TRANSVERSE GEOMETRY

for

TILING SPACES

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Sponsoring

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Mini-course delivered at SISSA, Trieste, Italy
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a synthesis on this growing set of research

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Content

Lecture I - The Fibonacci Tiling

The Fibonacci Sequence

The *Fibonacci sequence* is an infinite word generated by the substitution

$$\hat{\sigma} : \qquad a \longrightarrow ab\,, \qquad\qquad b \longrightarrow a$$

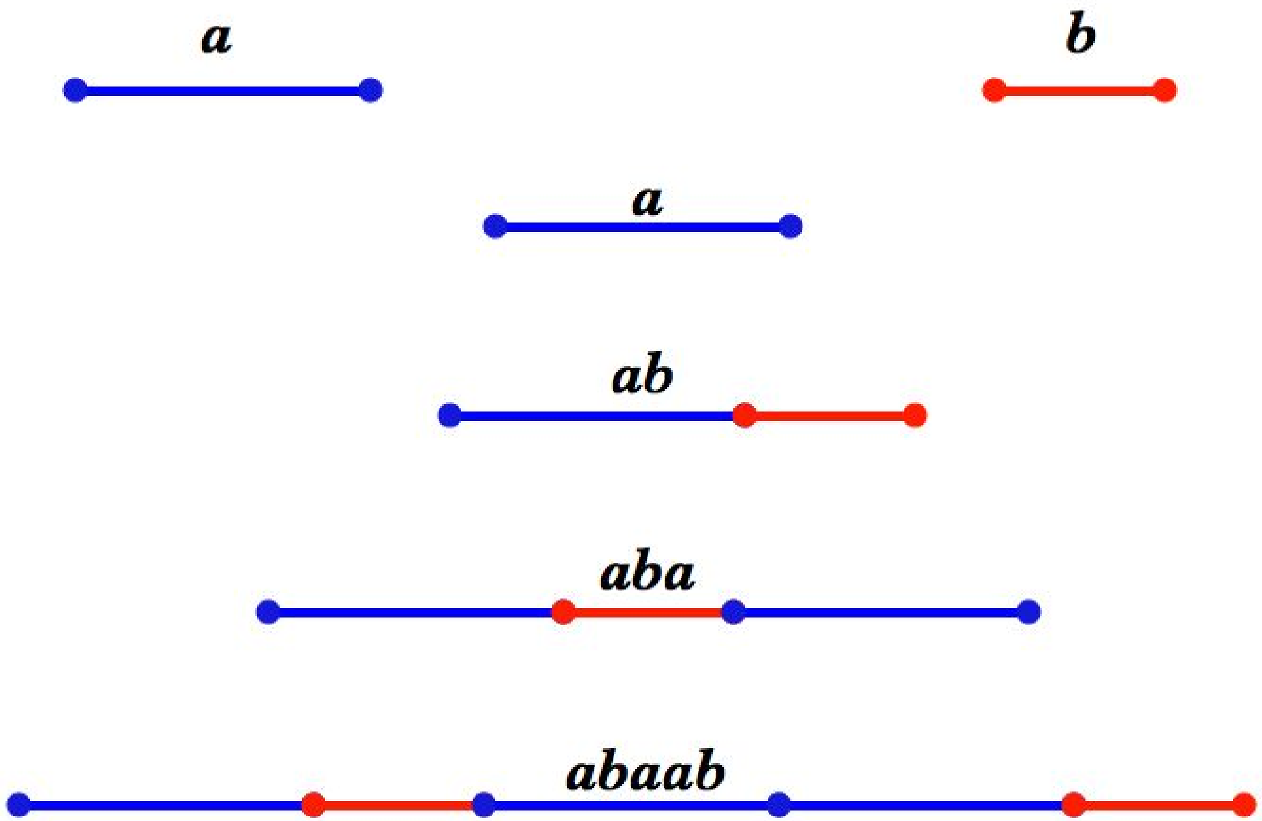
Iterating gives

$$\underbrace{a}_{a_0} \to \underbrace{ab}_{a_1} \to \underbrace{ab|a}_{a_2=a_1a_0} \to \underbrace{aba|ab}_{a_3=a_2a_1} \to \underbrace{abaab|aba}_{a_4=a_3a_2} \to \underbrace{abaababa|abaab}_{a_5=a_4a_3}$$

It can be represented by a *1D-tiling* if

$$a \to [0,1] \qquad b \to [0,\sigma] \qquad \sigma = \frac{\sqrt{5}-1}{2} \sim .618$$

The Fibonacci Sequence



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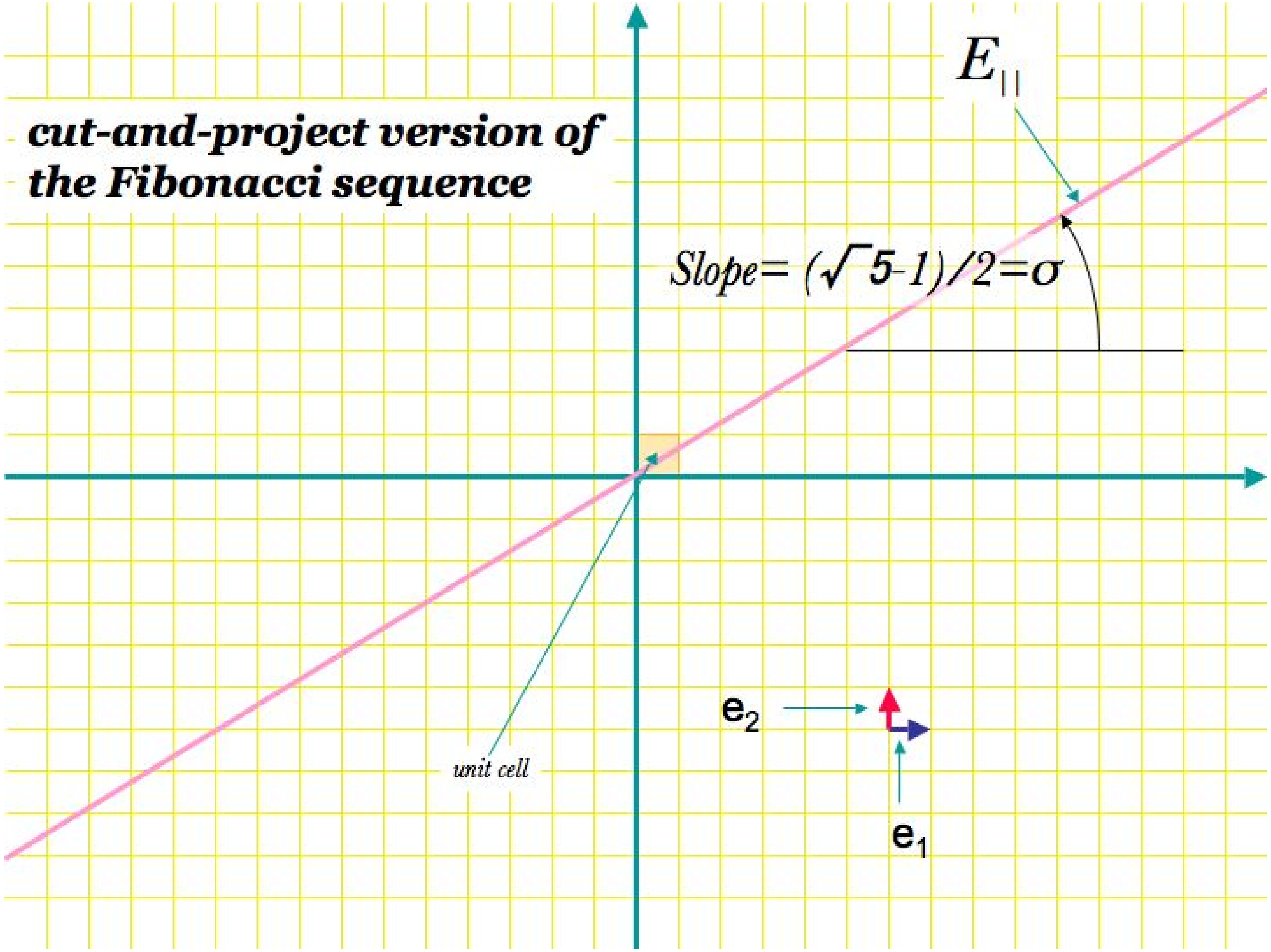

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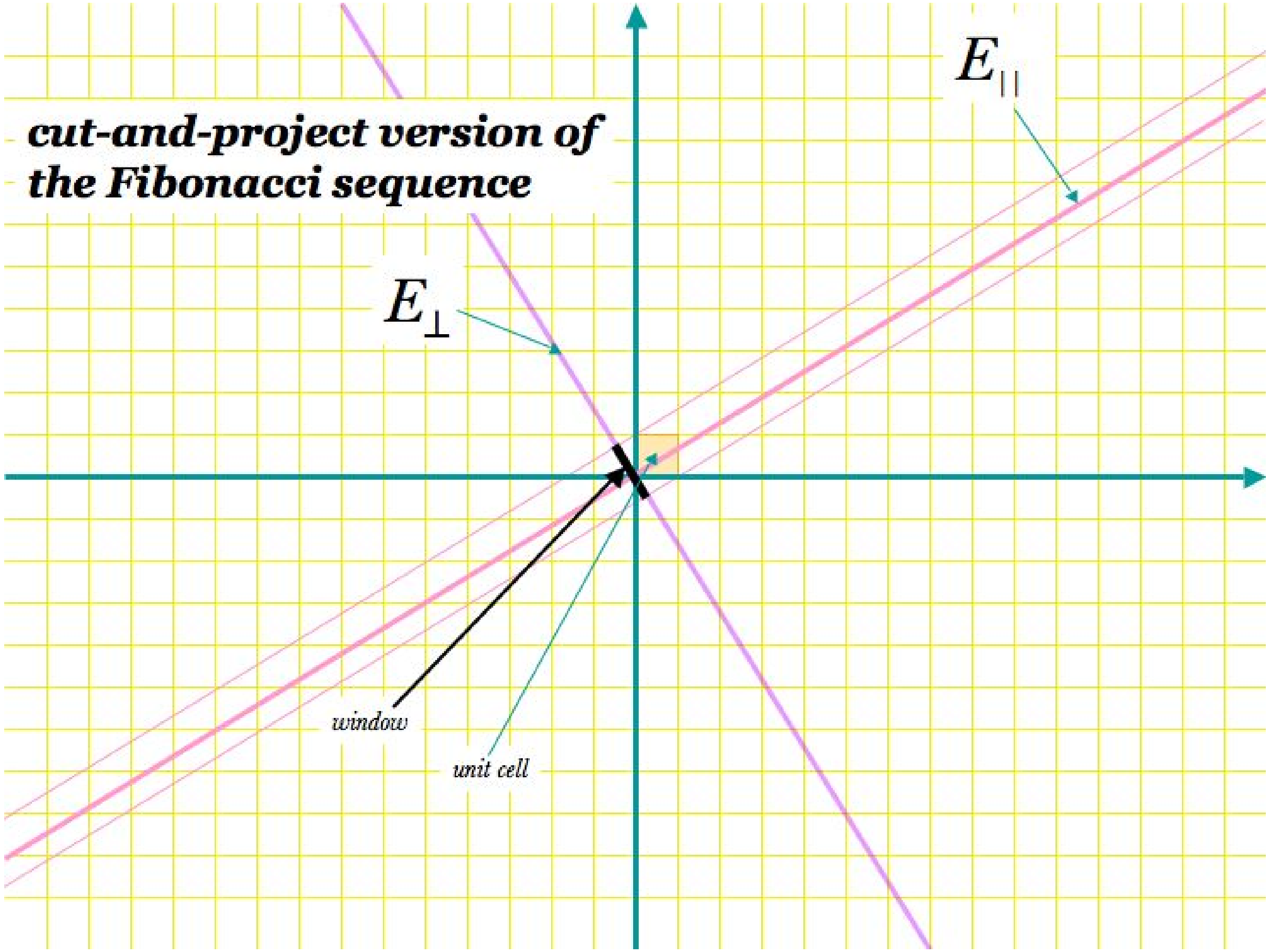

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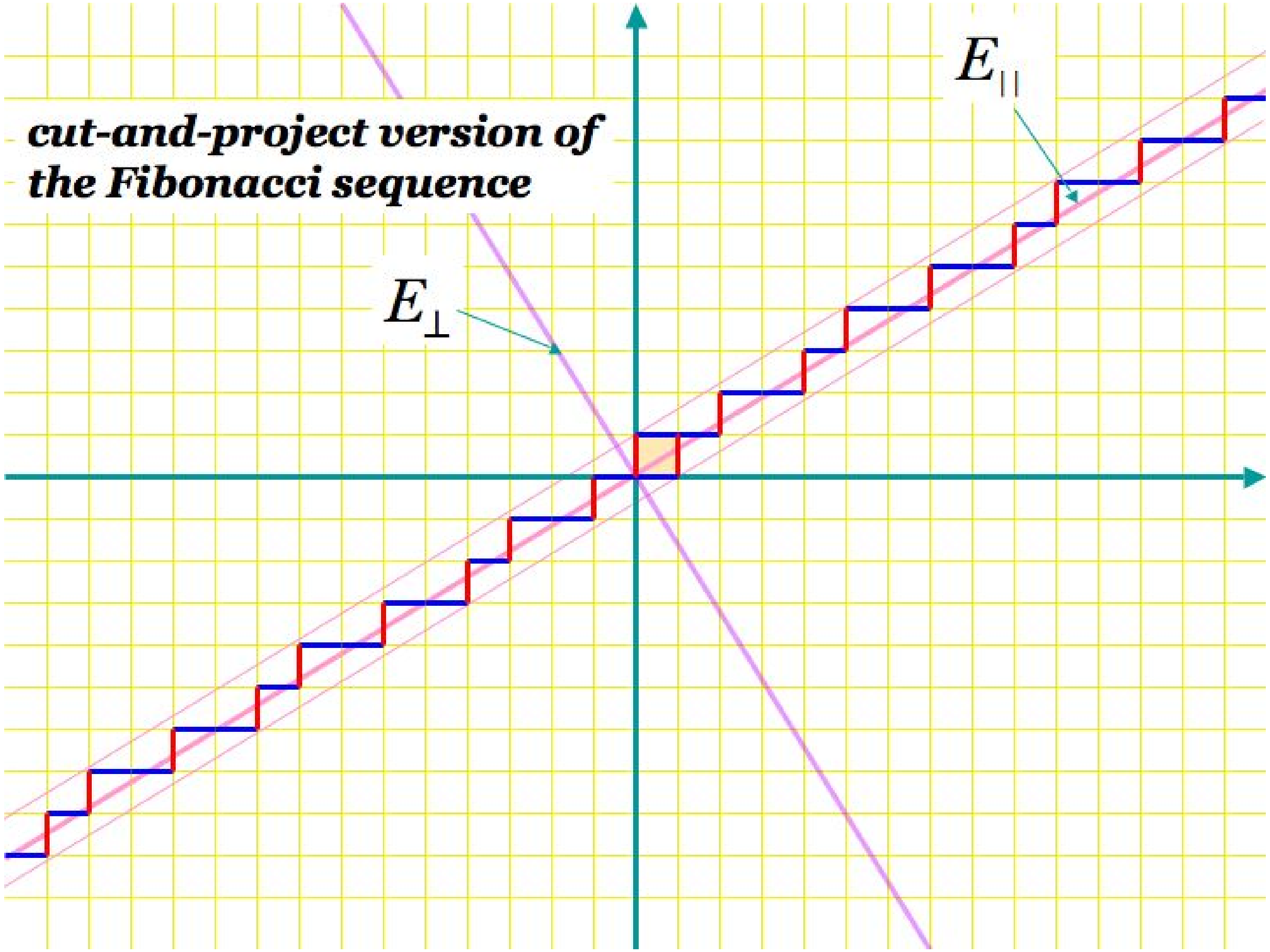

