# MATH 1501
# $\delta$,$\varepsilon$ PROOF

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**Problem.** Give a $\delta, \varepsilon$ proof that
$$\lim_{x \to 2} x^2 = 4.$$

*Solution.* Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|x^2 - 4| < \varepsilon$. To do this, we first note that we can write
$$|x^2 - 4| = |(x+2)(x-2)| = |x+2||x-2|. \tag{$*$}$$
We now would like to control $|x+2|$. We can do this by assuming that $\delta \leq 1$ (a restriction that we'll later combine to determine what $\delta$ needs to be). Then if $|x-2| < \delta \leq 1$, we have
$$-1 < x - 2 < 1 \Rightarrow 1 < x < 3 \Rightarrow 3 < x + 2 < 5,$$
so we can safely say that
$$|x+2| < 5. \tag{$**$}$$
Then using (*), we can see that
$$|x^2 - 4| = |x+2||x-2| < 5|x-2|.$$
Thus, if we have that $|x-2| < \varepsilon/5$, we will have $|x^2 - 4| < 5(\varepsilon/5) = \varepsilon$ like we want. Of course, we also need our previous restriction that $\delta \leq 1$, so we take $\delta = \min(1, \varepsilon/5)$.

We now show that this $\delta$ works. Suppose that $0 < |x-2| < \delta$. Then since $\delta \leq \varepsilon/5$ by definition, we have that $|x-2| < \varepsilon/5$ and hence $5|x-2| < \varepsilon$. However, we also have that $\delta \leq 1$, so we know by (**) that $|x+2| < 5$, so we now have
$$\varepsilon > 5|x-2| > |x+2||x-2| = |x^2-4|.$$
Therefore, we have established that $\lim_{x \to 2} x^2 = 4$ as desired. □

In 1501B6, Katrina commented that she got that $\delta = \varepsilon/3$ worked (in effect saying that we could use a larger $\delta$ than we got above). In quickly glancing over things, it looked like that worked. However, consider $\varepsilon = 1$. Then if we take $\delta = 0.33$ (less than 1 and $\varepsilon/3$), we should have that $|2.3^2 - 4| < \varepsilon = 1$ since $|2.3 - 2| = 0.3 < \delta = 0.33$ (if $\delta = \varepsilon/3$ works for $\varepsilon = 1$). Unfortunately, we have $2.3^2 = 5.29$ and $|5.29 - 4| = 1.29 \not< \varepsilon = 1$. Therefore, we cannot use $\delta = \varepsilon/3$.

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