MATH 1501
THE DERIVATIVE OF $\frac{1}{\sqrt{x}}$

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Problem. Find the derivative of $\dfrac{1}{\sqrt{x}}$ from the definition.

Solution. By definition, we know that

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}.$$

We now multiply numerator and denominator by the conjugate in order to eradicate the radicals and get

$$\begin{aligned}
\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} &= \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}\,\frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} \\
&= \frac{\frac{x-(x+h)}{x(x+h)}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} = \frac{\frac{-h}{x(x+h)}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} \\
&= \frac{\frac{-1}{x(x+h)}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}
\end{aligned}$$

Thus, we have

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\frac{-1}{x(x+h)}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}} = \frac{\frac{-1}{x^2}}{2\frac{1}{\sqrt{x}}} = -\frac{\sqrt{x}}{2x^2} = -\frac{1}{2}x^{-3/2}. \quad \square$$

What I did in class was not markedly different from this. However, my algebra in the last couple of steps got screwed up a bit, and so my final answer was not correct. The above is, however.

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Date: 9 September 2005 2005.