CONVERGENCE OF FOURIER SERIES

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ABSTRACT. We give a proof of Carleson's theorem on almost everywhere convergence of Fourier series.

1. INTRODUCTION

We define the Carleson operator $\mathcal{C}$ acting on a Schwartz function f on $\mathbf{R}$ by

$$\mathcal{C}f(x) = \sup_N \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi \quad , \tag{1}$$

where the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} \, dx \quad .$$

We give a simplified proof of the well known theorem [1],[2]:

Theorem 1.1. *The Carleson operator $\mathcal{C}$ is of weak type $(2,2)$, i.e.,*

$$\|\mathcal{C}f\|_{L^{2,\infty}} \leq C\|f\|_2$$

with a constant C not depending on f.

This theorem is the key ingredient in the proof of Carleson's celebrated theorem, which asserts that the Fourier series of a function in $L^2([0,1])$ converges pointwise almost everywhere.

We became interested in Carleson's theorem while studying the bilinear Hilbert transform [3], [4], [5], [6]. As it turns out, these two subjects are closely related. The purpose of the current article is to exhibit these connections by giving a proof of Carleson's theorem in the spirit of [3], [4], [5], [6]. In particular, the key Proposition 3.2 below is essentially taken from these papers.

While L. Carleson [1] uses a decomposition of the function f and C. Fefferman [2] features a decomposition of the Carleson operator guided by N (the function which picks the worst N for each x in the Carleson operator), we emphasize a symmetry between f and N, as expressed by the duality of Propositions 3.1 and 3.2. This symmetry is more perfect in the case of the bilinear Hilbert transform, where instead of f and N one has three Schwartz functions f_1, f_2, and f_3, and a variant of Proposition 3.2 is applied to all three of them.

In Section 2 we introduce most of the notation used in this paper, and we do a discretization of the Carleson operator. In Section 3 we prove boundedness of the discretized Carleson operator by taking for granted Propositions 3.1 and 3.2 and some technical inequality (6) from standard singular integral theory. These remaining items are proved in Sections 4, 5, and 6.

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2. Notation and preliminary reductions

Define translation, modulation, and dilation operators by

$$T_y f(x) = f(x-y) \quad ,$$
$$M_\eta f(x) = f(x)e^{2\pi i\eta x} \quad ,$$
$$D^p_\Lambda f(x) = \Lambda^{-\frac{1}{p}} f(\Lambda^{-1}x) \quad .$$

We write $|E|$ for the measure of a set $E \subset \mathbf{R}^n$. Unless otherwise specified, an *interval* I will be of the form $[x,y)$ with $x<y$. Let $c(I)$ denote the center of I, and αI for $\alpha > 0$ the interval with the same center and α times the length of I. Let 1_I be the characteristic function of I and define the weight functions

$$w(x) = (1+x)^{-\nu}, \quad w_I(x) := T_{c(I)} D^1_{|I|} w \quad ,$$

where the letter ν as in the rest of the paper is used for a large integer whose exact value is not important and may be different at different places of the argument.

Let ϕ be a Schwartz function such that $\widehat{\phi}$ is real, nonnegative, supported in $[-0.1, 0.1]$, and equal to 1 on $[-0.09, 0.09]$. For each rectangle $P = I_P \times \omega_P$ of area 1 in the (phase-) plane define

$$\phi_{1P} := M_{c(\omega_{1P})} T_{c(I_P)} D^2_{|I_P|} \phi \quad ,$$

where we have written ω_{1P} for the lower half $\omega_P \cap (-\infty, c(\omega_P))$ of ω_P. Similarly we write ω_{2P} for the upper half $\omega_P \setminus \omega_{1P}$. Observe that $\widehat{\phi}_{1P}$ is supported in $\frac{1}{2}\omega_{1P}$ and we have

$$|\phi_{1P}(x)| \le C|I_P|^{\frac{1}{2}} w_P(x) \quad ,$$

where $w_P := w_{I_P}$ and C denotes as in the rest of the paper a large number whose value depends only on the choice of ϕ and ν and may be different at different places of the argument.

