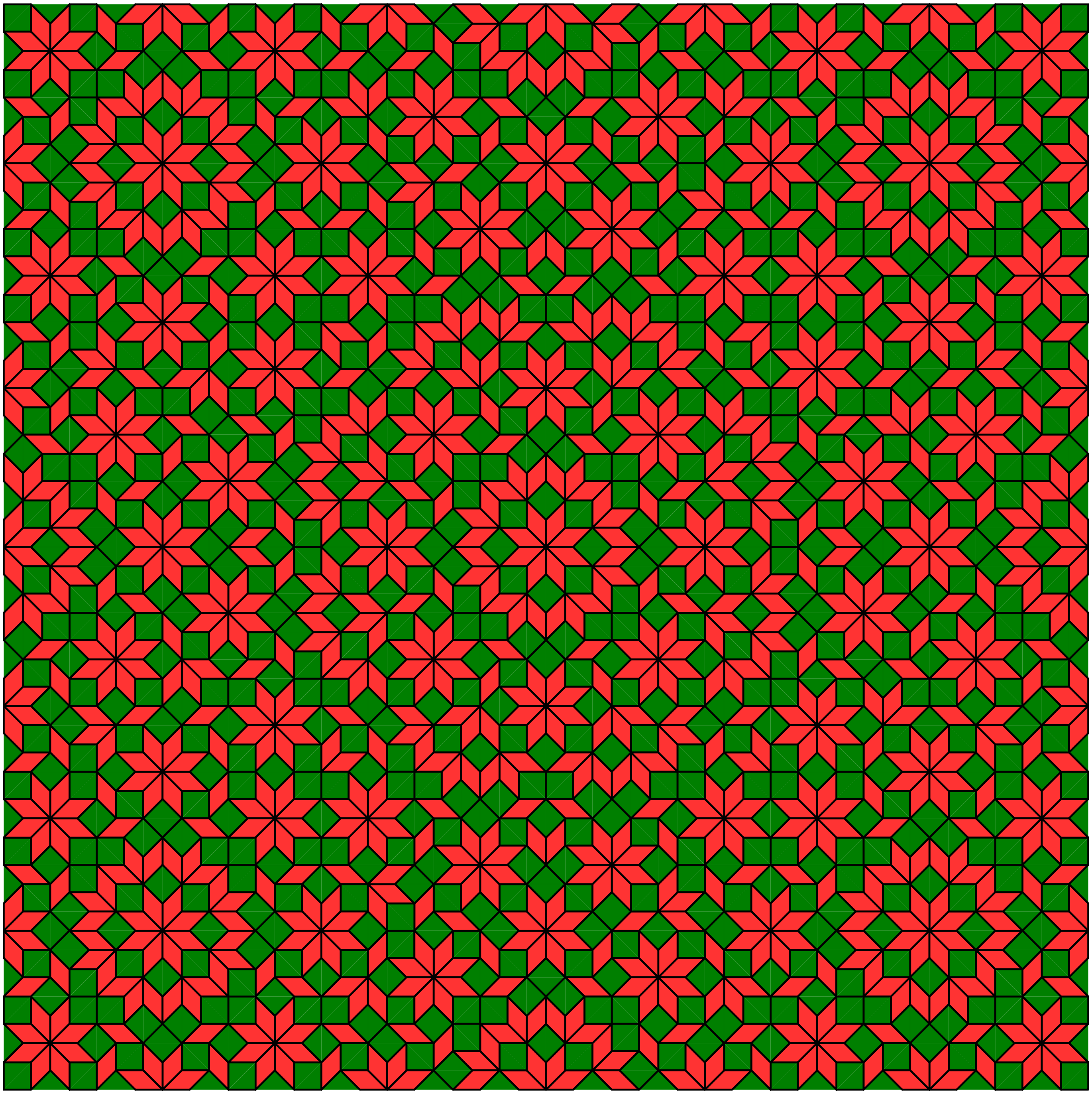




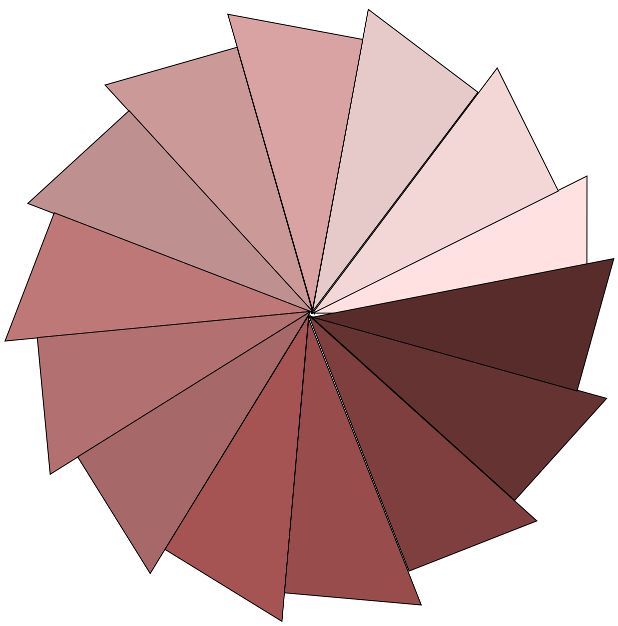
- Octagonal tiling: inflation rules -

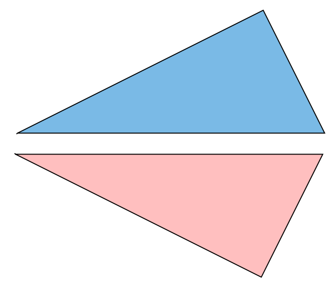


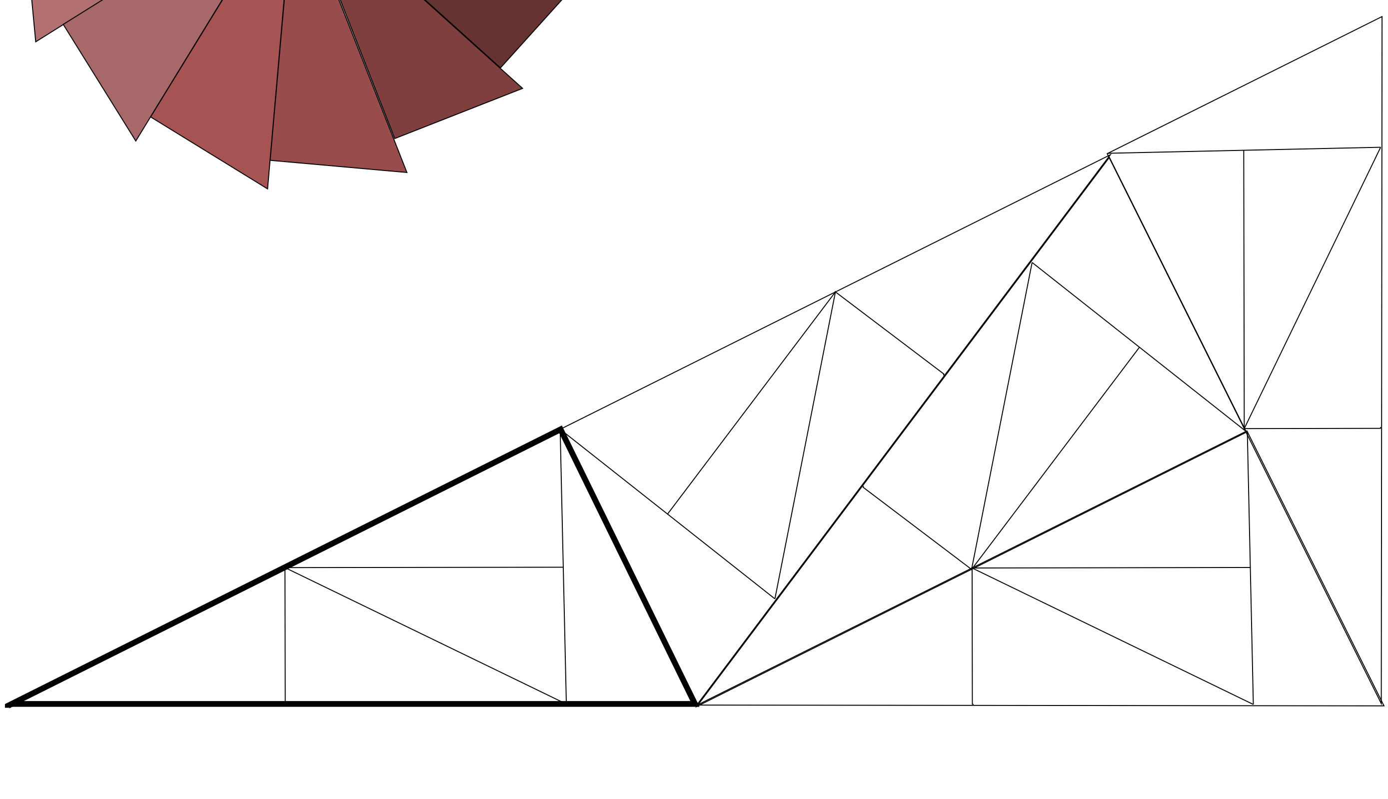
- Another octagonal tiling -



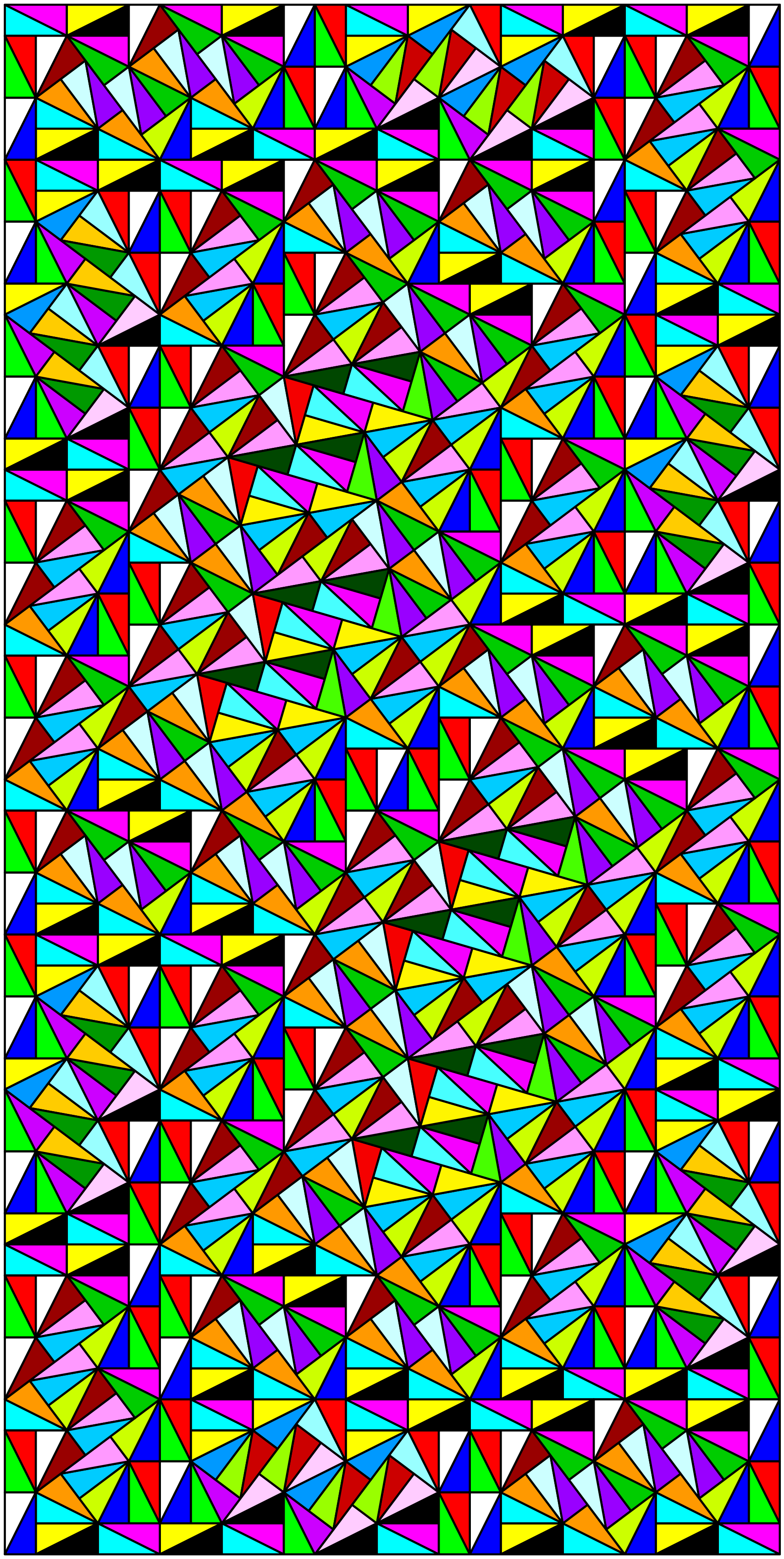
- Another octagonal tiling -