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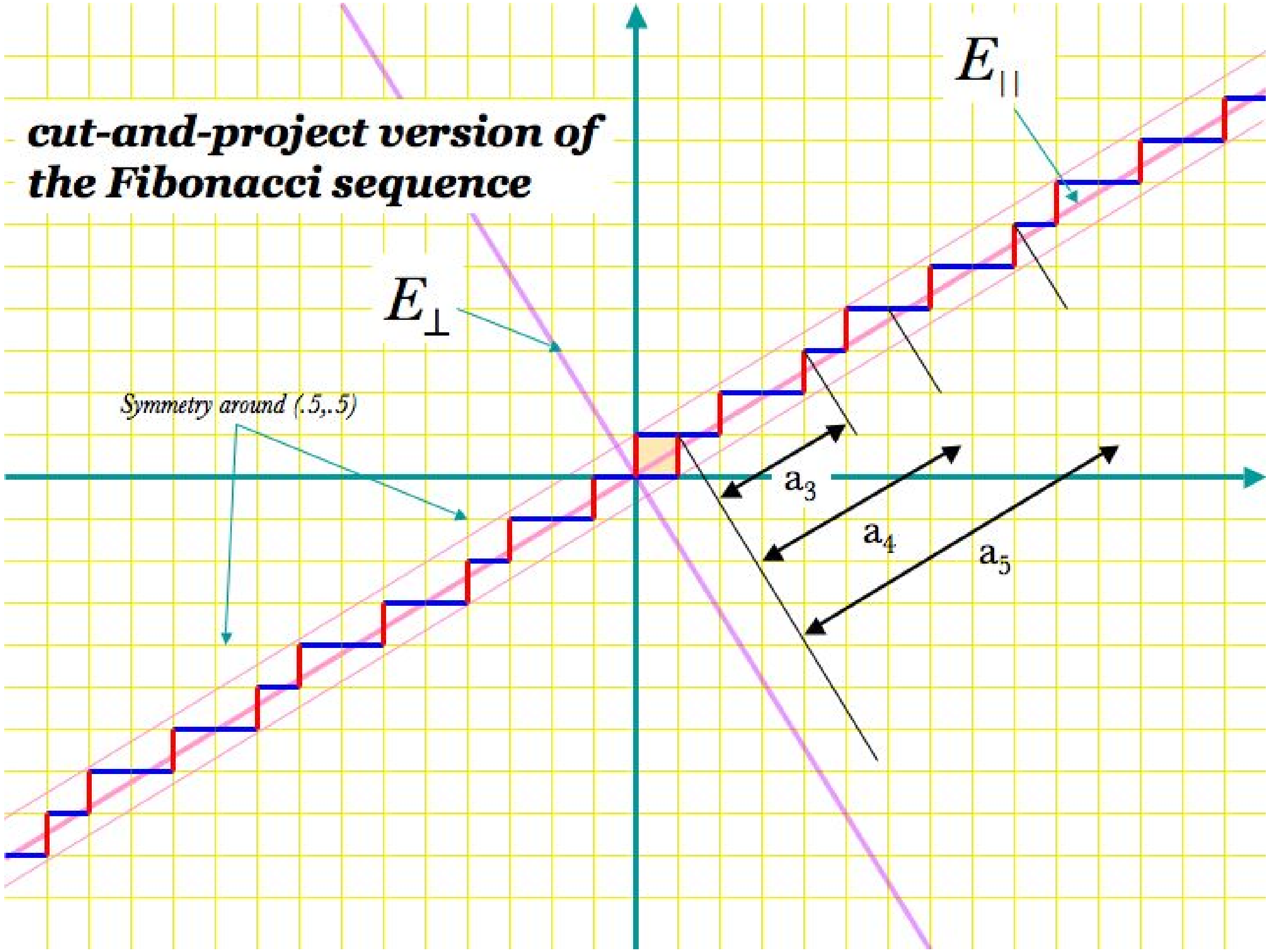

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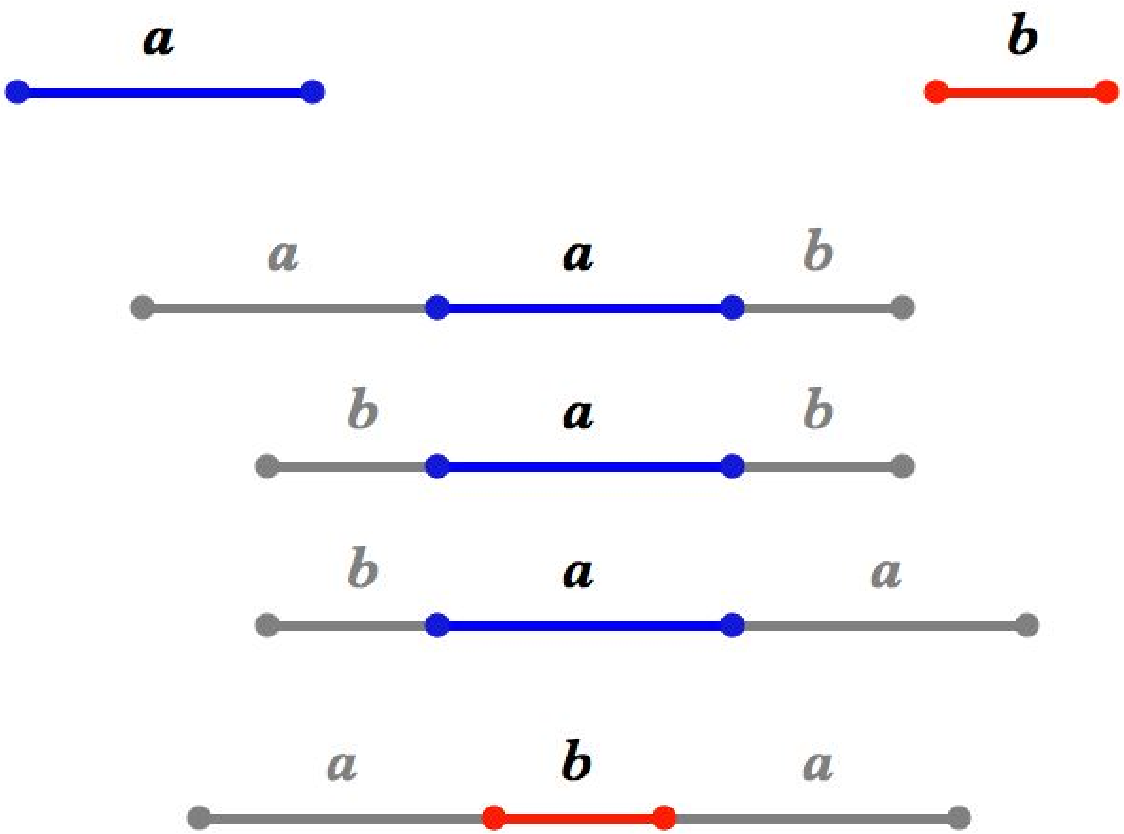
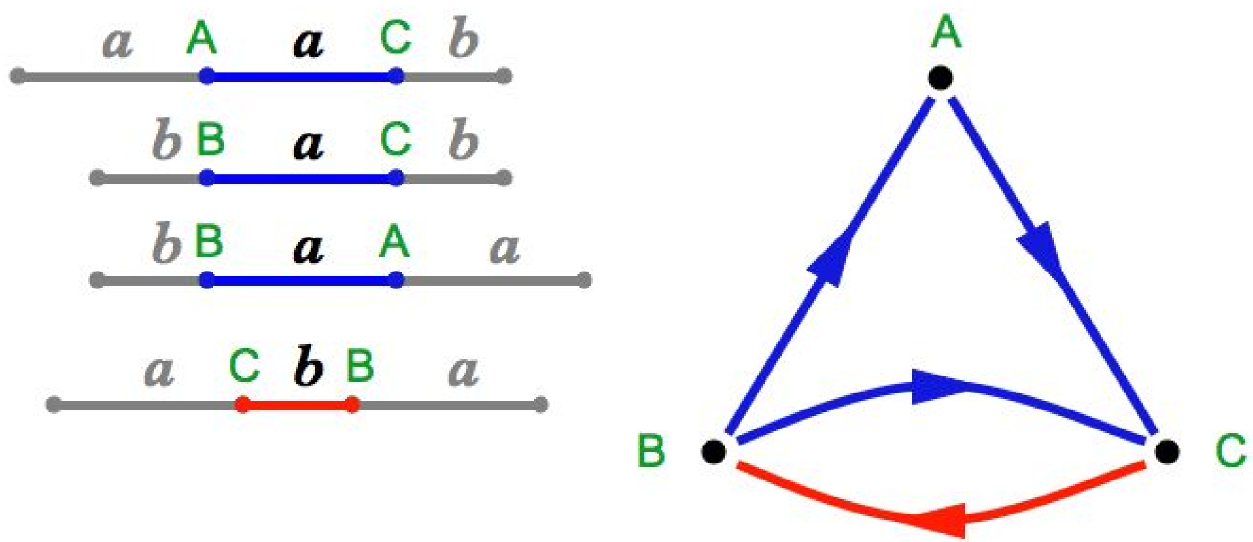


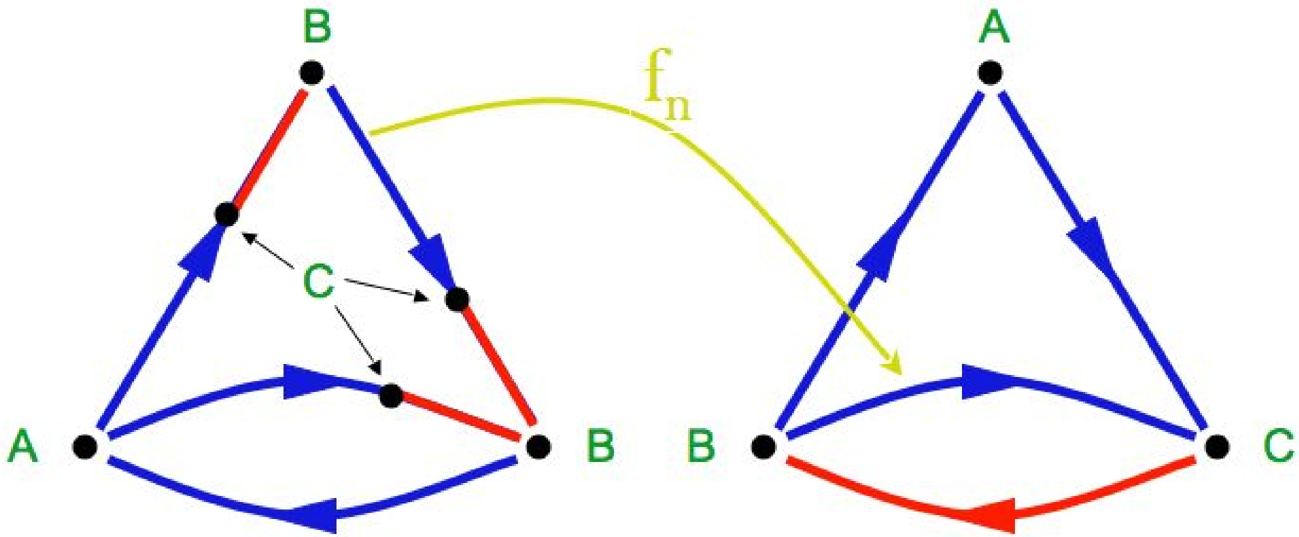
- Collared tiles in the Fibonacci tiling -

The Fibonacci Sequence



- The Anderson-Putnam complex for the Fibonacci tiling -

The Fibonacci Sequence



$$X_{n+1} \xrightarrow{f_n} X_n$$

- The substitution map -

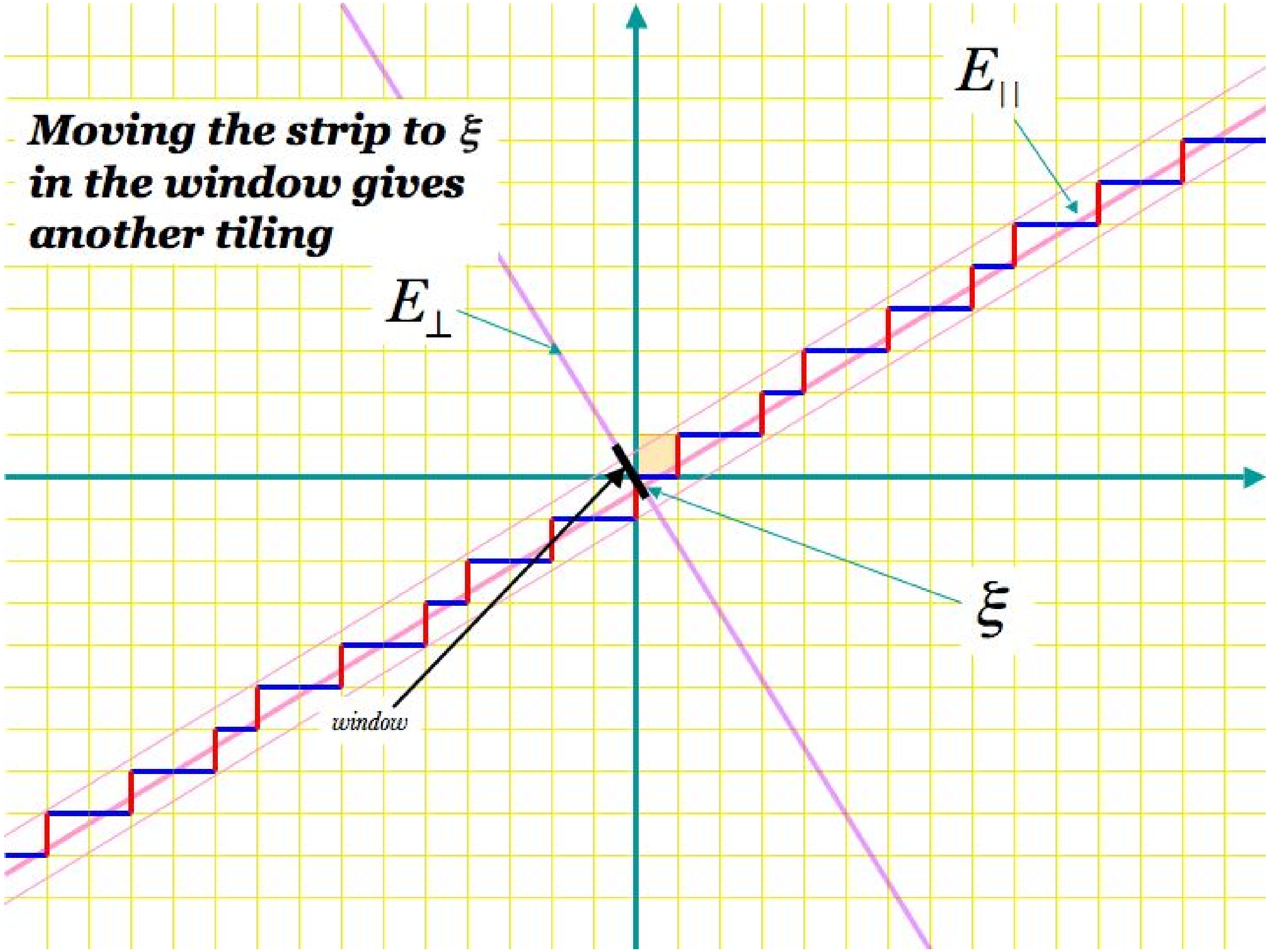
The Fibonacci Sequence

Let $\Xi_n \subset X_n$ be the set of *tile endpoints* (0-cells). The sequence of complexes $(X_n)_{n\in\mathbb{N}}$ together with the maps $f_n : X_{n+1} \mapsto X_n$ gives rise to inverse limits

$$\lim_{\leftarrow}(X_n, f_n) = \Omega \qquad \lim_{\leftarrow}(\Xi_n, f_n) = \Xi$$

- The space Ω is *compact* and is called the *Hull*.
- It is endowed with an *action* of $\mathbb{R}$ generated by infinitesimal translation on the X_n's
- The space Ξ is a Cantor set and is called the *transversal*
- Ξ parametrizes a the set of all tilings sharing the same words as the Fibonacci sequence with one tile endpoint at the origin.
- There is a *two-to one* correspondence between Ξ and the window.