Let G denote the set of all dyadic intervals, i.e., intervals $[n2^k, (n+1)2^k)$ with integers n and k. Let $\overline{\mathbf{P}}$ denote those rectangles in $G \times G$ of area one. Define

$$A_\eta f = \sum_{P \in \overline{\mathbf{P}}} \langle f, \phi_{1P} \rangle \, \phi_{1P} 1_{\omega_{2P}}(\eta) \quad ,$$

$$Af = \lim_{n\to\infty} \frac{1}{|K_n|} \int_{K_n \times [0,1]} \sum_{P \in \overline{\mathbf{P}}} M_{-\eta} T_{-y} D^2_{2^{-\kappa}} A_{2^{-\kappa}\eta} D^2_{2^\kappa} T_y M_\eta f \, dy\, d\eta\, d\kappa \quad ,$$

where K_n is any increasing sequence of rectangles $I_n \times \omega_n$ filling out $\mathbf{R}^2$. To see the pointwise convergence of the last expression, consider separately those rectangles $P \in \overline{\mathbf{P}}$ with $|I_P|$ fixed, then the integrand becomes periodic in y and η for fixed κ, and observe that for very large and very small values of $|I_P|$ the integrand becomes small. It is easy to verify that A extends to a bounded operator on L^2, is nonzero and positive semidefinite, commutes with T_y for all y and with D^2_λ for all $\lambda > 0$, and satisfies $Af = 0$ if f has only positive frequencies. This identifies A as

$$Af(x) = c \int_{-\infty}^0 \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

for some constant $c \neq 0$. Hence the Carleson operator is equal to $\mathcal{C}f(x) = c^{-1}\sup_N (M_N A M_{-N} f)(x)$. We will prove that

$$\|\sup\nolimits_N A_N f\|_{L^{2,\infty}} \le C\|f\|_2 \quad . \tag{2}$$

By averaging this proves the desired bound for the Carleson operator.

By duality and the triangle inequality estimate (2) follows from

$$\sum_{P\in\mathbf{P}} |\langle f,\phi_{1P}\rangle \langle \phi_{1P}(1_{\omega_{2P}}\circ N), 1_E\rangle| \le C\|f\|_2|E|^{\frac12} \tag{3}$$

for all Schwartz functions f, measurable functions N, measurable sets E, and finite subsets $\mathbf{P}$ of $\overline{\mathbf{P}}$. Since this estimate is homogeneous in f and invariant under appropriate simultaneous dilations of f, N and E , it suffices to prove the estimate for $\|f\|_2 = 1$ and $|E| \le 1$. Whenever a set E is specified, we write

$$E_P := E\cap\{x : N(x)\in\omega_P\},\quad E_{2P} := E\cap\{x : N(x)\in\omega_{2P}\}\quad .$$

3. The main argument

A rectangle $P = I_P \times \omega_P$ of $\overline{\mathbf{P}}$ will be called a *tile*. Each tile has area 1 and is the union of two *semitiles* $P_1 = I_P \times \omega_{1P}$ and $P_2 = I_P \times \omega_{2P}$. Observe that dyadic intervals such as I_P, ω_P, $\omega_{\mathrm{j},P}$ have the property that any two of them are either disjoint or one is contained in the other. Moreover, if ω_{1P} is strictly contained in a dyadic interval, then ω_{2P} is strictly contained in the same interval and vice versa. We will use these geometric properties without referring to them. We define a partial ordering on the set of tiles by $P < P'$ if $I_P \subset I_{P'}$ and $\omega_{P'} \subset \omega_P$.

A set T of tiles is called a *tree*, if there is a tile $P_T = I_T \times \omega_T$, the *top* of the tree, such that $P < P_T$ for all $P \in T$. Observe that the top is not necessarily an element of the tree. A tree is called j*-tree* if $\omega_{jP_T} \subset \omega_{jP}$ for all $P \in T$.

For a finite subset $\mathbf{P}\subset\overline{\mathbf{P}}$, define

$$\mathrm{mass}(\mathbf{P}) := \sup_{P\in\mathbf{P}}\ \sup_{P'\in\overline{\mathbf{P}}:P<P'} \int_{E_{P'}} w_{P'}(x)\,dx \quad ,$$

$$\mathrm{energy}(\mathbf{P}) := \sup\nolimits_T \left(|I_T|^{-1}\sum\nolimits_{P\in T}|\langle f,\phi_{1P}\rangle|^2\right)^{\frac12} \quad ,$$

where the sup is taken over all 2-trees $T \subset \mathbf{P}$.