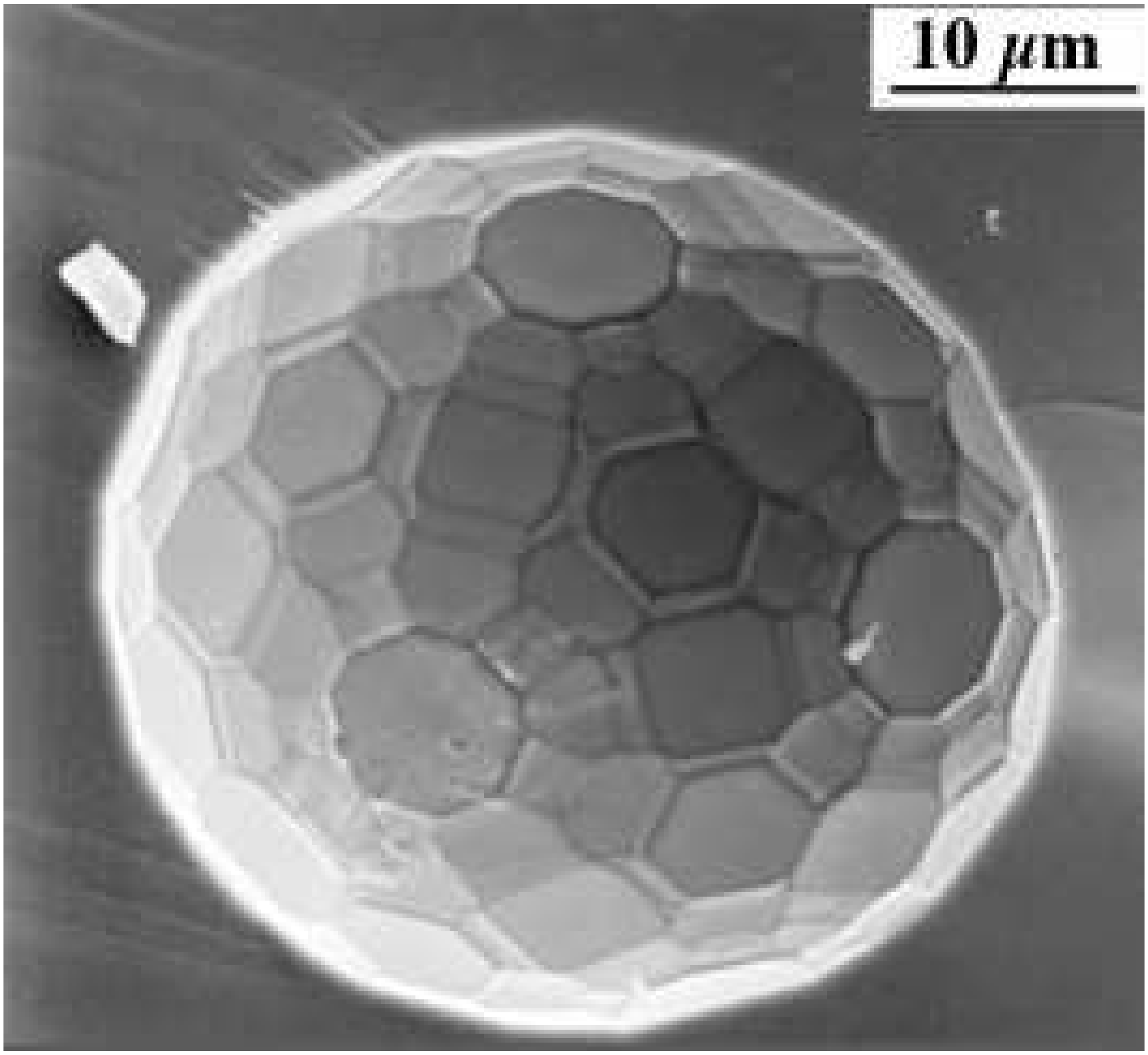




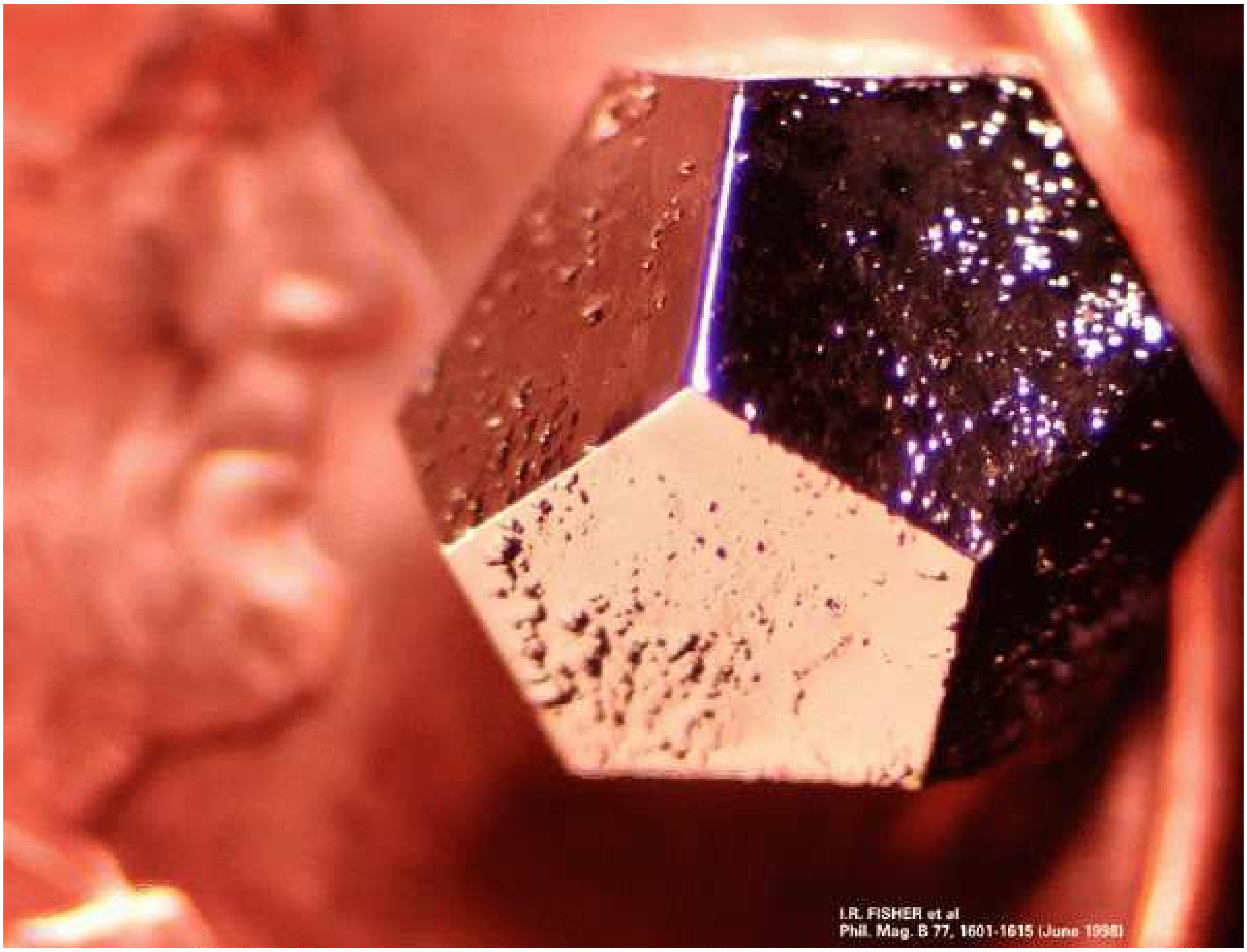
- Building the Pinwheel Tiling -



- The Pinwheel Tiling -



- The icosahedral quasicrystal $AlPdMn$ -



- The icosahedral quasicrystal $HoMgZn$-

II - The Hull as a Dynamical System

Point Sets

A subset $\mathcal{L} \subset \mathbb{R}^d$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.
4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.
5. *Finite type Delone* set: $\mathcal{L} - \mathcal{L}$ is discrete.
6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $\mathcal{C}_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x - y) \quad \in \mathfrak{M}(\mathbb{R}^d) \; .$$