The Fibonacci Sequence



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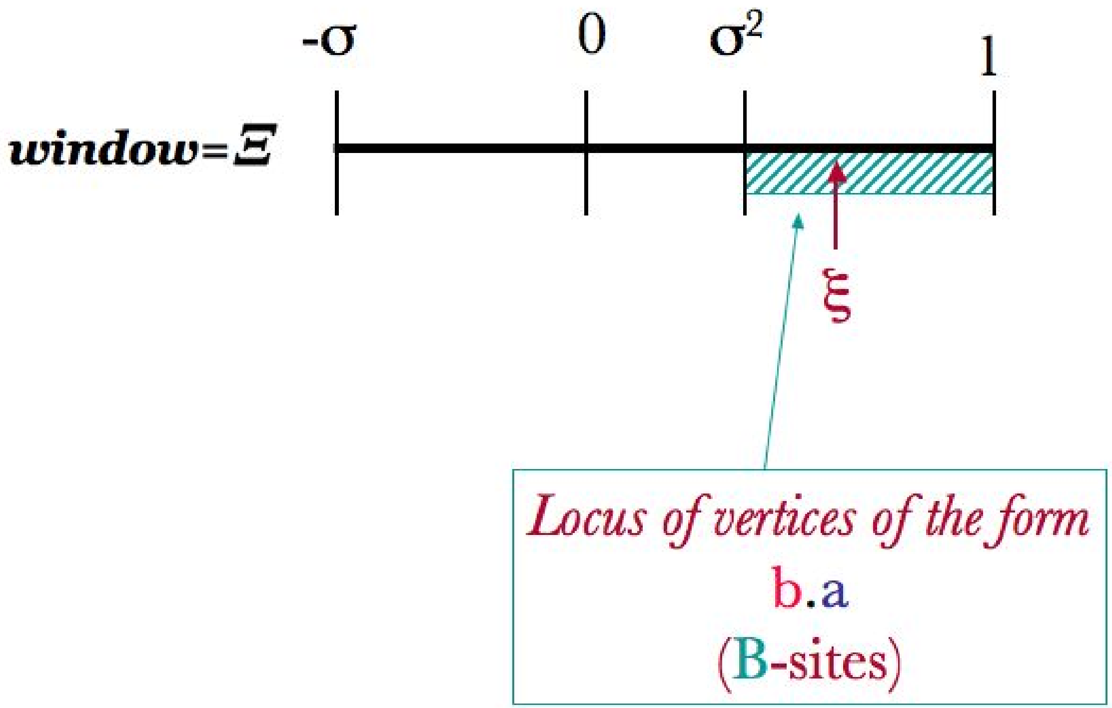

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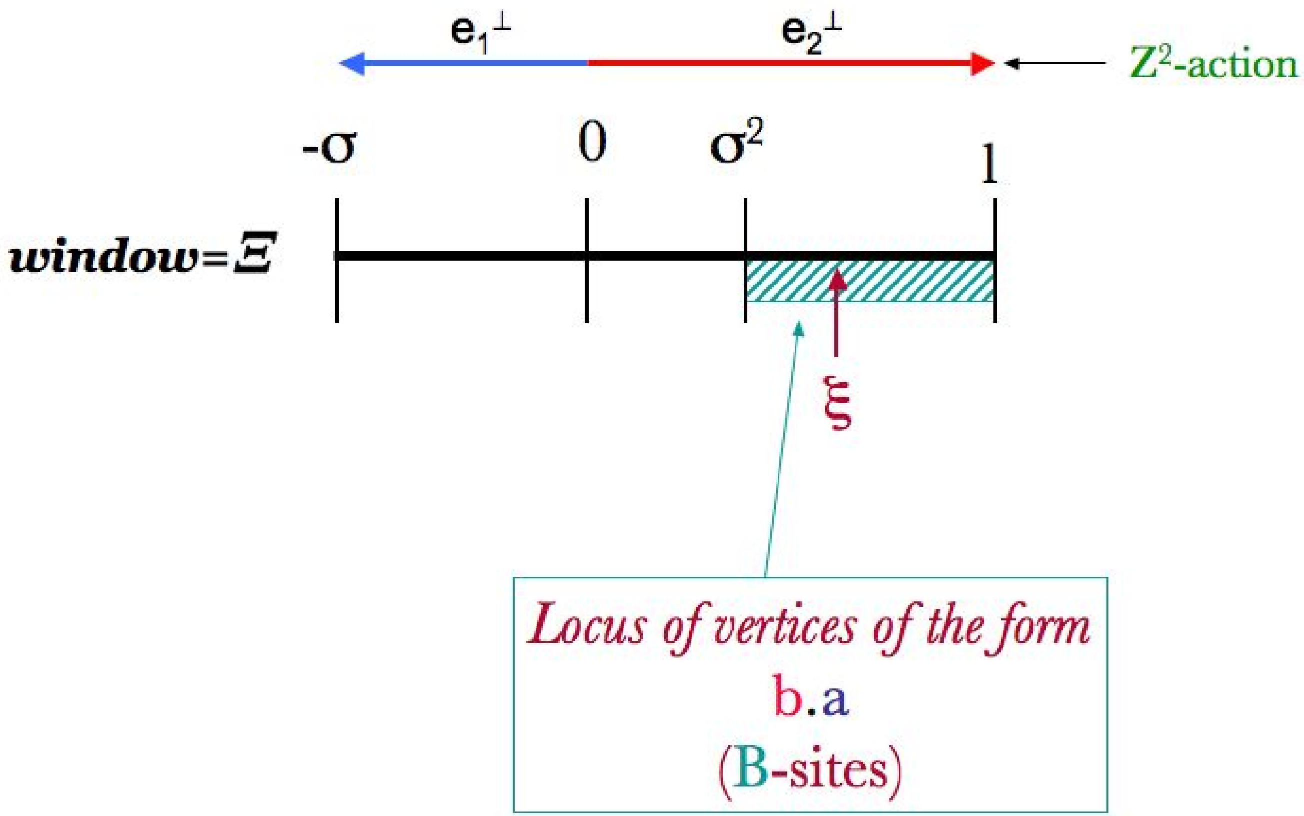

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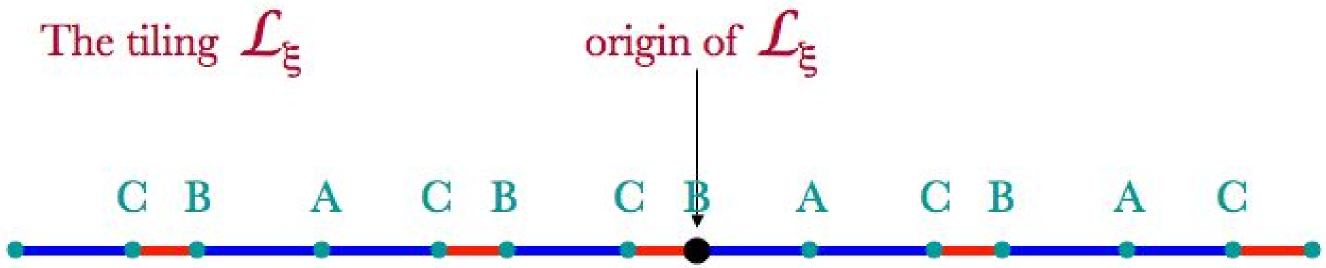

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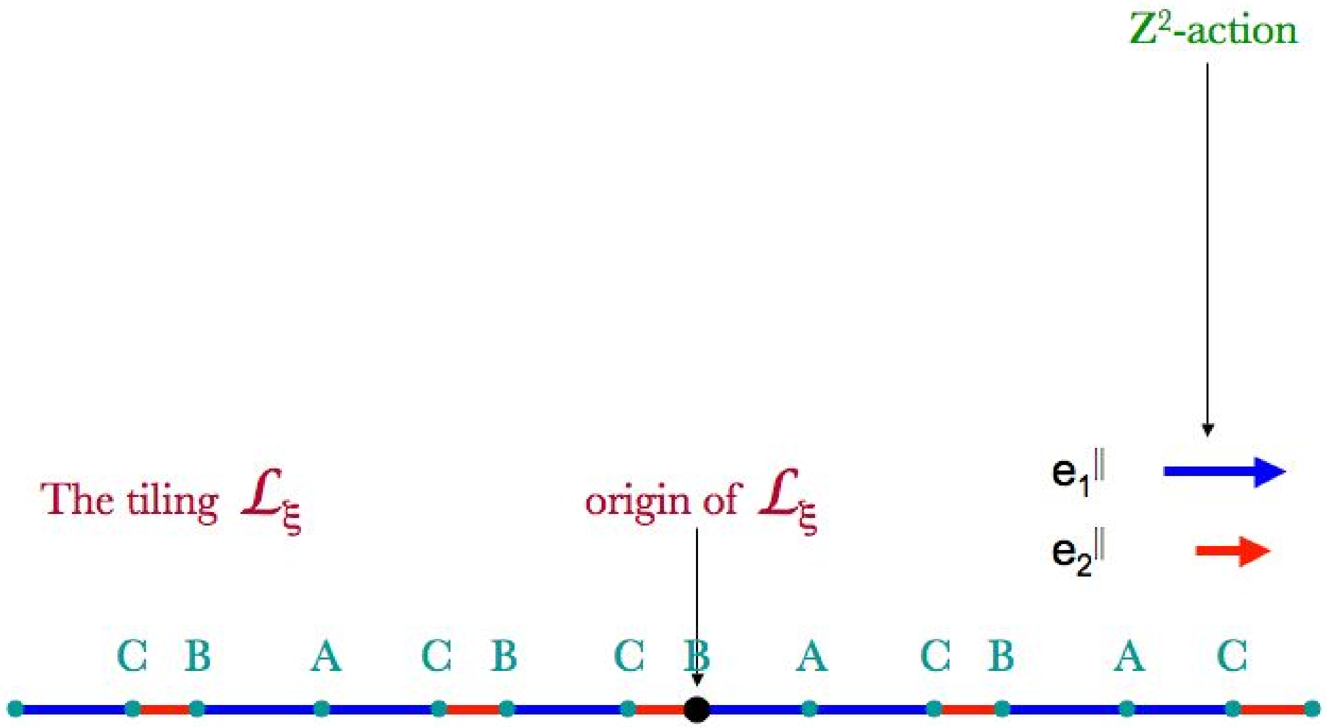

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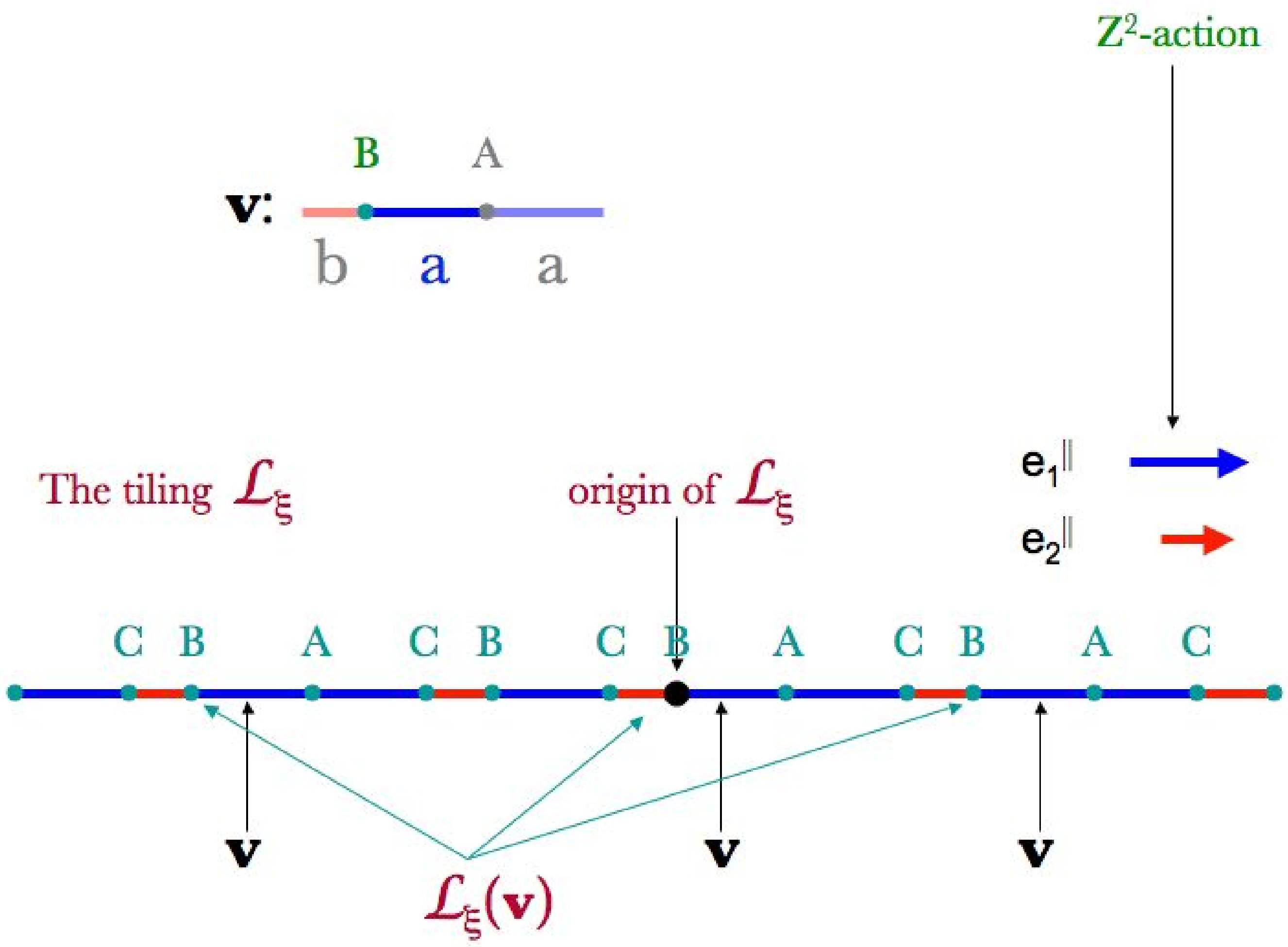

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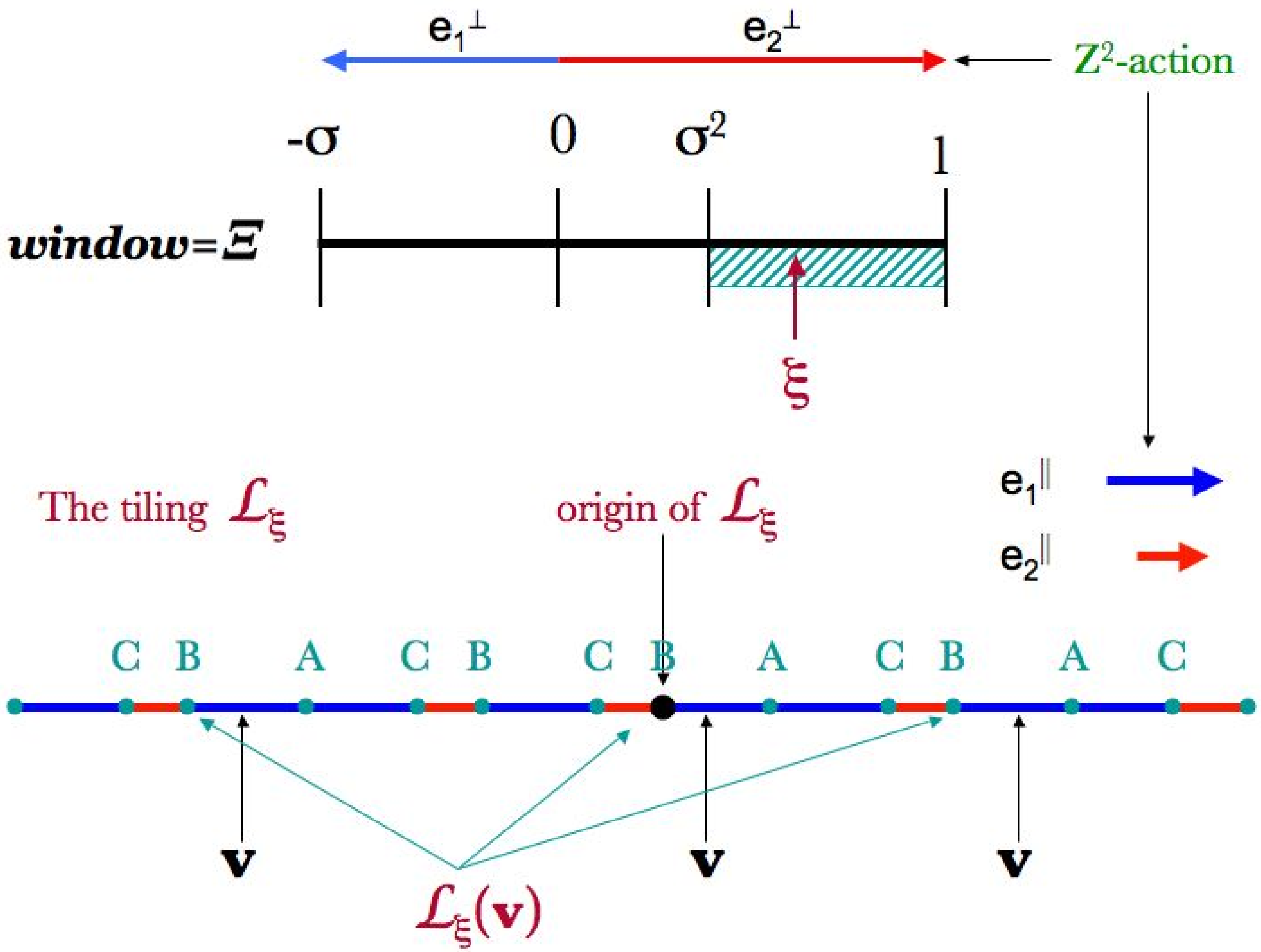

The Fibonacci Sequence: Groupoid

Γ_Ξ is the set of pairs (ξ, a) with $\xi \in \Xi$ and $a \in \mathcal{L}_\xi$.