Proposition 3.1. *Let $E\subset\mathbf{R}$ be a set of measure smaller or equal* 1. *Let $\mathbf{P}$ be a finite set of tiles. Then $\mathbf{P}$ is the union of $\mathbf{P}_1$ and $\mathbf{P}_2$ with*

$$\mathrm{mass}(\mathbf{P}_1) \le \tfrac12\mathrm{mass}(\mathbf{P}),$$

and $\mathbf{P}_2$ is the union of trees $T\in\mathbf{T}$ such that

$$\sum\nolimits_{T\in\mathbf{T}}|I_T| \le C\mathrm{mass}(\mathbf{P})^{-1} \quad . \tag{4}$$

Proposition 3.2. *Let $\|f\|_2 = 1$ and $\mathbf{P}$ be a finite set of tiles. Then $\mathbf{P}$ is the union of $\mathbf{P}_1$ and $\mathbf{P}_2$ with*

$$\mathrm{energy}(\mathbf{P}_1) \le \tfrac12\mathrm{energy}(\mathbf{P}),$$

and $\mathbf{P}_2$ is the union of trees $T\in\mathbf{T}$ such that

$$\sum\nolimits_{T\in\mathbf{T}}|I_T| \le C\mathrm{energy}(\mathbf{P})^{-2} \quad . \tag{5}$$

We postpone the proofs of these propositions to Sections 4 and 5 and show how the propositions imply (3). Observe that any collection $\mathbf{P}$ of tiles satisfies $\mathrm{mass}(\mathbf{P}) \le C$. We inductively apply Proposition 3.1 and 3.2 to $\mathbf{P}$, as needed, to achieve a decomposition of $\mathbf{P}$ into sets $\mathbf{P}_n$ with

$$\mathrm{mass}(\mathbf{P}_n) \le \min(C, 2^{2n}), \qquad \mathrm{energy}(\mathbf{P}_n) \le 2^n$$

and $\mathbf{P}_n$ is a union of trees $T \in \mathbf{T}_n$ with

$$\sum\nolimits_{T \in \mathbf{T}_n} |I_T| \leq C 2^{-2n} \quad .$$

In Section 6 we will prove for each tree T the inequality

$$\sum\nolimits_{P \in T} |\langle f, \phi_{1,P} \rangle \langle \phi_{1,P}, 1_{E_{2P}} \rangle| \leq C \text{energy}(T) \text{mass}(T) |I_T| \quad . \tag{6}$$

Hence for the collection $\mathbf{T}_n$ we will have the estimate

$$\sum\nolimits_{T \in \mathbf{T}_n} \sum\nolimits_{P \in T} |\langle f, \phi_{1,P} \rangle \langle \phi_{1,P}, 1_{E_{2P}} \rangle| \leq C \min(2^n, 2^{-n}) \quad .$$

This is summable in n and so completes the proof.

4. Proof of Proposition 3.1

Let $\mu = \text{mass}(\mathbf{P})$. Let $\mathbf{P}^+$ be the set of tiles $P \in \mathbf{P}$ with $\text{mass}(\{P\}) > \frac{1}{2}\mu$. To each such tile P we may associate a tile $P'(P)$ with $P < P'(P)$ and

$$\int_{E_{P'(P)}} w_{P'(P)} \, dx > \tfrac{1}{2}\mu \quad .$$

Then, let $\mathbf{P}'$ be those elements in $\{P'(P) \mid P \in \mathbf{P}^+\}$ which are maximal with respect to the partial order $<$ on tiles. It suffices to show that

$$\sum\nolimits_{P' \in \mathbf{P}'} |I_{P'}| < C \mu^{-1} \quad ,$$

because the tiles $P \in \mathbf{P}^+$ can be collected into trees with tops in $\mathbf{P}'$.

For $\kappa \in \mathbf{N}$ define $\mathbf{P}_\kappa$ to be the set of all $P \in \mathbf{P}'$ with

$$|E_P \cap 2^\kappa I_P| \geq c 2^{2\kappa} \mu |I_P|$$

for some constant c. If c is small enough, then one can conclude from the mass estimate that each element P of $\mathbf{P}'$ is contained in one of the sets $\mathbf{P}_\kappa$. Hence it suffices to show for every κ

$$\sum\nolimits_{P \in \mathbf{P}_\kappa} |I_P| \leq C 2^{-\kappa} \mu^{-1} \quad .$$