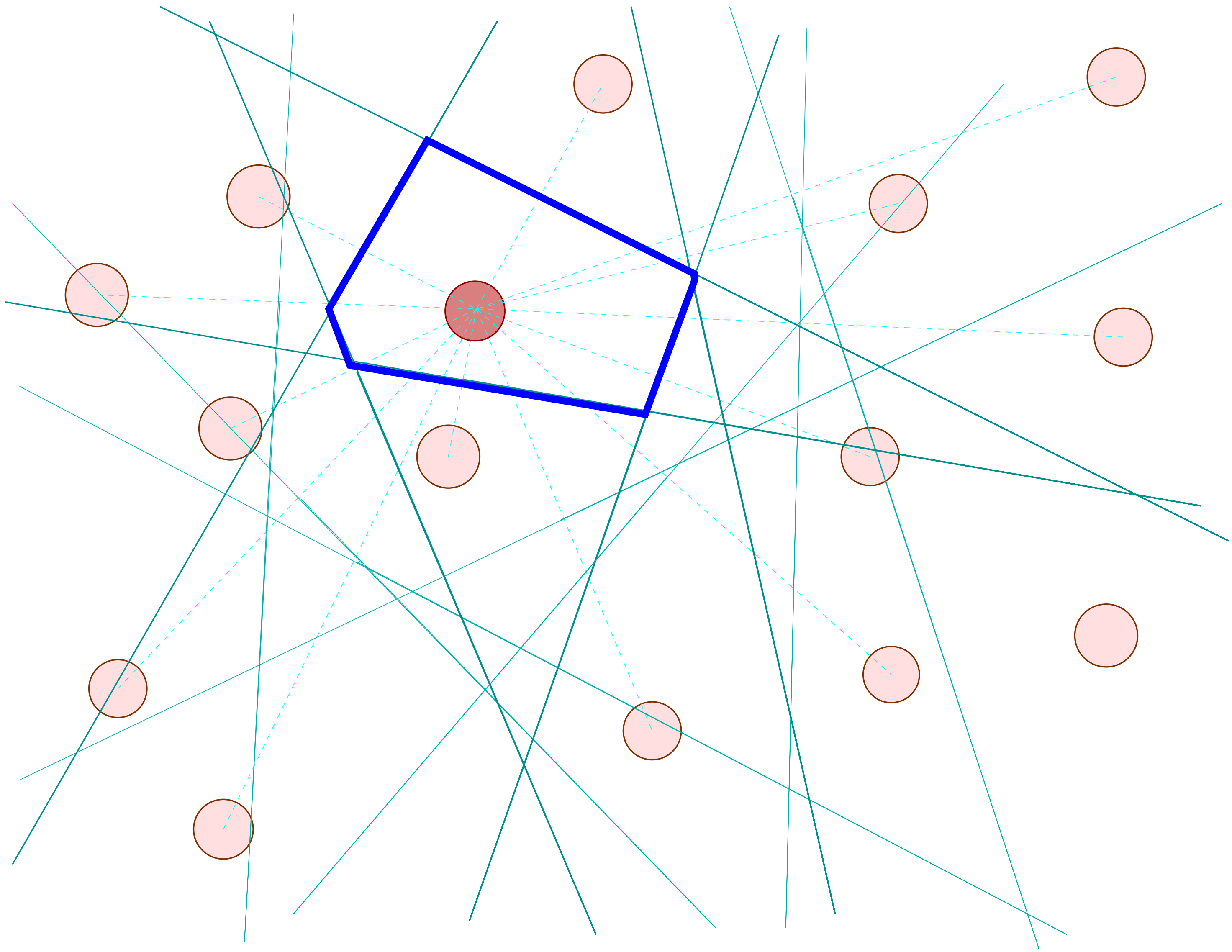
Point Sets and Tilings

Given a tiling with finitely many tiles *(modulo translations)*, a Delone set is obtained by defining a point in the interior of each *(translation equivalence class of)* tile.

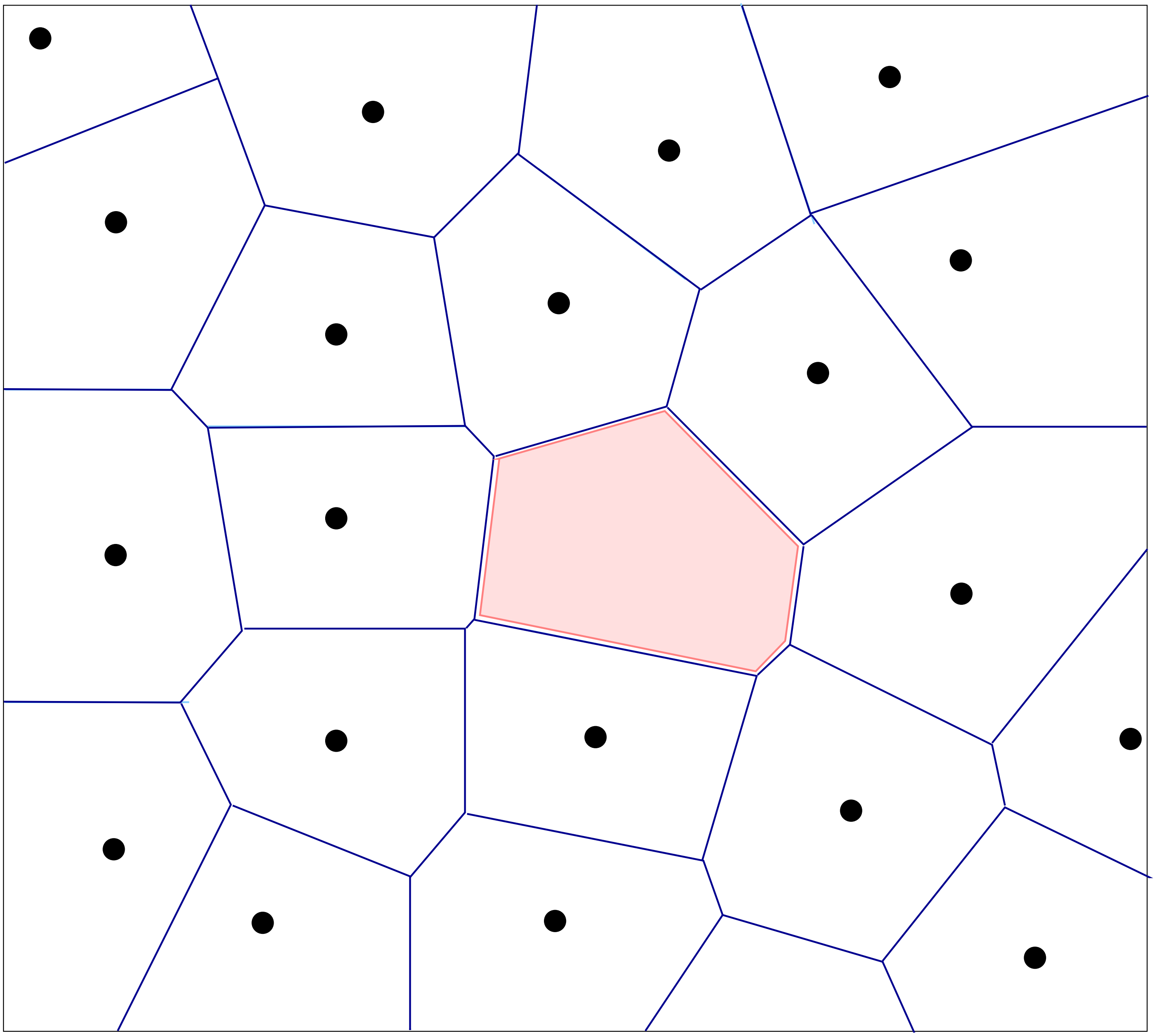
Conversely, given a Delone set, a tiling is built through the *Voronoi cells*