It is a *locally compact groupoid* when endowed with the following structure

- **Units:** Ξ,
- **Range and Source maps:** $r(\xi, a) = \xi$, $s(\xi, a) = \mathrm{T}^{-a}\xi$
- **Composition:** $(\xi, a) \circ (\mathrm{T}^{-a}\xi, b) = (\xi, a + b)$
- **Inverse:** $(\xi, a)^{-1} = (\mathrm{T}^{-a}\xi, -a)$
- **Topology:** induced by $\Xi \times \mathbb{R}$

Lecture II - Ultrametric Cantor Sets

I - Michon's Trees

G. Michon, "Les Cantors réguliers", *C. R. Acad. Sci. Paris Sér. I Math.*, (19), **300**, (1985) 673-675.

I.1)- Cantor sets

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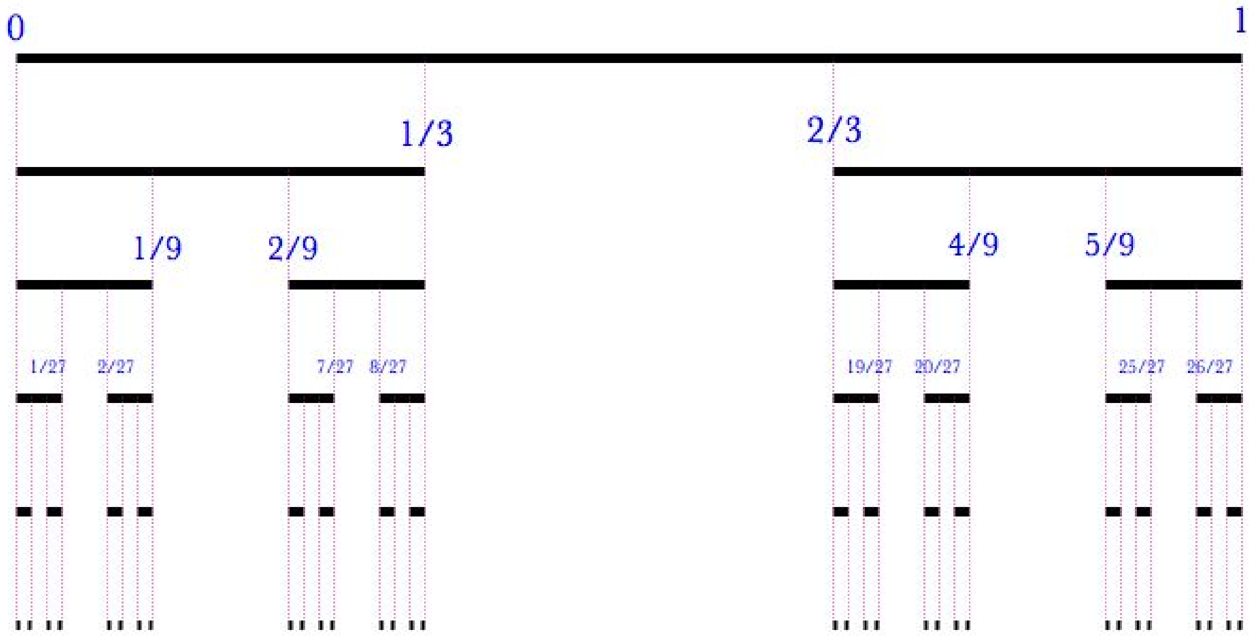

The triadic Cantor set

Definition *A Cantor set is a compact, completely disconnected set without isolated points*

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Theorem *Any Cantor set is homeomorphic to* $\{0,1\}^{\mathbb{N}}$.

L. Brouwer, "On the structure of perfect sets of points", *Proc. Akad. Amsterdam*, **12**, (1910), 785-794.

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Hence without extra structure there is only one Cantor set.

I.2)- Metrics

Definition *Let X be a set. A metric d on X is a map $d : X \times X \mapsto \mathbb{R}_+$ such that, for all $x, y, z \in X$*
(i) $d(x, y) = 0$ *if and only if* $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

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(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition *A metric d on a set X is an ultrametric if it satisfies*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all family x, y, z *of points of C.*

Given (C, d) a metric space, for $\epsilon > 0$ let $\overset{\epsilon}{\sim}$ be the equivalence relation defined by

$$x \overset{\epsilon}{\sim} y \quad \Leftrightarrow \quad \exists x_0 = x, x_1, \cdots, x_{n-1}, x_n = y \quad d(x_{k-1}, x_k) < \epsilon$$

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Theorem *Let (C, d) be a metric Cantor set. Then there is a sequence $\epsilon_1 > \epsilon_2 > \cdots \epsilon_n > \cdots \geq 0$ converging to 0, such that $\stackrel{\epsilon}{\sim} = \stackrel{\epsilon_n}{\sim}$ whenever $\epsilon_n \geq \epsilon > \epsilon_{n+1}$.*

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For each $\epsilon > 0$ there is a finite number of equivalence classes and each of them is close and open.

Moreover, the sequence $[x]_{\epsilon_n}$ of clopen sets converges to $\{x\}$ as $n \to \infty$.

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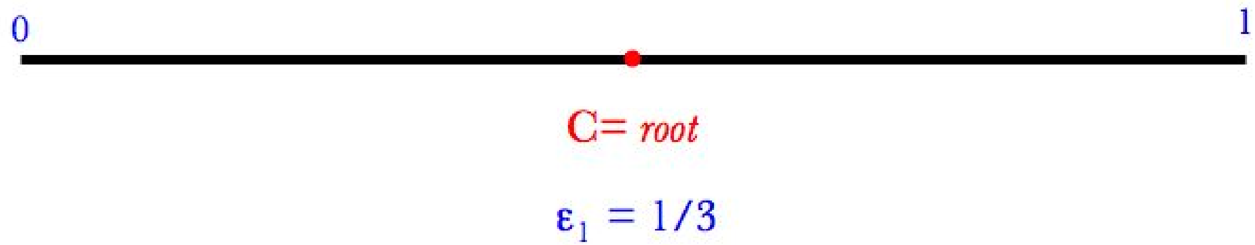
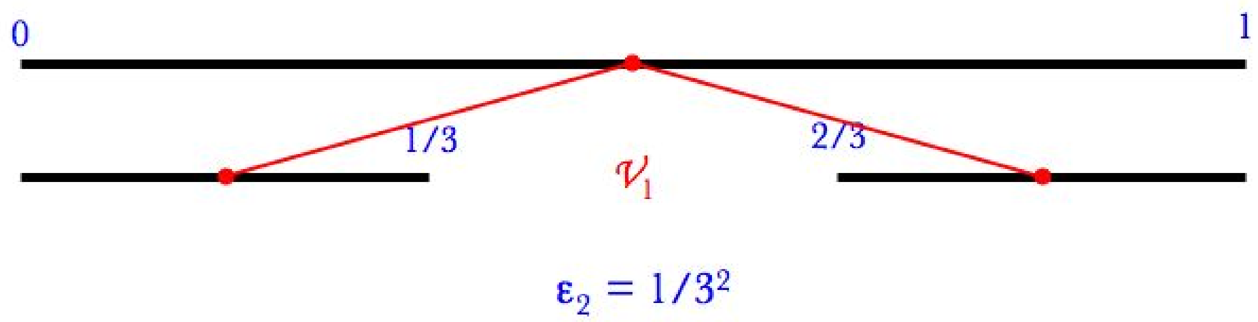
The family $\mathscr{T} = (C, \mathscr{V}, \mathscr{E}, \delta)$ defines a weighted rooted tree, with root C, set of vertices $\mathscr{V}$, set of edges $\mathscr{E}$ and weight δ



The Michon tree for the triadic Cantor set



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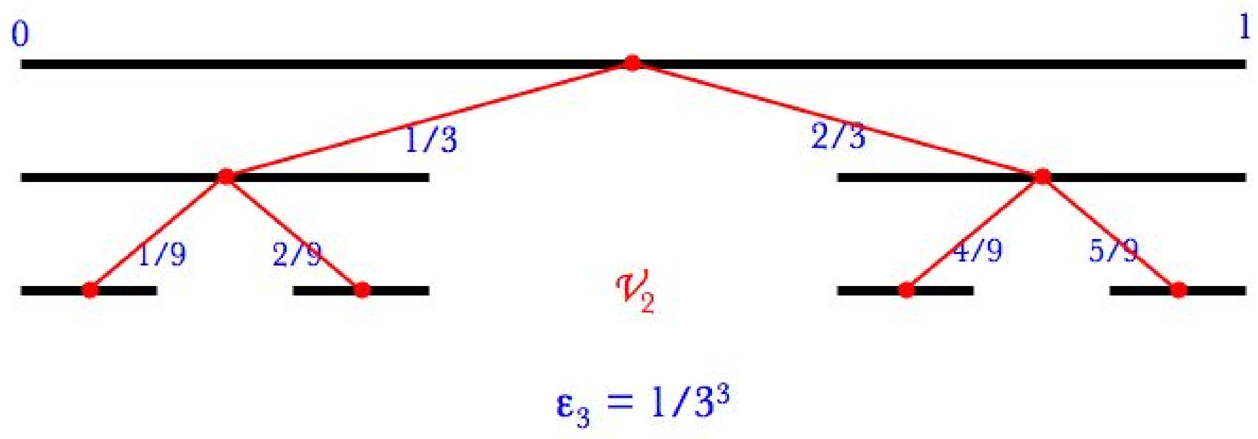

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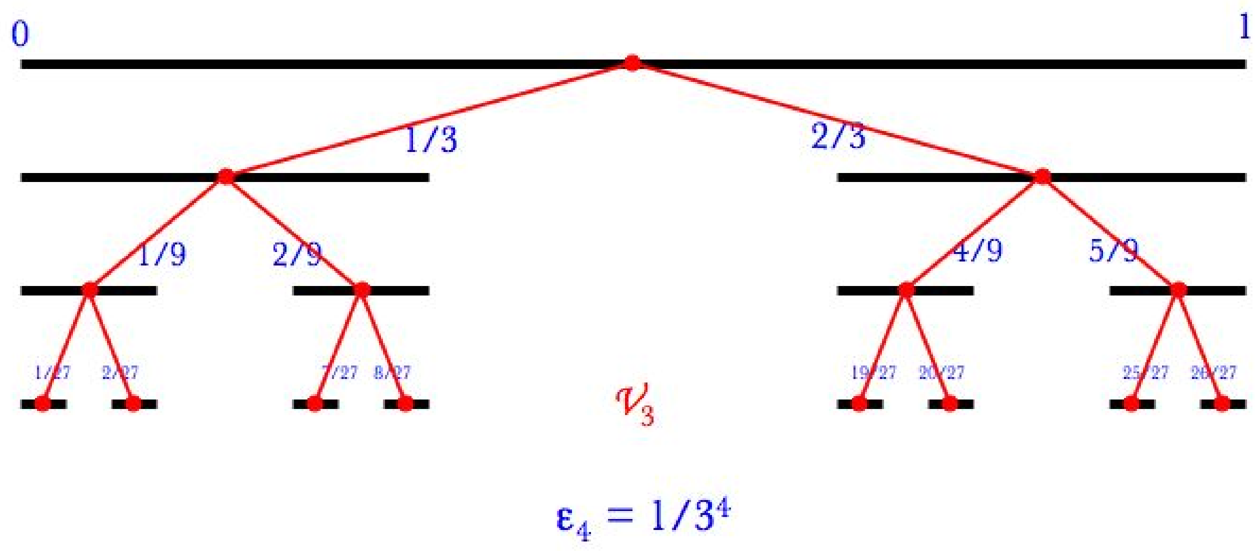

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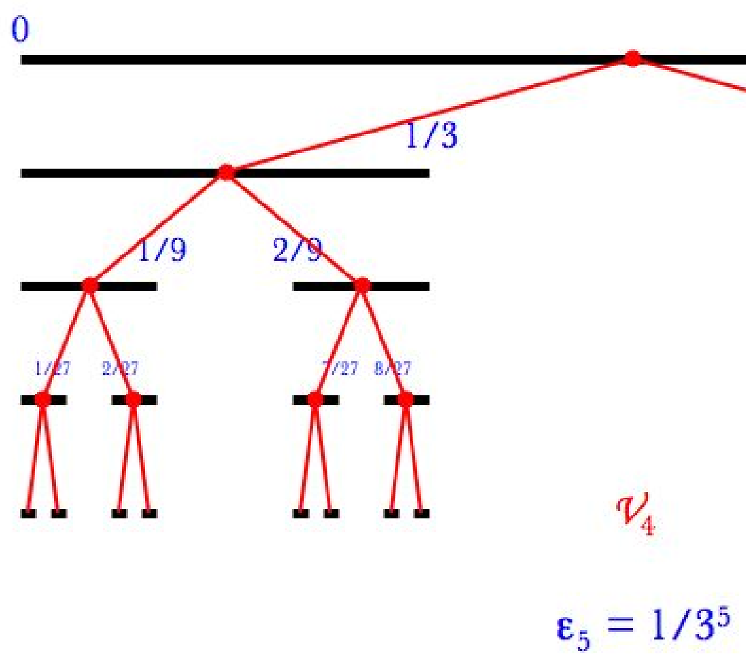