Fix κ. For each $P \in \mathbf{P}_\kappa$ we have an enlarged rectangle $(2^\kappa I_P) \times \omega_P$. We select successively elements $P \in \mathbf{P}_\kappa$ with maximal $|I_P|$ whose enlarged rectangles are disjoint from the enlarged rectangles of all previously selected elements. When no further element can be selected, then each rectangle $P' \in \mathbf{P}_\kappa$ can be associated to a selected rectangle P such that $|I_{P'}| < |I_P|$ and the enlarged rectangles of P and P' intersect. Let $\mathbf{P}_\kappa^{\text{sel}}$ be the set of selected elements. Since the rectangles in $\mathbf{P}_\kappa$ are pairwise disjoint, we see that the intervals $I_{P'}$ of the rectangles P' associated to a fixed $P \in \mathbf{P}_\kappa^{\text{sel}}$ are pairwise disjoint and contained in $2^{\kappa+2} I_P$. Hence

$$\sum\nolimits_{P \in \mathbf{P}_\kappa} |I_P| \leq C 2^\kappa \sum\nolimits_{P \in \mathbf{P}_\kappa^{\text{sel}}} |I_P|$$

$$\leq C 2^{-\kappa} \mu^{-1} \sum\nolimits_{P \in \mathbf{P}_\kappa^{\text{sel}}} |E_P \cap 2^\kappa I_P| \quad .$$

This is bounded by $C 2^{-\kappa} \mu^{-1}$ because the enlarged rectangles of the elements $P \in \mathbf{P}_\kappa^{\text{sel}}$ are pairwise disjoint and therefore the sets $E_P \cap 2^\kappa I_P$, which are contained in E, are pairwise disjoint. This finishes the proof of Proposition 3.1.

5. Proof of Proposition 3.2

Let $\varepsilon = \text{energy}(\mathbf{P})$. For a 2-tree T, let

$$\Delta(T)^2 = |I_T|^{-1} \sum\nolimits_{P \in T} |\langle f, \phi_{1P} \rangle|^2 \quad .$$

The important part is the construction of the collection $\mathbf{P}_2$, which is determined by the collection of trees $\mathbf{T}$.

Pick a 2-tree $T \in \mathbf{P}$ such that (1) $\Delta(T) \geq \varepsilon/2$ and (2) $c(\omega_T)$ is minimal among all 2-trees satisfying the first condition. Then let T^1 be the maximal (with respect to inclusion) tree in $\mathbf{P}$ with top $I_T \times \omega_T$.

Add T^1 to $\mathbf{T}$, add T to $\mathbf{T}_2$, which will be a collection of 2-trees we will work with in the sequel. And remove T^1 from $\mathbf{P}$. Then repeat the procedure above until there is no tree in $\mathbf{P}$ with $\Delta(T) \geq \varepsilon/2$. Then we can take the remaining tiles to be the collection $\mathbf{P}_1$.

The 2–trees in $\mathbf{T}_2$ have a disjointness property. Let $T, T' \in \mathbf{T}_2$ and let $P \in T$ and $P' \in T'$. If ω_P is contained in $\omega_{1P'}$, then $I_{P'} \cap I_T = \emptyset$. To see this, note that $c(\omega_T)$, which is contained in ω_P, is less than $c(\omega_{T'}) \in \omega_{2P'}$. Thus, T was selected before T'. But if $I_{P'}$ and I_T intersected, we would then have that P' would be in the tree T^1 which was removed from $\mathbf{P}$ before T' was selected.

It remains to show that

$$\varepsilon^2 \sum_{T \in \mathbf{T}_2} |I_T| \leq C \quad .$$

But, letting $\overline{\mathbf{P}}$ be the union of the 2–trees T with $T \in \mathbf{T}_2$, the left hand side is at most a constant times

$$\sum\nolimits_{P \in \overline{\mathbf{P}}} |\langle f, \phi_{1P} \rangle|^2 \leq \left\| \sum\nolimits_{P \in \overline{\mathbf{P}}} \langle f, \phi_{1P} \rangle \, \phi_{1P} \right\|_2 \quad .$$

This follows by rewriting the first term, applying Cauchy–Schwartz and using the fact that $\|f\| = 1$.

Therefore, it is sufficient to prove

$$\left\| \sum\nolimits_{P \in \overline{\mathbf{P}}} \langle f, \phi_{1P} \rangle \, \phi_{1P} \right\|_2^2 \leq C\varepsilon^2 \sum\nolimits_{T \in \mathbf{T}_2} |I_T| \quad . \tag{7}$$

We estimate the left hand side of (7) by

$$\sum_{P,P' \in \overline{\mathbf{P}} : \omega_P = \omega_{P'}} |\langle f, \phi_{1P} \rangle \, \langle \phi_{1P}, \phi_{1P'} \rangle \, \langle \phi_{1P'}, f \rangle| \tag{8}$$

$$+ \, 2 \sum_{P,P' \in \overline{\mathbf{P}} : \omega_P \subset \omega_{1P'}} |\langle f, \phi_{1P} \rangle \, \langle \phi_{1P}, \phi_{1P'} \rangle \, \langle \phi_{1P'}, f \rangle| \ . \tag{9}$$

Here we have used symmetry and the fact that $\langle f, \phi_{1P} \rangle = 0$ unless one of the intervals ω_{1P} and $\omega_{1P'}$ is contained in the other.