$$V(x) = \{a \in \mathbb{R}^d ; |a - x| < |a - y| , \forall y \mathcal{L} \setminus \{x\}\}$$

1. $V(x)$ is an *open convex polyhedron* containing $B(x;r)$ and contained into $\overline{B(x;R)}$.
2. Two Voronoi cells touch face-to-face.
3. If $\mathcal{L}$ has finite type, then the Voronoi tiling has finitely many tiles modulo translations.



- Building a Voronoi cell-



- A Delone set and its Voronoi Tiling-

The Hull

A point measure is $\mu \in \mathfrak{M}(\mathbb{R}^d)$ such that $\mu(B) \in \mathbb{N}$ for any ball $B \subset \mathbb{R}^d$. Its support is

1. *Discrete*.
2. *r-Uniformly discrete*: iff $\forall B$ ball of radius r, $\mu(B) \leq 1$.
3. *R-Relatively dense*: iff for each ball B of radius R, $\mu(B) \geq 1$.

$\mathbb{R}^d$ acts on $\mathfrak{M}(\mathbb{R}^d)$ by translation.

Theorem 1 *The set of r-uniformly discrete point measures is compact and $\mathbb{R}^d$-invariant.*
Its subset of R-relatively dense measures is compact and $\mathbb{R}^d$-invariant.

Definition 1 *Given $\mathcal{L}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{L}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $\nu^{\mathcal{L}}$.*

Hence the Hull is a *compact metrizable space* on which *$\mathbb{R}^d$ acts by homeomorphisms*.

Properties of the Hull

If $\mathcal{L} \subset \mathbb{R}^d$ is r-uniformly discrete with Hull Ω then using compactness

1. each point $\omega \in \Omega$ *is an r-uniformly discrete* point measure with support $\mathcal{L}_\omega$.
2. if $\mathcal{L}$ is *(r, R)-Delone*, so are all $\mathcal{L}_\omega$'s.
3. if $\mathcal{L}$ has *finite type*, so are all the $\mathcal{L}_\omega$'s.
   Moreover then $\mathcal{L} - \mathcal{L} = \mathcal{L}_\omega - \mathcal{L}_\omega$ $\forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{L}_\omega$.*

Theorem 2 *If $\mathcal{L}$ has finite type, then its transversal is completely discontinuous.*

Minimality

A *patch* is a finite subset of $\mathcal{L}$ of the form

$$p = (\mathcal{L} - x) \cap \overline{B(0, r_1)} \qquad x \in \mathcal{L}\,,\ r_1 \geq 0$$

$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

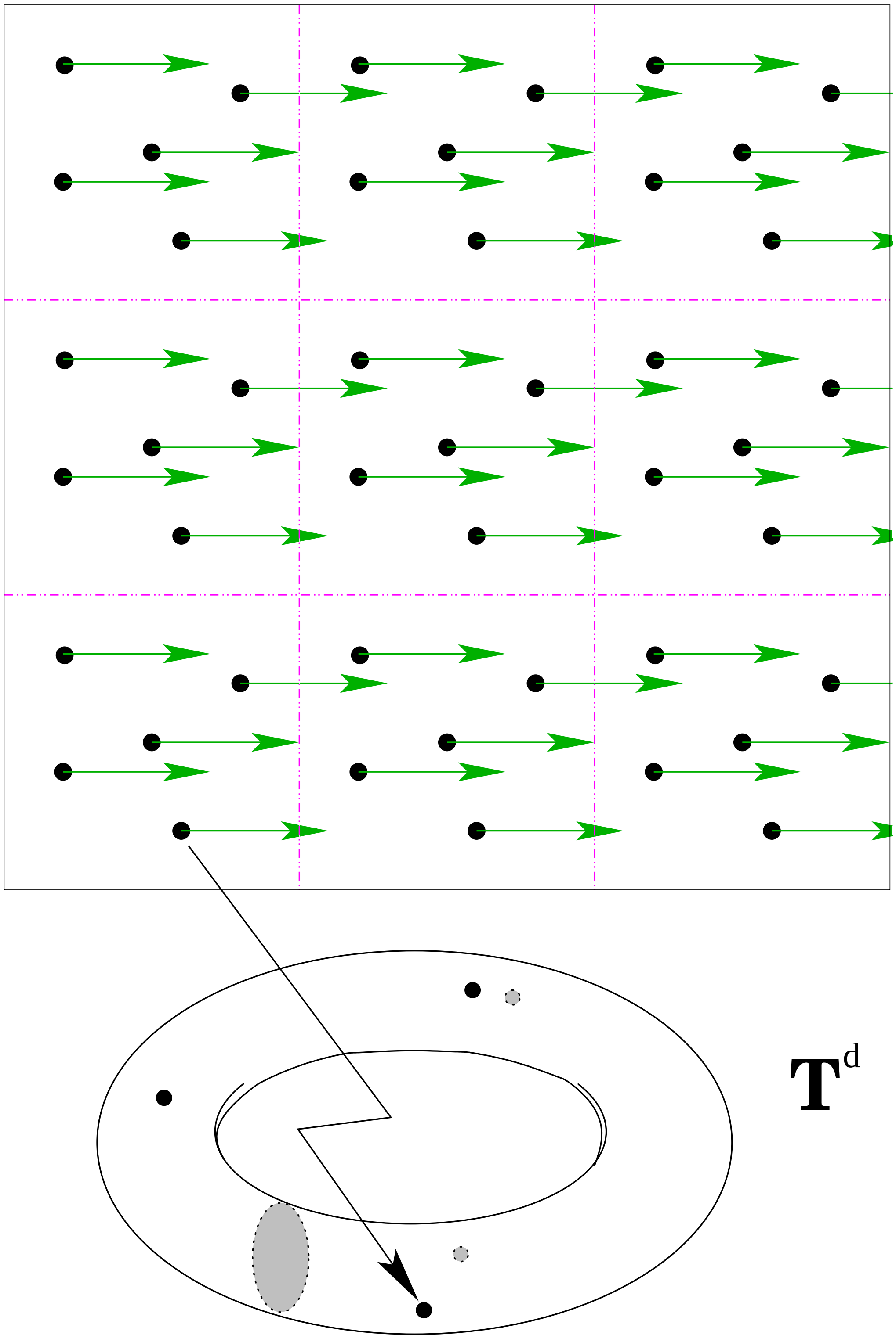
Theorem 3 *$\mathbb{R}^d$ acts minimaly on Ω if and only if $\mathcal{L}$ is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Quasicrystals :* $\Omega \simeq \mathbb{T}^n,\ n > d$ with an irrational action of $\mathbb{R}^d$ and a completely discontinuous topology in the transverse direction to the $\mathbb{R}^d$-orbits. The transversal is a Cantor set.

3. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for Si atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

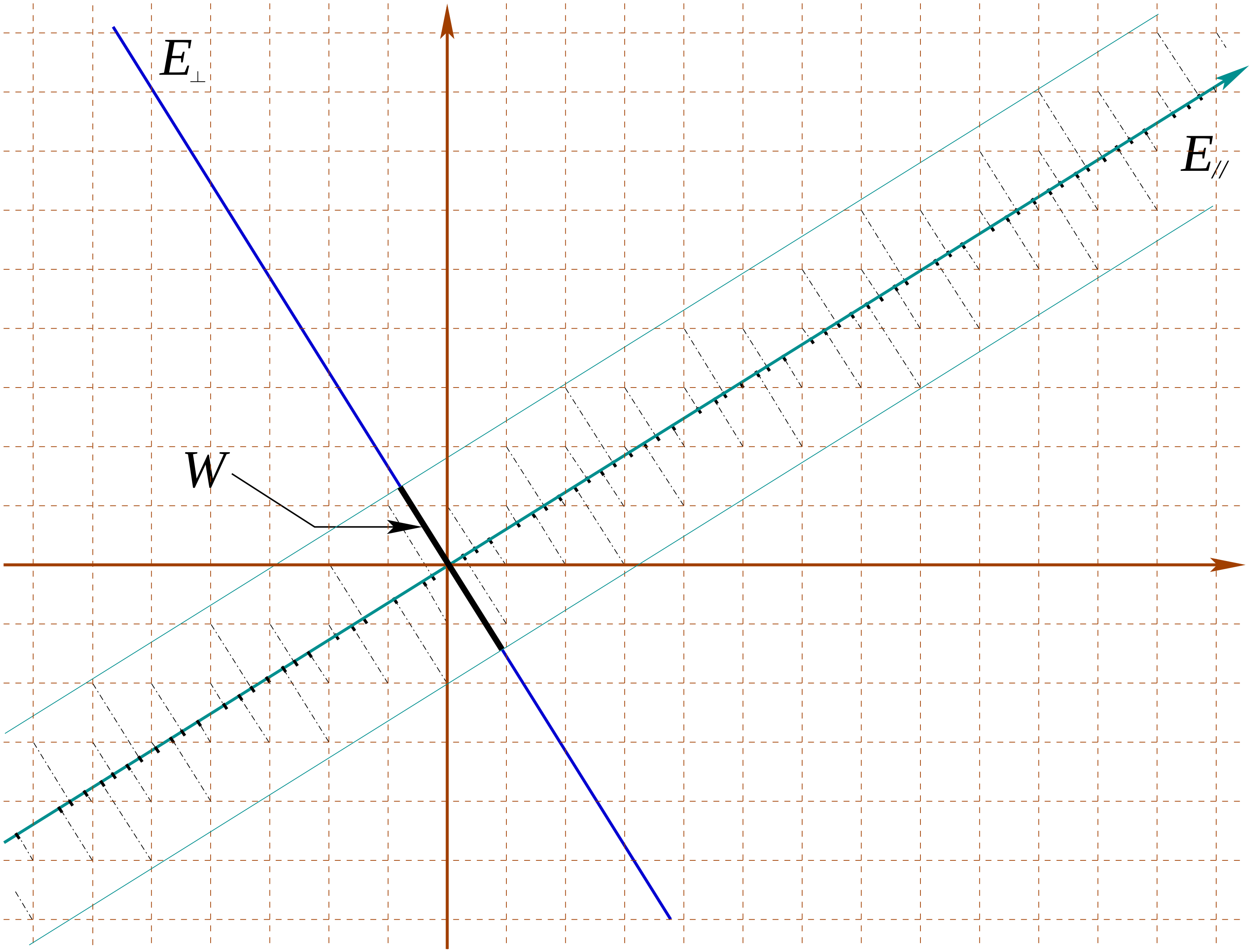
$$\mathbb{R}^d \simeq \mathcal{E}_{\|} \xleftarrow{\pi_{\|}} \mathbb{R}^n \xrightarrow{\pi_{\perp}} \mathcal{E}_{\perp} \simeq \mathbb{R}^{n-d}$$

$$\mathcal{L} \xleftarrow{\pi_{\|}} \tilde{\mathcal{L}} \xrightarrow{\pi_{\perp}} W \quad ,$$