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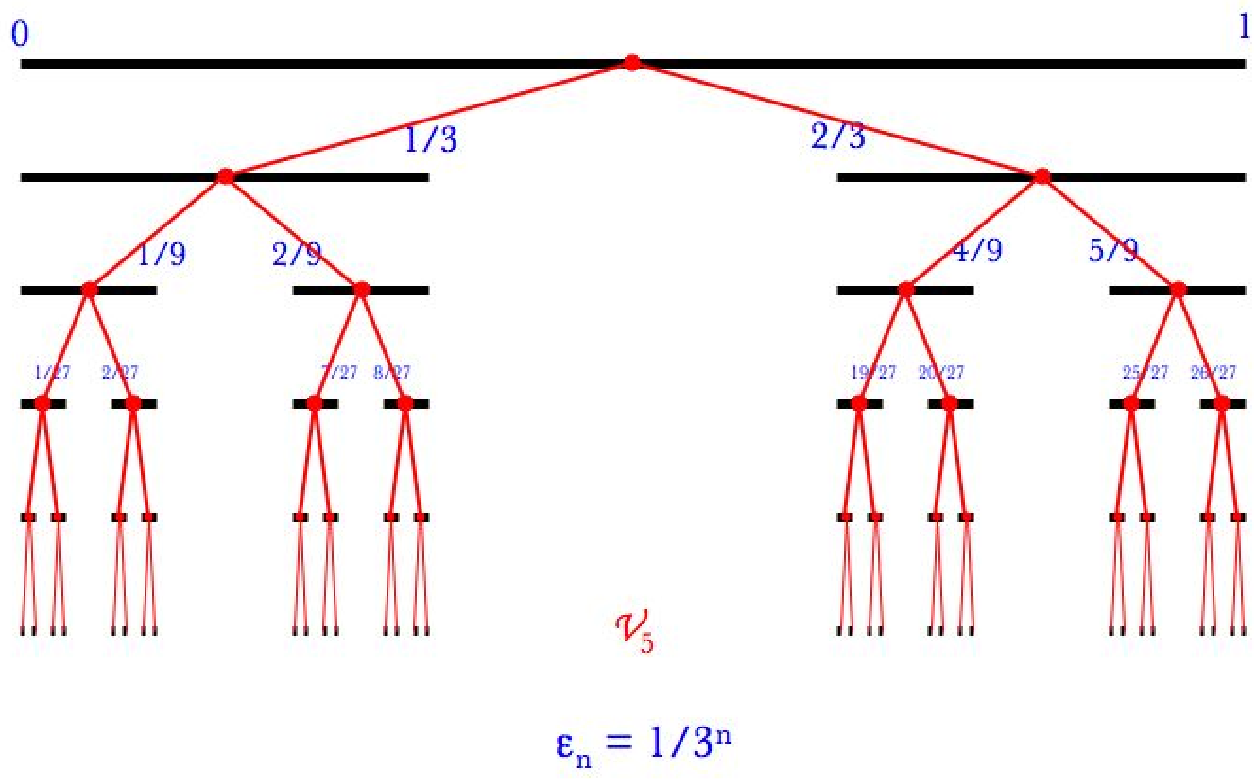

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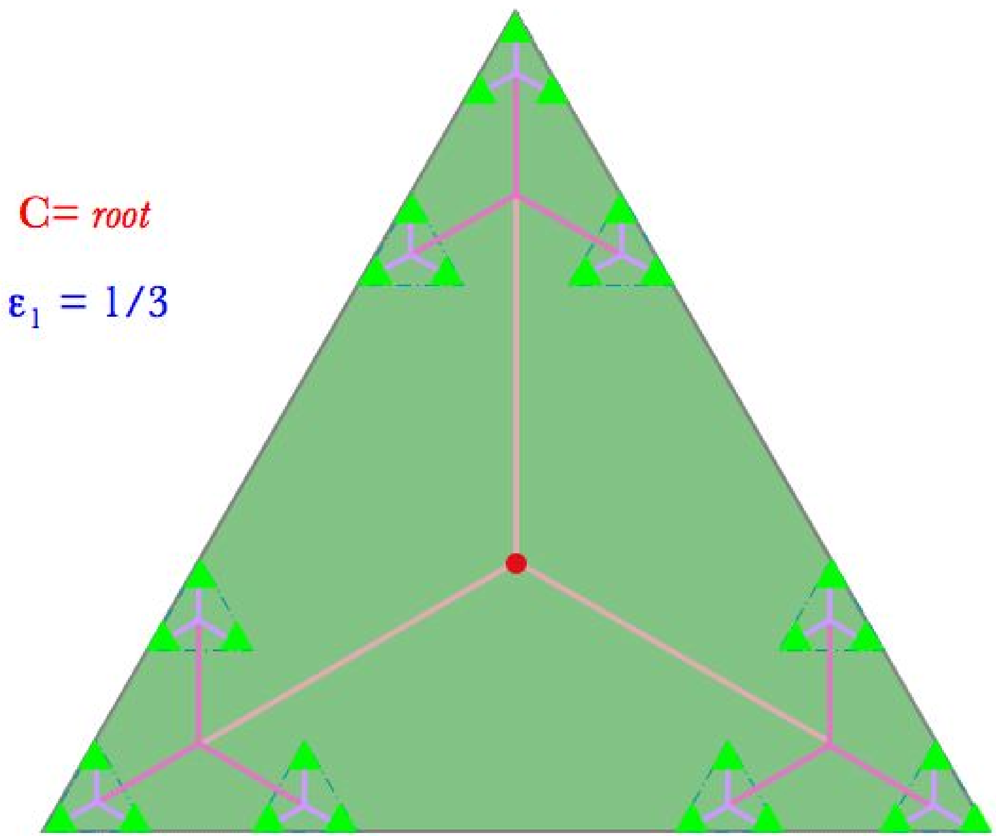
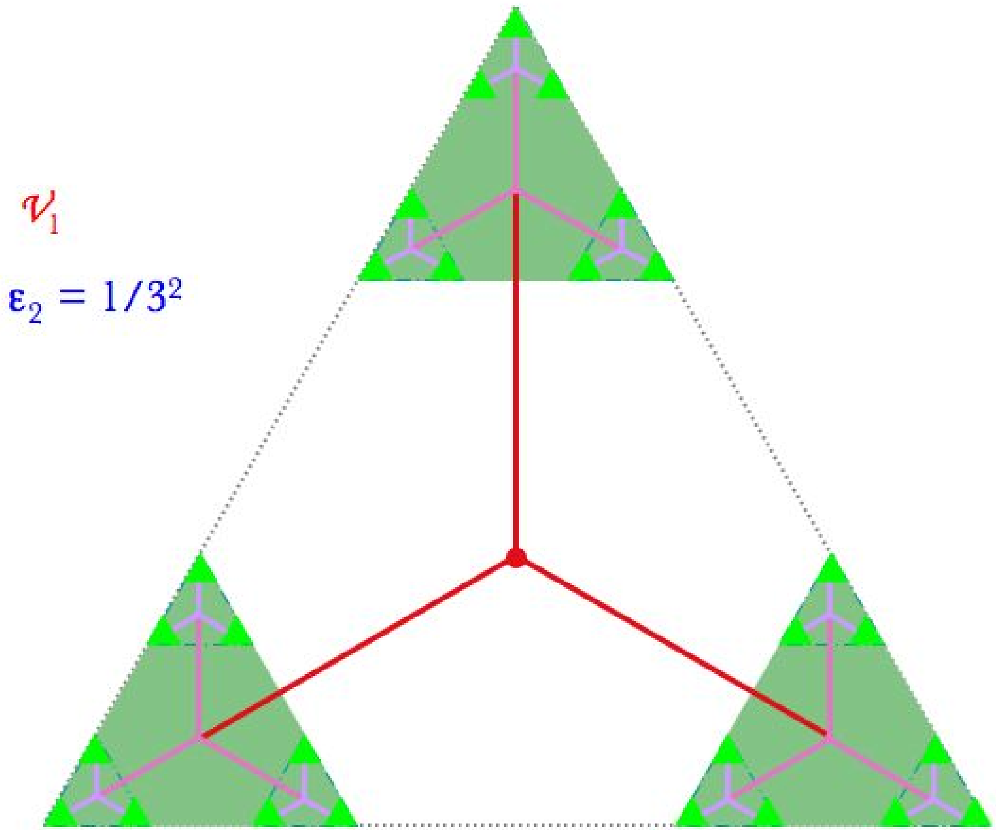


The Michon tree for the triadic ring $\mathbb{Z}(3)$



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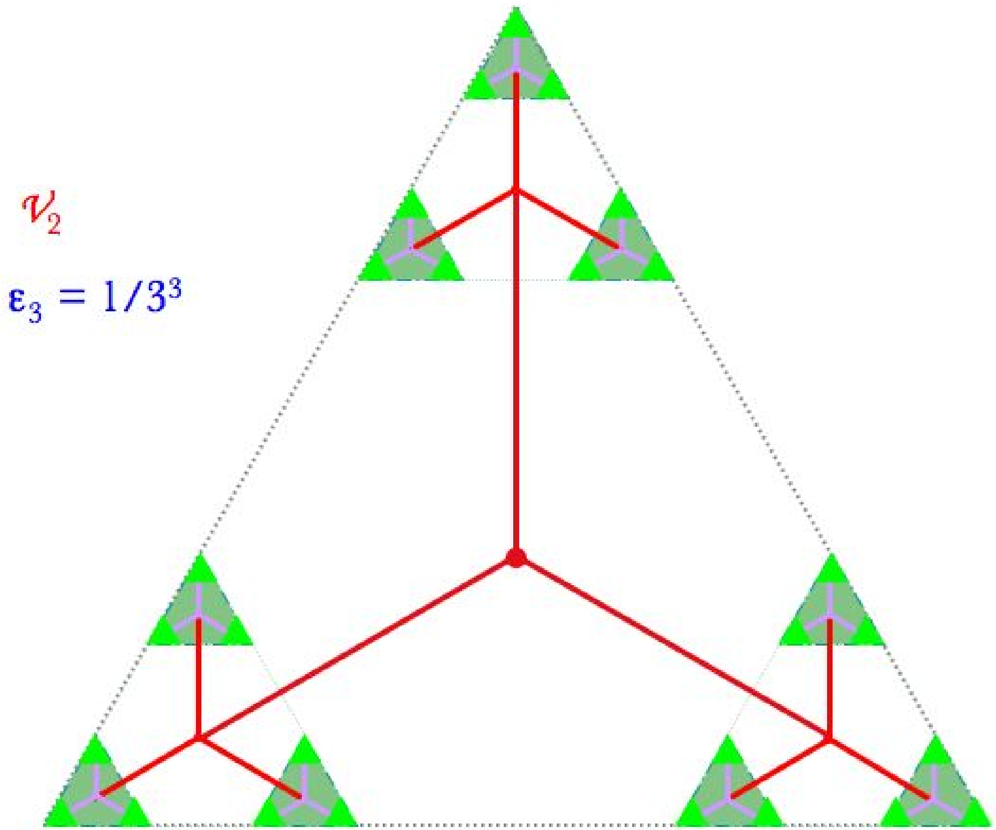

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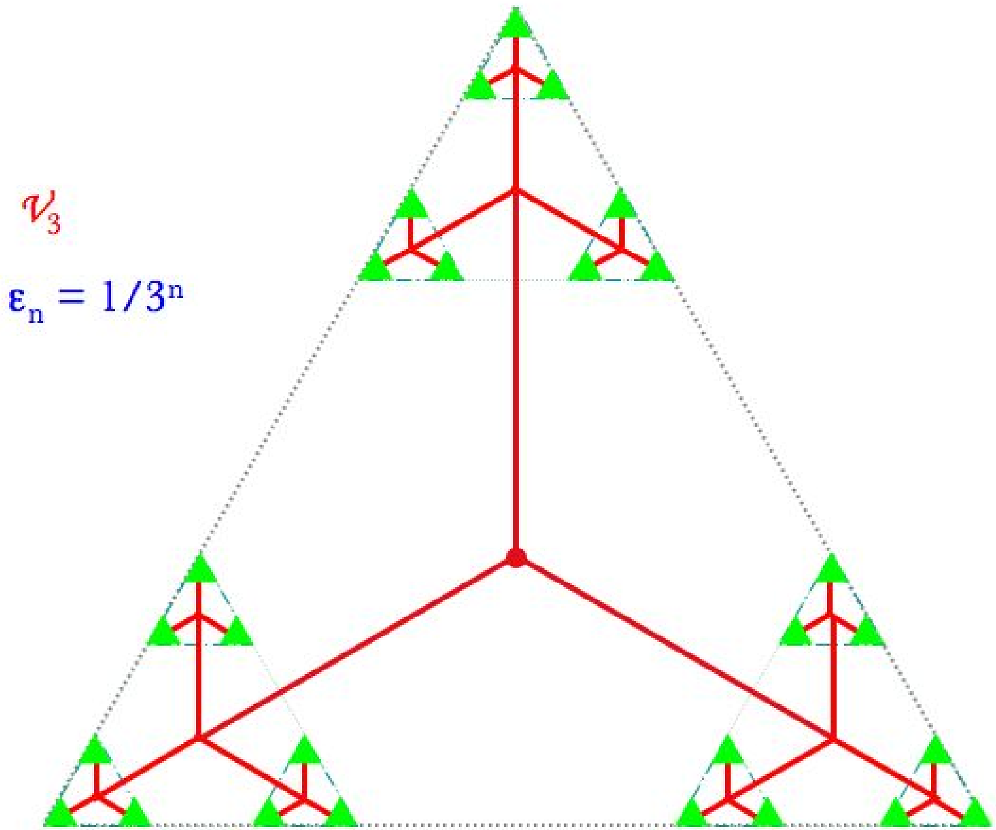

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Theorem *The family $\{[v]\,;\, v \in \mathscr{V}\}$ is the basis of a topology making $\partial\mathscr{T}$ a Cantor set.*

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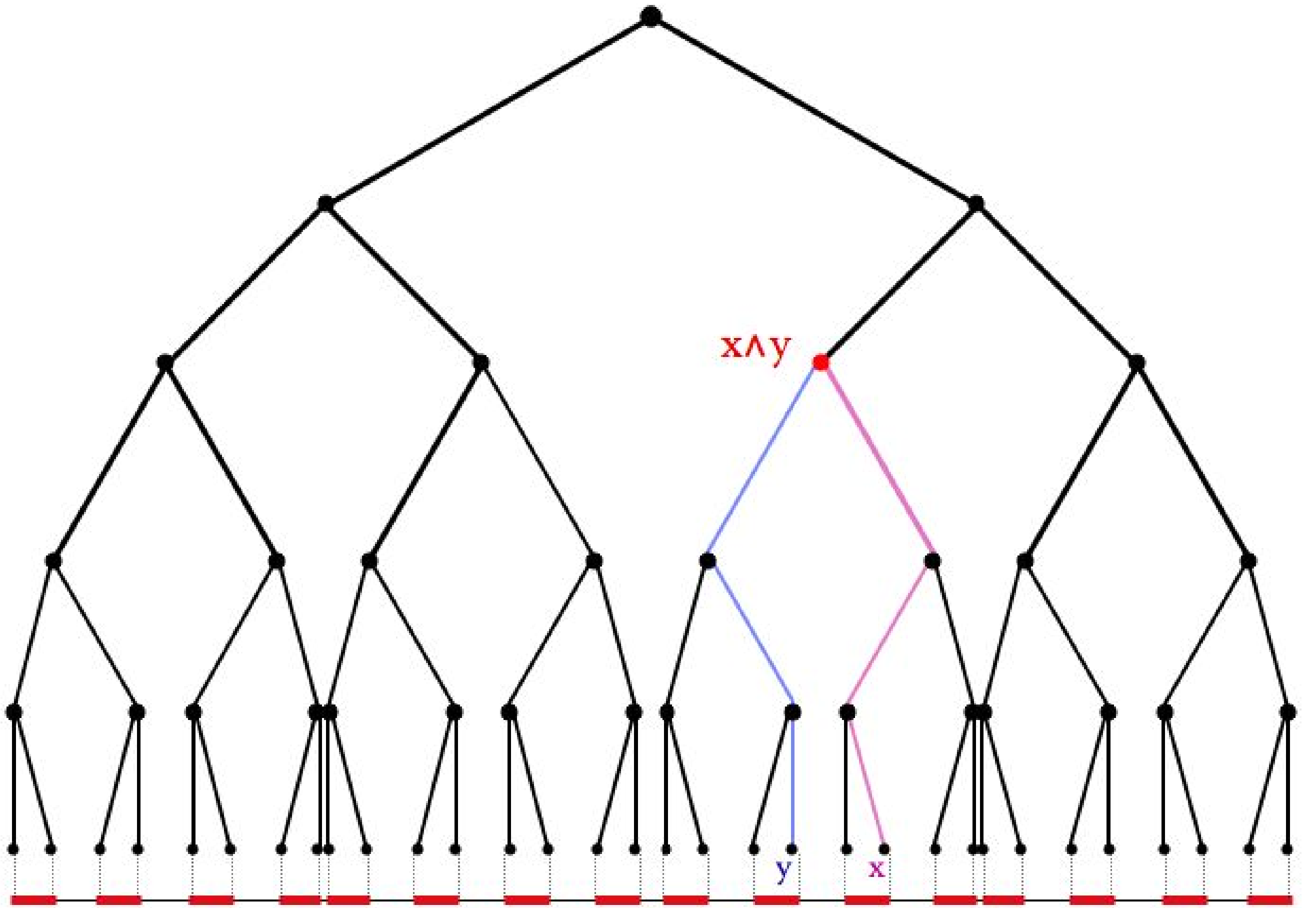
Theorem *If $\mathscr{T}$ is a Cantorian rooted tree with a weight δ, then $\partial\mathscr{T}$ admits a canonical* ultrametric *d_δ defined by.*

$$d_\delta(x, y) = \delta([x \wedge y])$$

where $[x \wedge y]$ is the least common ancestor of x and y.



