School of Mathematics
Georgia Tech

Math 1501, QUIZ 4

Date: September 21, 2005 Name (printed; last name first) and section: ____________________

There is one problem on this quiz that is worth eight points. Two points are awarded solely for taking the quiz. Motivate your answers. Partial credit will be awarded.

(a) (6 points) Is the sequence whose general term is given by

$$a_n = \frac{n^2}{n+1}, \ n \geq 1$$

monotonic? If so, what kind of monotonicity does it have?

(b) (2 points) Is the sequence defined above *convergent*? If so, find its limit.

Solution.

a. Yes, this sequence is monotonic. There are a few ways that we can see this. One way is to define the continuous extension $f(x) = x^2/(x+1)$ and find its derivative

$$f'(x) = \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

which is positive for $x \geq 1$ so we know that the sequence is *increasing.* Another way is to consider the ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{n+2}}{\frac{n^2}{n+1}} = \frac{(n+1)^3}{n^3 + 2n^2} = \frac{n^3 + 3n^2 + 3n + 1}{n^3 + 2n^2}.$$

This ratio is greater than 1 since the numerator is greater than the denominator. Therefore, $a_{n+1} > a_n$.

b. No, this sequence is not convergent. Probably the easiest way to see this is that

$$a_n = \frac{n^2}{n+1} = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \frac{n^2}{n+1} = \frac{1}{\frac{1}{n} + \frac{1}{n^2}},$$

so as n gets arbitrarily large, the denominator gets arbitrarily small (both $1/n$ and $1/n^2$ tend to 0 as $n \to \infty$), and thus a_n gets arbitrarily large. When a sequence does not have a finite limit, we say that it diverges, not that it "converges to ∞", as our definition of convergence only works when the limit is a real number. □

Comments: There were several common mistakes on this quiz, and so I'd like to provide a brief rundown.

First, do not differentiate a_n. You can't differentiate a sequence, as it's only defined for positive integers, and $\lim_{h\to 0}[(f(x+h)-f(x))/h]$ doesn't make sense if the only valid inputs to your function are positive integers. You can, however, define the continuous extension and differentiate that. The fact that sequences are only defined for n being a positive integer also means that you should not use $n = 0$, $n = -1$, or any other negative value for n.

Second, the fact that a sequence is always positive does not guarantee that it is monotone. Consider the sequence

$$b_n := 2 + \frac{(-1)^n}{n}.$$

We have that $b_n > 0$ for all n, however it is not monotone, as $b_1 = 1$, $b_2 = 5/2$, $b_3 = 5/3$, etc. Also, a sequence doesn't have to oscillate to fail to be monotone. The sequence $-1, 1, 1/2, 1/3, 1/4, \ldots$ is not monotone because the second term is greater than the first but then (assuming the suggested pattern holds) each successive term is less than the previous. I wouldn't say that this sequence oscillates, however.

Third, ∞ is not a number! You cannot just "plug in" ∞ into a formula. Things such as 2∞ and $\infty + 1$ and ∞^2 **do not make sense**.

Fourth, when testing for monotonicity, you cannot just test a few values of n and see what happens. For example, if you didn't check the change between the first and second terms of the sequence $-1, 1, 1/2, 1/3, 1/4, \ldots$, you would be led to think that it's decreasing. If you just checked the first pair, you'd think that it's increasing. In fact, it's *neither*!

Fifth, a sequence does not fail to have a limit or fail to be bounded from above simply because it is increasing. Consider the sequence $c_n := -1/n$. This sequence is increasing, but all the terms are negative, so it's bounded above by 0 and in fact $\lim_{n\to\infty} c_n = 0$.