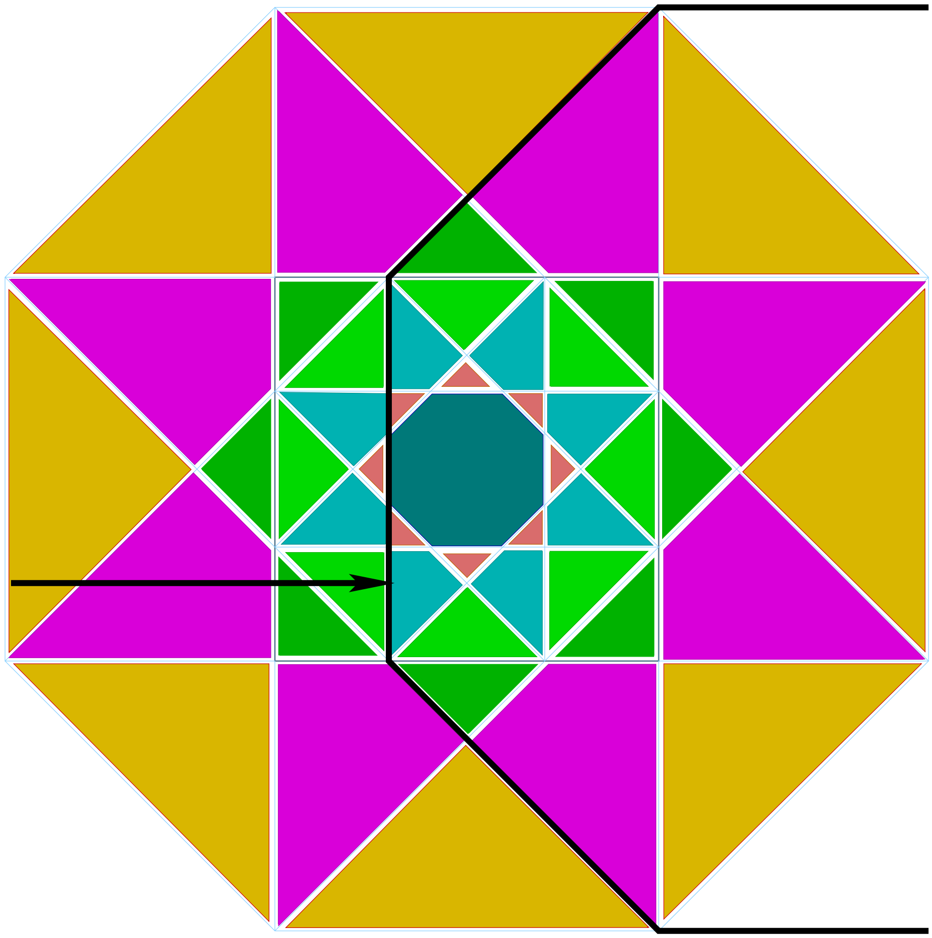
Here

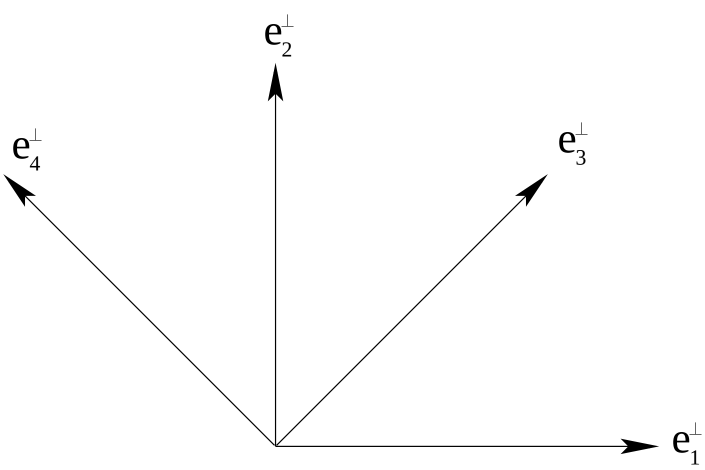
1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{L}$ is the *quasilattice* in $\mathcal{E}_{\|}$ defined as

$$\mathcal{L} \;=\; \{\pi_{\|}(m) \in \mathcal{E}_{\|}\,;\, m \in \tilde{\mathcal{L}}\,,\, \pi_{\perp}(m) \in W\}$$



− The cut−and−project construction −





- The transversal of the Octagonal Tiling -

- is completely disconnected -

III - Branched Oriented Flat Riemannian Manifolds

Laminations and Boxes

A *lamination* is a foliated manifold with $\mathcal{C}^\infty$-structure along the leaves but only finite $\mathcal{C}^0$-structure transversally. The *Hull of a Delone set is a lamination* with $\mathbb{R}^d$-orbits as leaves.

If $\mathcal{L}$ is a *finite type, repetitive, Delone* set, with Hull Ω A *box* is the homeomorphic image of

$$\phi : (\omega, x) \in F \times U \mapsto \mathrm{T}^{-x}\omega \in \Omega$$

if F is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and T denotes the $\mathbb{R}^d$-action on Ω.

A quasi-partition is a family $(B_i)_{i=1}^n$ of boxes such that $\bigcup_i \overline{B_i} = \Omega$ and $B_i \cap B_j = \emptyset$.

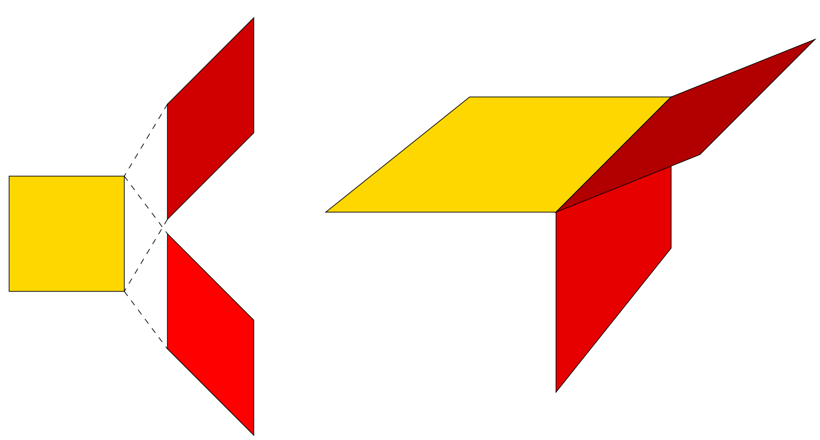
Theorem 4 *The Hull of a finite type, repetitive, Delone set admits a finite quasi-partition. It is always possible to choose these boxes as $\phi(F \times U)$ with U a d-rectangle.*