The least common ancestor of x and y

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Conversely, if $\mathscr{T}$ *is the Michon tree of a metric Cantor set* (C, d)*, with weight* $\delta(v) = \mathrm{diam}(v)$*, then there is a contracting homeomorphism from* (C, d) *onto* $(\partial\mathscr{T}, d_\delta)$ *and* d_δ *is the smallest ultrametric dominating* d.

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In particular, if d *is an ultrametric, then* $d = d_\delta$ *and the homeomorphism is an isometry.*

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Conversely, if $\mathcal{T}$ is the Michon tree of a metric Cantor set (C, d), with weight $\delta(v) = \operatorname{diam}(v)$, then there is a contracting homeomorphism from (C, d) onto $(\partial\mathcal{T}, d_\delta)$ and d_δ is the smallest ultrametric dominating d.

In particular, if d is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.

This gives a representation of all ultrametric Cantor sets together with a parametrization of the space of ultrametrics.

II - Spectral Triples

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

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- G, D are defined by

$$(D\psi)_v = \frac{1}{\delta(v)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_v \qquad (G\psi)_v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi_v$$

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- $\mathcal{A} = C_{\text{Lip}}(C)$ is the space of Lipshitz continuous functions on (C, d)

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Let $\mathrm{Ch}(v)$ be the set of children of v. Consequently, the set $\Upsilon(\mathsf{C})$ of choices is given by

$$\Upsilon(\mathsf{C}) = \prod_{v \in \mathscr{V}} \Upsilon_v \qquad \Upsilon_v = \bigsqcup_{w \neq w' \in \mathrm{Ch}(v)} [w] \times [w']$$

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Within this interpretation, the set $\Upsilon(C)$ can be seen as the unit sphere bundle inside the tangent bundle.

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Let $\tau \in \Upsilon(C)$ be a choice. If $v \in \mathscr{V}$ write $\tau(v) = (\tau_+(v), \tau_-(v))$. Then π_τ is the representation of $C_{\rm Lip}(C)$ into $\mathcal{H}$ defined by

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Theorem *The distance d on C can be recovered from the following Connes formula*

$$d(x,y) = \sup\left\{ \left| f(x) - f(y) \right| \; ; \; \sup_{\tau \in \Upsilon(C)} \|[D, \pi_\tau(f)]\| \leq 1 \right\}$$

Remark: the commutator $[D, \pi_\tau(f)]$ is given by

$$([D, \pi_\tau(f)]\psi)_v = \frac{f(\tau_+(v)) - f(\tau_-(v))}{d_\delta(\tau_+(v), \tau_-(v))} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \psi_v$$

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In particular $\sup_\tau \|[D, \pi_\tau(f)]\|$ is the Lipshitz norm of f

$$\|f\|_{\text{Lip}} = \sup_{x \neq y \in C} \left| \frac{f(x) - f(y)}{d_\delta(x, y)} \right|$$

III - ζ-function and Metric Measure

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons 1990.

G.H. Hardy & M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press (1915).

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Theorem *Let (C, d) be an ultrametric Cantor set. The abscissa of convergence of the ζ-function of the corresponding Dirac operator coincides with the* upper box dimension *of (C, d).*

- The *upper box dimension* of a compact metric space (X, d) is defined by

$$\overline{\dim}_B(C) = \limsup_{\delta \downarrow 0} \frac{\log N_\delta(C)}{-\log \delta}$$

where $N_\delta(X)$ is the least number of sets of diameter at most δ that cover X.

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- Thanks to the definition of the Dirac operator

$$\zeta(s) = 2 \sum_{v \in \mathcal{V}} \delta(v)^s$$

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$$\zeta(s) = 2 \sum_{v \in \mathcal{V}} \delta(v)^s$$

- There are examples of metric Cantor sets with *infinite upper box dimension*. This is the case for the transversal of tilings with positive entropy.

III.2)- Dixmier Trace & Metric Measure

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If the abscissa of convergence is finite, then a *probability measure μ* on (C, d) can be defined as follows (if the limit exists)

$$\mu(f) = \lim_{s \downarrow s_0} \frac{\mathrm{Tr}\ (|D|^{-s} \pi_\tau(f))}{\mathrm{Tr}\ (|D|^{-s})} \qquad f \in \mathcal{C}_{\mathrm{Lip}}(C)$$

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This limit coincides with the *normalized Dixmier trace*

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Theorem *The definition of the* Metric Measure *μ is independent of the choice τ.*

- If ζ admits an *isolated simple pole at $s = s_0$*, then $|D|^{-1}$ belongs to the *Mačaev ideal $\mathcal{L}^{s_0+}(\mathcal{H})$*. Therefore the measure μ is well defined.

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- In particular μ is the *metric analog of the Lebesgue measure class* on a Riemannian manifold, in that the measure of a ball of radius r behaves like r^{s_0} for r small

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- μ is the analog of the *volume form* on a Riemannian manifold.

As a consequence μ defines a *canonical probability measure* ν on the space of choices Υ as follows

$$\nu = \bigotimes_{v \in \mathscr{V}} \nu_v \qquad\qquad \nu_v = \frac{1}{Z_v} \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu \otimes \mu|_{[w]\times[w]}$$

where Z_v is a normalization constant given by

$$Z_v = \sum_{w \neq w' \in \mathrm{Ch}(v)} \mu([w])\mu([w'])$$

IV - The Laplace-Beltrami Operator

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IV.1)- Dirichlet Forms

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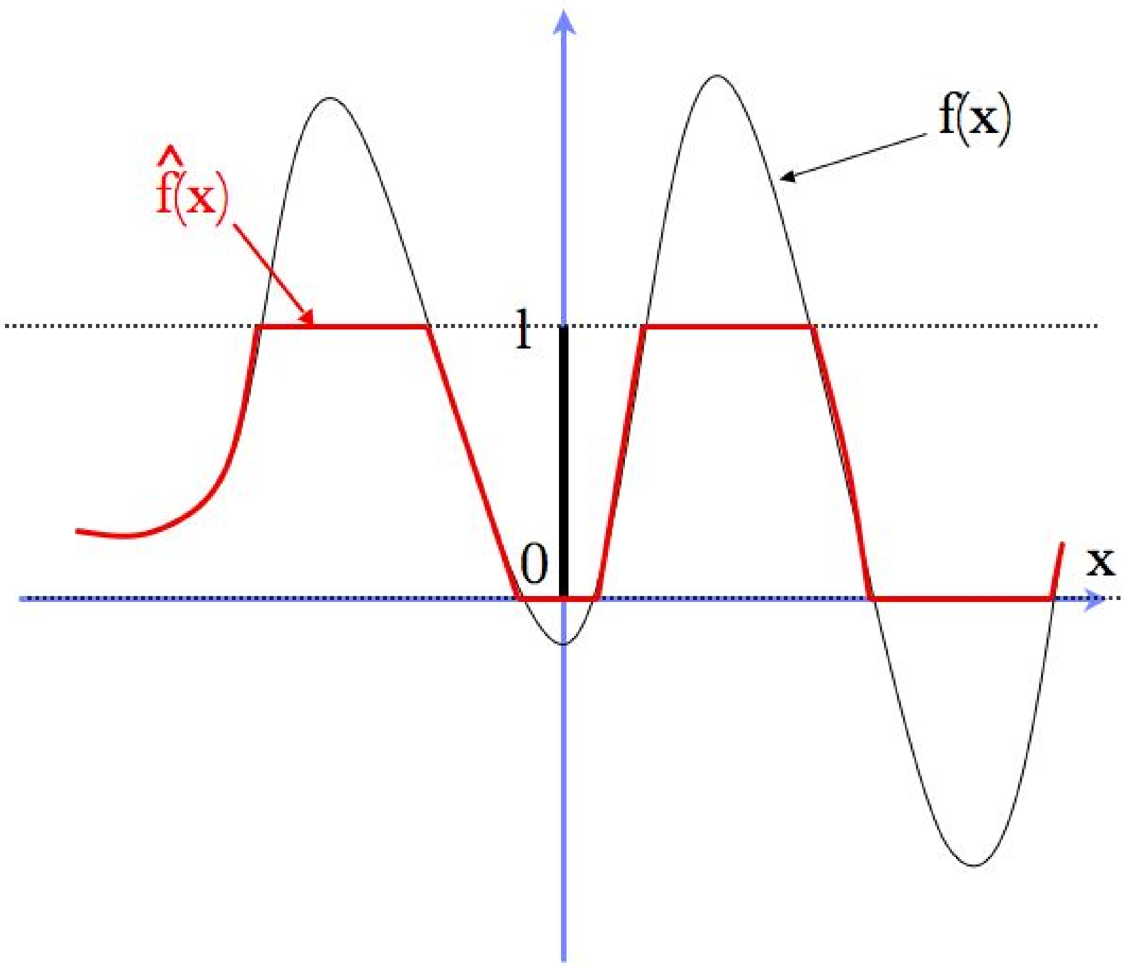
Let (X, μ) be a probability space space. For f a *real valued* measurable function on X, let $\hat{f}$ be the function obtained as

$$\hat{f}(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq 1 \\ f(x) & \text{if} \quad 0 \leq f(x) \leq 1 \\ 0 & \text{if} \quad f(x) \leq 0 \end{cases}$$



Markovian cut-off of a real valued function

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- Q is densely defined with domain $\mathscr{D} \subset L^2(X, \mu)$
- Q is closed
- Q is *Markovian*, namely if $f \in \mathscr{D}$, then $Q(\hat{f}, \hat{f}) \leq Q(f, f)$

The simplest typical example of Dirichlet form is related to the Laplacian Δ_Ω on a bounded domain $\Omega \subset \mathbb{R}^D$

$$Q_\Omega(f,g) = \int_\Omega d^D x \ \overline{\nabla f(x)} \cdot \nabla g(x)$$

with domain $\mathscr{D} = C_0^1(\Omega)$ the space of continuously differentiable functions on Ω vanishing on the boundary.

This form is closeable in $L^2(\Omega)$ and its closure defines a Dirichlet form.

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If Q is a Dirichlet form on X, then the contraction semigroup $\Phi = (\Phi_t)_{t\geq 0}$ is a *Markov semigroup*.

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Theorem (Fukushima) *A contraction semi-group on $L^2(X,\mu)$ is a Markov semi-group if and only if its generator is defined by a Dirichlet form.*

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where g is the metric. Equivalently (in local coordinates)

$$Q_M(f,g) = \int_M d^D x \, \sqrt{\det(g(x))} \int_{S(x)} d\nu_x(u) \, \overline{u \cdot \nabla f(x)} \, u \cdot \nabla g(x)$$

IV.2)- The Laplace-Beltrami Form

Let M be a *Riemannian manifold* of dimension D. The *Laplace-Beltrami operator* is associated with the Dirichlet form

$$Q_{\mathrm{M}}(f,g) = \sum_{i,j=1}^{D} \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))} \, g^{ij}(x) \, \overline{\partial_i f(x)} \, \partial_j g(x)$$

where g is the metric. Equivalently (in local coordinates)

$$Q_{\mathrm{M}}(f,g) = \int_M d^{\mathrm{D}}x \, \sqrt{\det(g(x))} \int_{S(x)} d\nu_x(u) \, \overline{u \cdot \nabla f(x)} \, u \cdot \nabla g(x)$$

where $S(x)$ represent the *unit sphere* in the tangent space whereas ν_x is the *normalized Haar measure* on $S(x)$.

Similarly, if (C, d) is an ultrametric Cantor set, the expression

$$[D, \pi_\tau(f)]$$

can be interpreted as a *directional derivative*, analogous to $u \cdot \nabla f$, since a choice τ has been interpreted as a unit tangent vector.

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The *Laplace-Pearson operators* are defined, by analogy, by

$$Q_s(f, g) = \int_\Upsilon d\nu(\tau) \, \mathrm{Tr} \left\{ \frac{1}{|D|^s} [D, \pi_\tau(f)]^* \, [D, \pi_\tau(g)] \right\}$$

for $f, g \in C_{\mathrm{Lip}}(C)$ and $s > 0$.

Let $\mathscr{D}$ be the linear subspace of $L^2(C, \mu)$ generated by the *characteristic functions* of the clopen sets $[v]$, $v \in \mathscr{V}$. Then

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Theorem *For any $s \in \mathbb{R}$, the form Q_s defined on $\mathscr{D}$ is closeable on $L^2(C, \mu)$ and its closure is a Dirichlet form.*

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The corresponding operator $-\Delta_s$ leaves $\mathscr{D}$ invariant, has a discrete spectrum.

For $s < s_0 + 2$, $-\Delta_s$ is unbounded with compact resolvent.

