School of Mathematics
Georgia Tech

Math 1502, QUIZ 11

Date: April 09, 2008 Name (printed; last name first) and section: SOLUTION

There is one problem on this quiz that is worth eight points. Two points are awarded solely for taking the quiz. Motivate your answers. Partial credit will be awarded.

Perform the QR-factorization for the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and find an orthonormal basis for $Img(A)$.

We first find an orthonormal basis for Img(A), as then the QR-factorization will follow. Let

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Then $\vec{u}_1 = \frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\tfrac{1}{\sqrt{2}}\right)\cdot\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1/2 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{|\vec{w}_2|}\vec{w}_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{4}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\sqrt{\frac{2}{3}}\cdot 3\right)\left(\sqrt{\frac{2}{3}}\right)\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0.$$

Thus $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for Img A.

We have $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$ and $R = \begin{bmatrix} \vec{u}_1\cdot\vec{v}_1 & \vec{u}_1\cdot\vec{v}_2 & \vec{u}_1\cdot\vec{v}_3 \\ \vec{u}_2\cdot\vec{v}_1 & \vec{u}_2\cdot\vec{v}_2 & \vec{u}_2\cdot\vec{v}_3 \end{bmatrix}$

$\left(\sqrt{\frac{2}{3}}\cdot\frac{1}{2} = \frac{1}{\sqrt{6}}\right)$

$$= \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 2\sqrt{2} \\ 0 & \frac{\sqrt{6}}{2} & \sqrt{6} \end{bmatrix}.$$