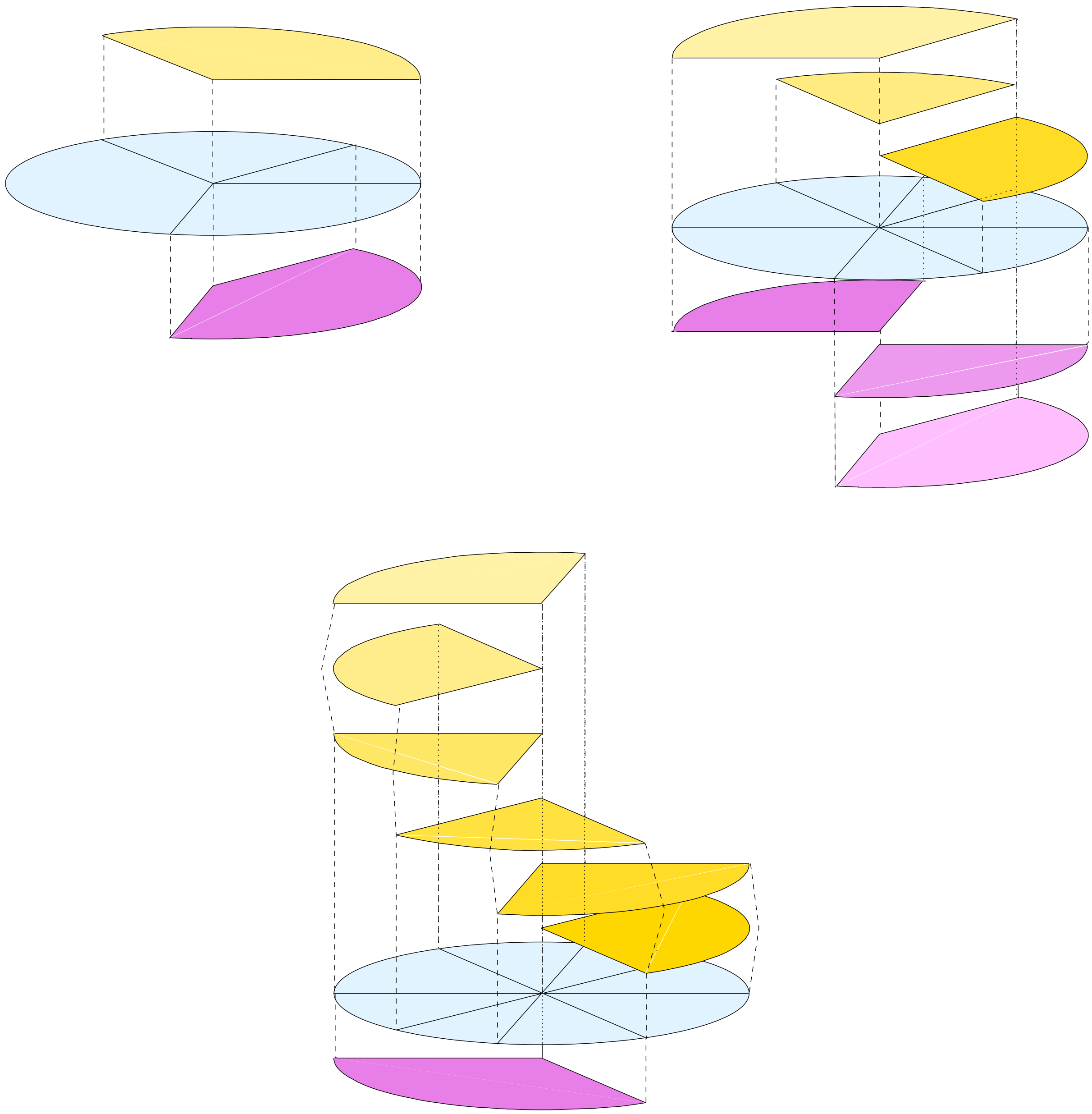
Branched Oriented Flat Manifolds

Flattening a box decomposition along the transverse direction leads to a *Branched Oriented Flat manifold*. Such manifolds can be built from the tiling itself as follows

Step 1:

1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.





- Vertex branching for the octagonal tiling -

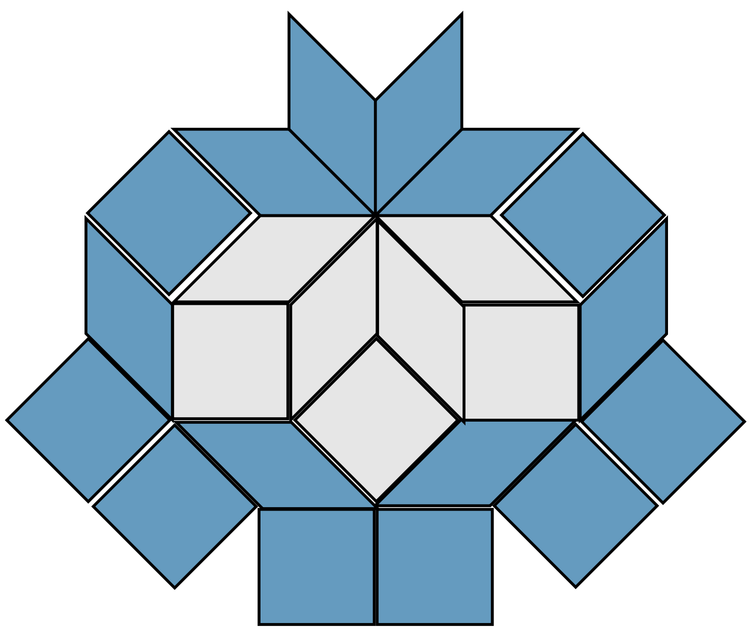
Step 2:

1. Having defined the patch p_n for $n \geq 0$, let $\mathcal{L}_n$ be the subset of $\mathcal{L}$ of points centered at a translated of p_n. By repetitivity this is a finite type repetitive Delone set too. Its prototiles are tiled by tiles of $\mathcal{L}$ and define a finite family $\mathfrak{P}_n$ of patches.

2. Color each patch in $\mathcal{T} \in \mathfrak{P}_n$ by the tiles touching it from outside along its frontier. Call such a patch *modulo translation* a *a colored patch* and $\mathfrak{P}_n^c$ their set.

3. Proceed then as in Step 1 by replacing prototiles by colored patches to get the BOF-manifold B_n.

4. Then choose for p_{n+1} as the colored patch in $\mathfrak{P}_n^c$ containing p_n.



Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n. Note that $Df_n = \mathbf{1}$.
2. Call Ω the *projective limit* of the sequence

$$\cdots \stackrel{f_{n+1}}{\rightarrow} B_{n+1} \stackrel{f_n}{\rightarrow} B_n \stackrel{f_{n-1}}{\rightarrow} \cdots$$

3. $X_1, \cdots X_d$ are the commuting constant vector fields on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action T on Ω.

Theorem 5 *The dynamical system*

$$(\Omega, \mathbb{R}^d, \mathrm{T}) = \varprojlim (B_n, f_n)$$

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

Longitudinal (co)-Homology

The Homology groups are defined by the inverse limit

$$H_*(\Omega, \mathbb{R}^d) \;=\; \lim_{\longleftarrow} \left(H_*(B_n, \mathbb{R}), f_n^*\right)$$

Theorem 6 *The homology group $H_d(\Omega, \mathbb{R}^d)$ admits a canonical positive cone induced by the orientation of $\mathbb{R}^d$, isomorphic to the affine set of positive $\mathbb{R}^d$-invariant measures on Ω.*

The cohomology groups are defined by the direct limit

$$H^*(\Omega, \mathbb{R}^d) \;=\; \lim_{\longrightarrow} \left(H^*(B_n, \mathbb{R}), f_n^*\right)$$

Theorem 7 *If $\mathbb{P}$ is an $\mathbb{R}^d$-invariant probability on Ω, then the pairing with $H^d(\Omega, \mathbb{R}^d)$ satisfies*

$$\langle \mathbb{P} | H^d(\Omega, \mathbb{R}^d) \rangle \;=\; \int_\Xi d\mathbb{P}_{\mathrm{tr}} \;\; \mathcal{C}(\Xi, \mathbb{Z})$$

where Ξ is the transversal, $\mathbb{P}_{\mathrm{tr}}$ is the probability on Ξ induced by $\mathbb{P}$ and $\mathcal{C}(\Xi, \mathbb{Z})$ is the space of integer valued continuous functions on Ξ.

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.
2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.
3. This dynamical system can be seen as a *lamination* and admits *finite box decompositions*.
4. Such box decompositions gives rise to *inverse limits* of *Branched Oriented Flat Riemannian Manifolds*, an equivalent description.
5. Such inverse limit gives the *longitudinal Homology* and *Cohomology* groups of this tiling space. In maximum degree, they give the family of *invariant measures* and the *Gap Labelling Theorem*.