IV.3)- Jumps Process over Gaps

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Δ_s generates a Markov semigroup,
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Then if χ_v is the characteristic function of $[v]$

$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v,w)(\chi_w - \chi_v)$$

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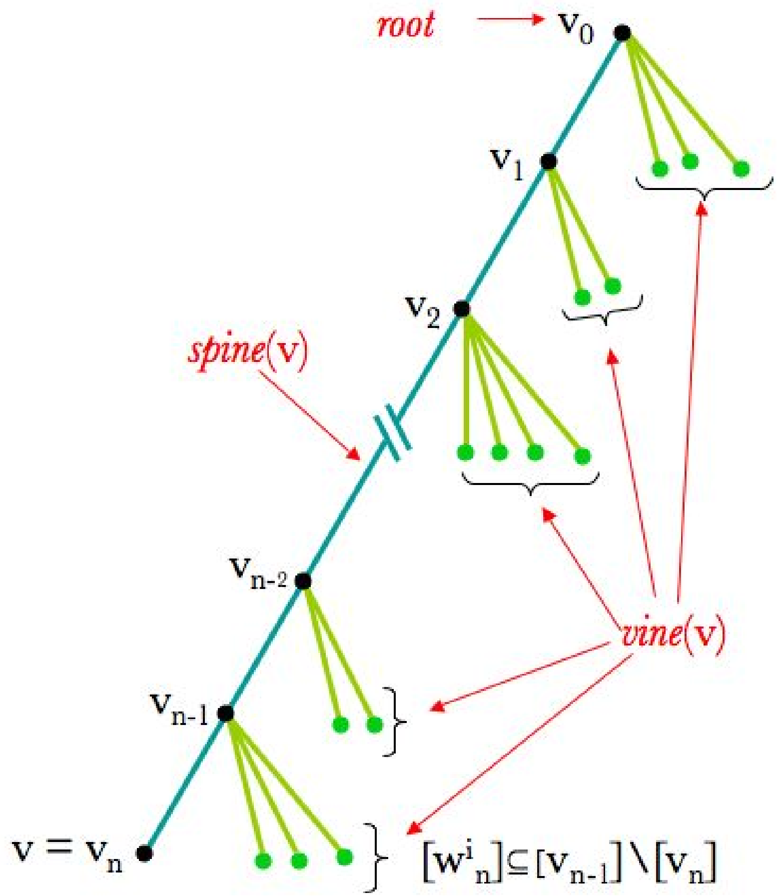
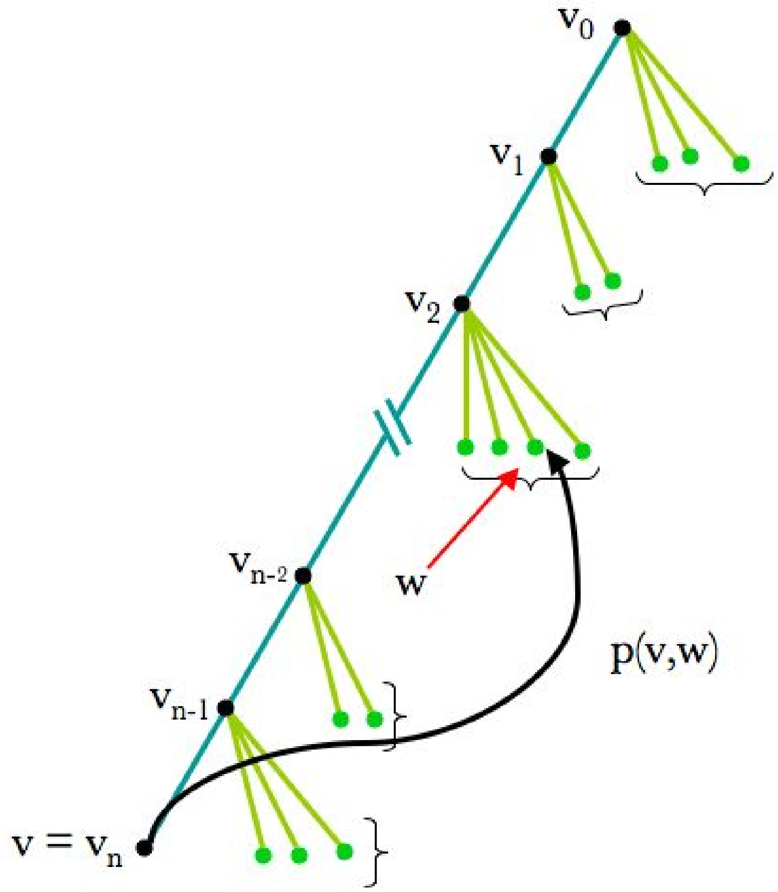
Then if χ_v is the characteristic function of $[v]$

$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v,w)(\chi_w - \chi_v)$$

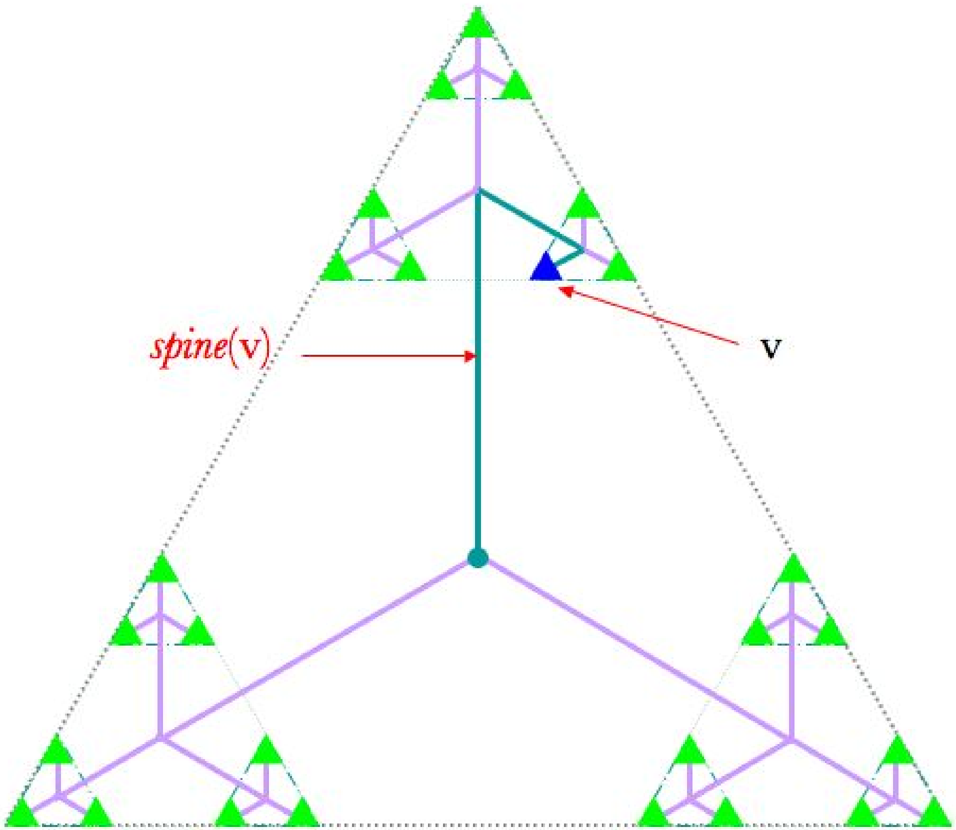
where $p(v,w) > 0$ represents the *probability for X_t to jump from v to w per unit time*.



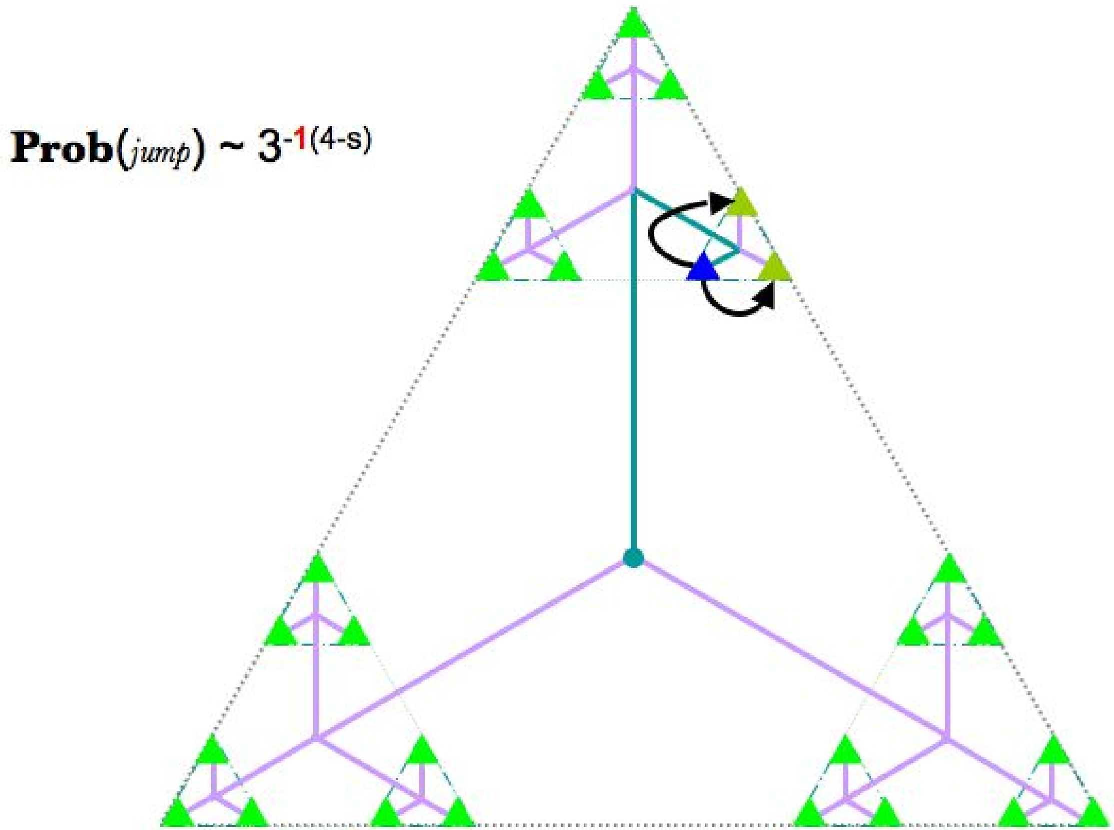
The vine of a vertex v



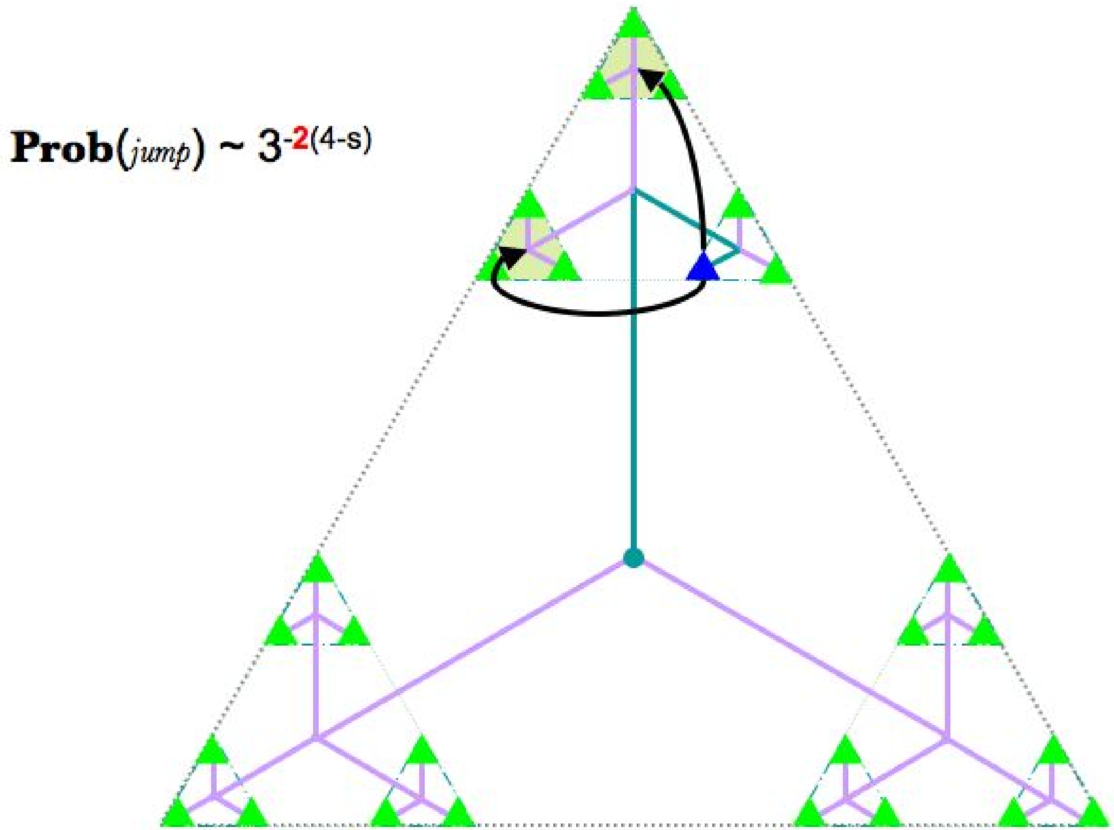
Jump process from v to w



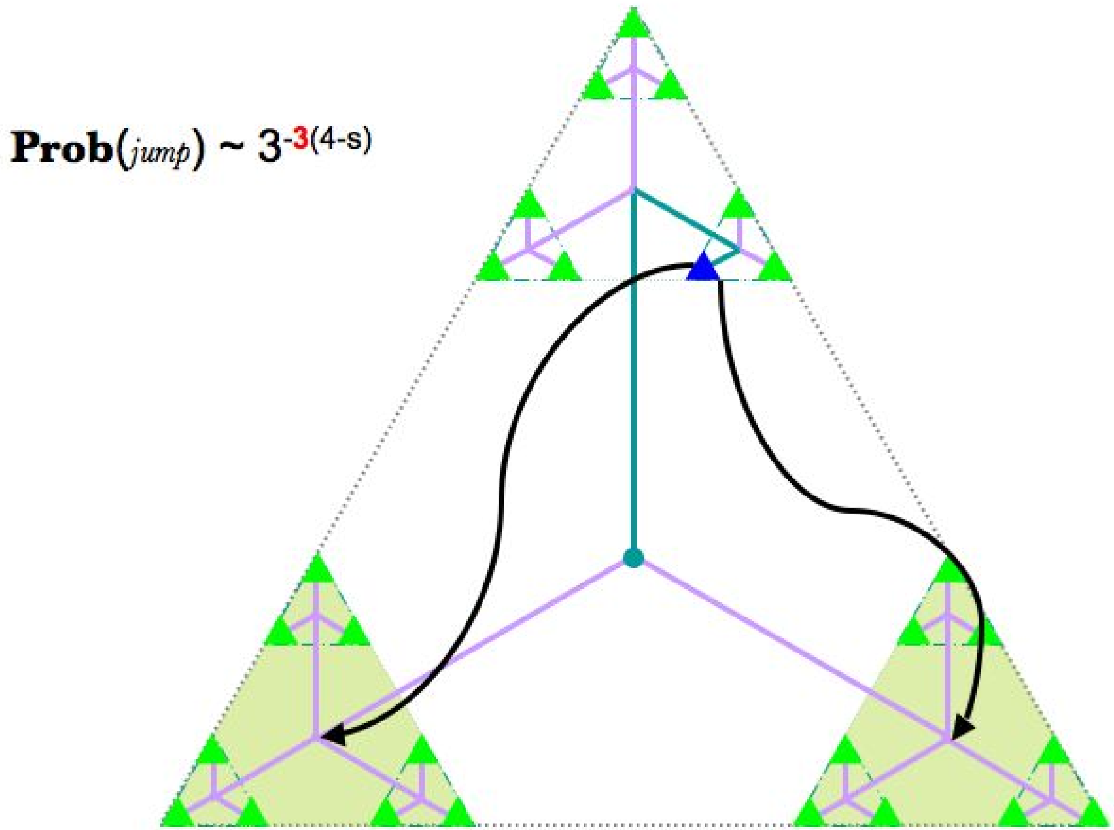
The tree for the triadic ring $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$



Jump process in $\mathbb{Z}(3)$

Concretely, if $\hat{w}$ denotes the *father* of w (which belongs to the spine)

$$p(v, w) \;=\; 2\delta(\hat{w})^{s-2}\; \frac{\mu([v])}{Z_{\hat{w}}}$$

where $Z_{\hat{w}}$ is the normalization constant for the measure $\nu_{\hat{w}}$ on the set of choices at $\hat{w}$, namely

$$Z_{\hat{w}} = \sum_{u \neq u' \in \mathrm{Ch}(\hat{w})} \mu([u])\mu([u'])$$

IV.4)- Eigenspaces

Let v be a vertex of the Michon graph with $\mathrm{Ch}(v)$ as its set of children.

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$$\chi_v = \sum_{w \in \mathrm{Ch}(v)} \chi_w \quad \in \mathcal{E}_v \, .$$

Theorem *For any $s \in \mathbb{R}$, the eigenspaces of $-\Delta_s$ are the spaces of the form $\{\chi_v\}^{\perp} \subset \mathcal{E}_v$, namely, the orthogonal complement of χ_v is $\mathcal{E}_v$.*

V - To conclude

- Ultrametric Cantor sets can be described as *Riemannian manifolds*, through Noncommutative Geometry.
- An analog of the *tangent unit sphere* is given by *choices*
- The *upper box dimension* plays the role of the dimension
- A *volume measure* is defined through the Dixmier trace
- A *Laplace-Beltrami operator* is defined with compact resolvent and Weyl asymptotics
- It generates a *jump process* playing the role of the *Brownian motion*.
- This process exhibits *anomalous diffusion*.

Recent Progress

I. Palmer, *Noncommutative Geometry and Compact Metric Spaces*, PhD Thesis, Georgia Tech, May 2010.

J. Cheeger, *Differentiability of Lipschitz continuous Functions on Metric Measure Spaces*
GAFA, Geom. funct. anal., **9**, 428-517, (1999).

- The construction of a spectral triple can be extended to any *compact metric space* if the partitions by clopen sets are replaces by suitable *open covers*.
- If the compact metric space (X, d) has *finite Hausdorff dimension* then the spectral triple can be chosen to admits $\dim_H(X)$ as *abscissa of convergence*.
- If (X, d) admits a *positive finite Hausdorff measure* the spectral triple can be constructed so as to have the measure μ, defined by the Dixmier trace, equal to the *normalized Hausdorf measure*.
- Under some extra local regularity property on (X, d) a Laplace-Beltrami operator be defined (J. Cheeger).

Lecture III - Spectral Metric Spaces

I - Spectral Triples and Dynamics

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

- there is a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D is selfadjoint with compact resolvent (*Dirac operator*)
- there is a core $\mathcal{D} \subset \mathcal{H}$ for D and a *-invariant subset $\mathscr{A} \subset \mathcal{A}$, generating $\mathcal{A}$, such that any element $a \in \mathscr{A}$ leaves $\mathcal{D}$ invariant and such that $[D, a]$ is bounded.

Remark: *Then the set* $C^1(X) = \{a \in \mathcal{A} ; \|[D, a]\| < \infty\}$ *is a dense* $*$*-subalgebra of* $\mathcal{A}$*, invariant under the holomorphic functional calculus.*

A $*$-automorphism α on $\mathcal{A}$ is a *quasi-isometry* on X if α and α^{-1} leave $C^1(X)$ invariant. Then (X, α) is called a *metric dynamical system*.

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

$$d(x,y) = \sup\{|f(x) - f(y)|\,;\ f \in \mathcal{A},\ \|[D,f]\| \leq 1\}$$

Actually $\|[D,f]\| = \|\nabla f\|_{L^\infty} = \|f\|_{C_{\mathrm{Lip}}}$ and $C^1(X) = \mathrm{Lip}(M)$.

The *geodesic flow* defines a one-parameter group of quasi-isometries (actually isometries) on $\mathcal{A}$.

Problem

Let $(X, \alpha) = (\mathcal{A}, \mathcal{H}, D, \alpha)$ *be a metric dynamical system.*

Is there a canonical *spectral triple* $Y = (\mathcal{A} \rtimes_\alpha \mathbb{Z}, \mathcal{K}, \hat{D})$*, based on the crossed product algebra induced by the dynamics, inducing on* X *an equivalent metric structure ?*

It will be shown that the answer is *YES* only when α is equivalent to an *isometry*.

Problem

If α cannot be reduced to an isometry, then, following the Connes-Moscovici approach, the analog of the *metric bundle* construction gives a way to change X into a new spectral triple $\hat{X}$ on which α induces a dynamic $\hat{\alpha}$ which becomes an *isometry* and allows to make the construction.

The latter construction comes with a *price*: $\hat{X}$ is *no longer compact* on which the *metric* is *unbounded* in general.

This is a source of technical difficulties that are not understood *fully yet*.

II - The Basic Construction

Compact Spectral Metric Spaces

Let $X = (\mathcal{A}, \mathcal{H}, D)$ be a spectral triple.
It will be called *compact* whenever $\mathcal{A}$ is unital.
It will be called a *spectral metric space* if

- The *D-commutant* $\mathcal{A}'_D = \{a \in \mathcal{A}\,;\, [D, a] = 0\}$ is reduced to $\mathbb{C}\mathbf{1}$
- The *Lipshitz ball* $B_{Lip} = \{a \in \mathcal{A}\,;\, \|[D, a]\| \leq 1\}$ has a precompact image in $\mathcal{A}/\mathcal{A}'_D$.

Theorem (Pavlovic, Rieffel) *A compact spectral triple is a spectral metric space if and only if the Connes distance on the state space*

$$d(\rho, \omega) = \sup\{|\rho(a) - \omega(a)|\,;\, a \in \mathcal{A},\ \|[D, a]\| \leq 1\}$$

is bounded and generates the weak-topology.*

Quasi-isometries

Let $\mathrm{Qiso}(X)$ be the set of quasi-isometries of the compact spectral metric space $X = (\mathcal{A}, \mathcal{H}, D)$. Then

Proposition *A $*$-automorphism of $\mathcal{A}$ is a quasi-isometry if and only if it generates a bi-Lipshitz transformation of the state space, namely there is $C > 0$ such that*

$$\frac{1}{C}\, d(\rho, \omega) \leq d(\rho \circ \alpha, \omega \circ \alpha) \leq C\, d(\rho, \omega)$$

for every pair of states (ρ, ω).

Equicontinuity

Let $X = (\mathcal{A}, \mathcal{H}, D)$ be a compact spectral metric space. A quasi-isometry $\alpha \in \mathrm{Qiso}(X)$ is called *equicontinuous* whenever

$$\sup_{n \in \mathbb{Z}} \| [D, \alpha^n(a)] \| < \infty \qquad \forall a \in C^1(X)$$

Theorem *A quasi-isometry is equicontinuous if and only if the group it generates in the set of $*$-automorphism of $\mathcal{A}$ has a compact closure*

$\alpha \in \mathrm{Qiso}(X)$ is called an *isometry* whenever

$$\| [D, a] \| = \| [D, \alpha(a)] \| \qquad \forall a \in C^1(X)$$

Proposition (Rieffel) $\alpha \in \mathrm{Qiso}(X)$ *is an isometry if and only if it defines an isometry in the state space for the Connes metric.*

Main Result

Let $\mathcal{A}$ be a unital separable C*-algebra.
Let α be a $*$-automorphism of $\mathcal{A}$.
Then, let u denotes the unitary implementing α in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$.

Theorem *There is a spectral metric space* $X = (\mathcal{A}, \mathcal{H}, D)$ *based on* $\mathcal{A}$ *for which* α *is equicontinuous if and only if there is a spectral metric space* $Y = (\mathcal{A} \rtimes_\alpha \mathbb{Z}, \mathcal{K}, \hat{D})$*, based on the crossed product, such that*

- *The dual action on* $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ *is equicontinuous*
- $u^{-1}[\hat{D}, u]$ *is bounded and commutes to the elements of* $\mathcal{A}$
- *The Connes metrics induced by* X *and by* Y *on the state space of* $\mathcal{A}$ *are equivalent*

Constructing Υ

- **Hilbert space:** $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$
  Then $f \in \mathcal{K} \Leftrightarrow f = (f_{n+}, f_{n-})_{n\in\mathbb{Z}}$ with $f_{n\pm} \in \mathcal{H}$
- **Representation:** left regular representation $\hat{\pi}$ of $\mathcal{A} \rtimes_\alpha \mathbb{Z}$

$$(\hat{\pi}(a)f)_n = \alpha^{-n}(a) f_n \qquad (\hat{\pi}(u)f)_n = f_{n-1} \qquad a \in \mathcal{A}$$

- **Dirac operator:**

$$\left(\widehat{D}f\right)_n = \begin{bmatrix} 0 & D - \imath n \\ D + \imath n & 0 \end{bmatrix} f_n$$

Properties of Υ

- **Commutator with $\widehat{D}$:**

$$\left([\widehat{D}, \hat{\pi}(a)]f\right)_n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [D, \alpha^{-n}(a)]\, f_n$$

  Hence $[\widehat{D}, \hat{\pi}(a)]$ is bounded if and only if $\alpha \in \mathrm{Qiso}(X)$.

$$\left(\hat{\pi}(u^{-1})[\widehat{D}, \hat{\pi}(u)]f\right)_n = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} f_n$$

  Hence $u^{-1}[\widehat{D}, u]$ commutes with the elements of $\mathcal{A}$.

- **Dual action:**

$$(v_k f)_n = e^{-\imath k n} f_n \qquad\qquad k \in \mathbb{T}$$

  commutes with $\widehat{D}$, thus is *isometric*

Properties of Υ

Lemma: ***(difficult)*** *The Lipshitz Ball of Υ is precompact modulo the $\widehat{D}$-commutant*

Lemma: *The metric induced on the state space of $\mathcal{A}$ by $\widehat{D}$ is equivalent to the metric induced by X and makes α an isometry*

The last result shows that the basic construction is the noncommutative analog of the construction of an *invariant metric* on a classical metric space when the action is provided by an equicontinuous bi-Lipshitz homeomorphism.

Examples

Crossed product algebra $C(M) \rtimes_\phi \mathbb{Z}$ if M is a *compact metric space* and ϕ an *isometry* or, more generally, an *homeomorphism* satisfying

$$\sup_{n\in\mathbb{Z}} \left(\sup_{x\neq y} \frac{d(\phi^n(x), \phi^n(y))}{d(x,y)} \right) < \infty$$

- For instance the action of an odometer on the Cantor set can be seen in this way.
- Any Kronecker flow on a torus (leading to a noncommutative torus)
- The geodesic flow at time $t = 1$ on a compact spin^c Riemannian manifold

III - The Metric Bundle

Examples

Arnold's cat map: $\mathcal{A} = C(\mathbb{T}^2)$, $\mathcal{H} = L^2(\mathbb{T}^2) \otimes \mathbb{C}^2$, and

$$D = \begin{bmatrix} 0 & -\imath\partial_1 - \partial_2 \\ -\imath\partial_1 + \partial_2 & 0 \end{bmatrix}, \qquad \phi(x) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x \, ,$$

with $\alpha(f) = f \circ \phi^{-1}$. Then α is a *quasi-isometry* that is *not equicontinuous*

$$\| [D, \alpha^n(f)] \| \overset{|n|\uparrow\infty}{\sim} \left(\frac{\sqrt{5}+1}{2} \right)^{|n|}$$

More generally any *strictly hyperbolic map* on a compact metric space (*Smale spaces*) will give rise to a similar situation.

The Metric Bundle

If M is a smooth manifold, the *metric bundle* is a principle bundle over M such that the fiber over each point is the cone of possible positive definite metrics on the tangent space.

Connes and Moscovici have shown that this bundle admits a tautological *Riemannian structure* that is *invariant by the diffeomorphisms* of M. In particular each diffeomorphim becomes an *isometry* for this structure.

The Metric Bundle

If ϕ is a diffeomorphism of M, it is sufficient to restrict this bundle to the *orbits* of ϕ with its Riemannian structure.

The *C^*-algebra of this orbit* is the tensor product $C(M) \otimes c_0(\mathbb{Z})$. The action of ϕ on the $\mathbb{Z}$-part is reduced to the *shift*.

Metric on $\mathbb{Z}$

Let $d_{\mathbb{Z}}$ be a *bounded translation invariant metric* on $\mathbb{Z}$. Then a spectral triple, based on $c_0(\mathbb{Z})$, can be defined as follows

- **Clifford matrices:** $\gamma_1, \cdots, \gamma_4$ acting on the Hilbert space $\mathcal{E}$
- **Hilbert Space:** $\ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathcal{E}$
- **Operators:**,

$$(\nabla f)_{n,r} = \frac{f_{n,r} - f_{n-r,r}}{d_{\mathbb{Z}}(n, n-r)}, \qquad (X f)_{n,r} = \left(n\,\gamma_3 + \frac{1}{d_{\mathbb{Z}}(0,r)^2}\gamma_4\right) f_{n,r}$$

- **Dirac operator:**

$$D_{\mathbb{Z}} = \frac{\gamma_1 + \imath\gamma_2}{2}\,\nabla + \frac{\gamma_1 - \imath\gamma_2}{2}\,\nabla^* + X.$$

Metric on $\mathbb{Z}$

Ref.: F. Latrémolière, *Taiwanese J. of Math.*, **11**, (2007), 447-469.

Proposition: $(c_0(\mathbb{Z}), \ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathcal{E}, D_{\mathbb{Z}})$ *is a spectral triple.*

Its Lipshitz Ball B_{Lip} *is bounded and, for any strictly positive sequence* $h \in c_0(\mathbb{Z})$, $hB_{Lip}h$ *is precompact.*

In particular, while the state space of $c_0(\mathbb{Z})$ *is not weak*-compact, the Connes distance is bounded and generates the weak*-topology.*

The Spectral Metric Bundle

Theorem: *Let* $X = (\mathcal{A}, \mathcal{H}, D)$ *be a compact spectral metric space. Let* $\alpha \in \mathrm{Qiso}(X)$ *be non-equicontinuous.*

Then there is a spectral triple $Y = (\mathcal{A} \otimes c_0(\mathbb{Z}), \mathcal{K}, D_{\mathcal{K}})$ *which is a non-compact spectral metric space for which the Connes metric is bounded on which* α *can be extended as an isometry.*

Moreover, $\mathcal{K}$ *support a representation of* $\mathcal{C} = \mathcal{A} \otimes c_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ *which makes* $Z = (\mathcal{C}, \mathcal{K}, D_{\mathcal{K}})$ *a spectral metric space on which the dual action is equicontinuous with respect to the weak-uniform topology.*

The Spectral Metric Bundle

Let $X = (\mathcal{A}, \mathcal{H}, D)$ be a compact spectral metric space and let $\alpha \in \mathrm{Qiso}(X)$. If α is *not equicontinuous*, then Y will be the spectral triple built as follows

- $\mathcal{A}$ is replaced by $\mathcal{A} \otimes c_0(\mathbb{Z})$. Then α is extended as

$$\hat{\alpha}(b)_n = \alpha(b_{n-1}), \qquad b \in \mathcal{A} \otimes c_0(\mathbb{Z})$$

- **Hilbert space:** $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathcal{E}$, where now, $\mathcal{E}$ is the representation space for five Clifford matrices.
- **Representation:**

$$(bf)_{n,r} = \alpha^{-n}(b_n)\, f_{n,r}, \qquad b \in \mathcal{A} \otimes c_0(\mathbb{Z})$$

The Spectral Metric Bundle

- **Dirac operator:** $D_{\mathcal{K}} = D_{\mathbb{Z}} + \gamma_5 D$
- **The action** $\hat{\alpha}$

$$(uf)_{n,r} = f_{n-1,r}\,, \qquad \Rightarrow \qquad ubu^{1} = \hat{\alpha}(b)$$

- Then $u^{-1}[D_{\mathcal{K}}, u]$ is bounded and commutes with the elements of $\mathcal{A} \otimes c_0(\mathbb{Z})$.
- In particular $\hat{\alpha}$ is *isometric* on Y.
- Moreover, $\mathcal{K}$ supports a representation of the crossed product $\mathcal{C} = \mathcal{A} \otimes c_0(\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z}$.

The Spectral Metric Bundle

- **Dual action:**

$$(v_k f)_{n,r} = e^{\imath kn} f_{n,r}\,, \qquad \Rightarrow \qquad v_k u v_k^{-1} = e^{\imath k} u$$

This dual action is NOT *equicontinuous* for the norm topology. However it is equicontinuous for the *weak-uniform topology*.

- If C_{Lip} is the Lipshitz ball in the crossed product, then there is h *strictly positive* in $\mathcal{C} = \mathcal{A} \otimes c_0(\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z}$ such that $hC_{Lip}h$ is norm compact.
- this is enough to show that the Connes metric associated with the triple $(\mathcal{C}, \mathcal{K}, D_{\mathcal{K}})$ generates the weak*-topology in the state space.

IV - Conclusion and Remarks

To Conclude

1. *Equicontinuity* of a quasi-isometry is necessary and sufficient to built a spectral metric space over the crossed product algebra.

2. If equicontinuity fails, the *metric bundle* construction, restricted to the orbit of the dynamical system, provides a way to make the dynamics isometric.

3. As long as the metric chosen along this orbit is *bounded* the construction is under control: the Connes metric generates the weak* topology on the state space.

Open Problems

1. Can this construction be extended to the case of a *group action* ? Say for discrete groups with a length function ?

2.
   - What if the metric on $\mathbb{Z}$ chosen along the orbit in the metric bundle, is *unbounded* (like the usual metric on $\mathbb{Z}$) ?
   - More generally, is there an analog of the Rieffel-Pavlovič result for spectral triples for which the Lipshitz ball is unbounded ? Namely, what are the condition for the Connes metric to generate the weak*-topology ?
   - This is the noncommutative analog of the *Wasserstein distance* on the set of probabilities on a complete (unbounded) metric space.