Observe that for $|I_{P'}| \leq |I_P|$ we have

$$|\langle \phi_{1P}, \phi_{1P'} \rangle| \leq C |I_P|^{\frac{1}{2}} |I_{P'}|^{-\frac{1}{2}} \left\| w_P 1_{I_{P'}} \right\|_1 \quad .$$

We estimate the smaller one of $|\langle f, \phi_{1P} \rangle|$ and $|\langle f, \phi_{1P'} \rangle|$ by the larger one and use symmetry to obtain for (8) the upper bound

$$C \sum_{P \in \overline{\mathbf{P}}} |\langle f, \phi_{1P} \rangle|^2 \sum_{P' \in \overline{\mathbf{P}} : \omega_P = \omega_{P'}} \left\| w_P 1_{I_{P'}} \right\|_1 \quad .$$

The interior sum we can estimate by $|I_P|^{-1}\|w_P\|_1 \leq C$, because the intervals $I_{P'}$ with $\omega_{P'} = \omega_P$ are pairwise disjoint. This proves the desired bound for (8).

The second summand (9) we estimate by

$$\sum_{P \in \overline{\mathbf{P}}} |\langle f, \phi_{1P}\rangle| \sum_{P' \in \overline{\mathbf{P}}: \omega_P \subset \omega_{1P'}} |\langle \phi_{1P}, \phi_{1P'}\rangle \langle \phi_{1P'}, f\rangle|$$

$$\leq \sum_{T \in \mathbf{T}_2} \left(\sum_{P \in T} |\langle f, \phi_{1P}\rangle|^2 \right)^{\frac{1}{2}} H(T)^{\frac{1}{2}} \leq C\varepsilon \sum_{T \in \mathbf{T}_2} |I_T|^{\frac{1}{2}} H(T)^{\frac{1}{2}} \quad ,$$

where

$$H(T) := \sum_{P \in T} \left(\sum_{P' \in \overline{\mathbf{P}}: \omega_P \subset \omega_{1P'}} |\langle \phi_{1P}, \phi_{1P'}\rangle \langle \phi_{1P'}, f\rangle| \right)^2 \quad .$$

It remains to show that $H(T) \leq C\varepsilon |I_T|$ for each tree $T \in \mathbf{T}_2$.

But,

$$H(T) \leq C\varepsilon \sum_{P \in T} |I_P| \left(\sum_{P' \in \overline{\mathbf{P}}: \omega_P \subset \omega_{1P'}} \left\| w_P 1_{I_{P'}} \right\|_1 \right)^2 \quad ,$$

where we have used the upper energy estimate for each individual P' (which is a 2-tree by itself), and the estimate on $\langle \phi_{1P}, \phi_{1P'}\rangle$. Fix P, then the intervals $I_{P'}$ with $\omega_P \subset \omega_{1,P'}$ are pairwise disjoint and disjoint from I_T by the above disjointness properties. Hence we have

$$\sum_{P' \in \overline{\mathbf{P}}: \omega_P \subset \omega_{1P'}} \left\| w_P 1_{I_{P'}} \right\|_1 \leq C \left\| w_P 1_{I_T^c} \right\|_1 \quad .$$

For each $x \in I_T$ there is at most one $P \in T$ of each scale with $x \in I_P$. Hence we have:

$$\sum_{P \in T} |I_P| \left\| w_P 1_{I_T^c} \right\|_1^2 \leq C \sum_{P \in T} \left\| w_P 1_{I_T^c} \right\|_1$$

$$\leq C \sum_{k \in \mathbf{N}} \left\| (1_{I_T} * D^1_{2^{-k}|I_T|} w) 1_{I_T^c} \right\|_1 \leq C|I_T| \quad .$$

This gives the appropriate bound for $H(T)$ and thus finishes the proof of (7).

6. Proof of estimate (6)

Let $\mathcal{J}$ be the collection of all maximal dyadic intervals such that $3J$ does not contain any I_P with $P \in T$. Then $\mathcal{J}$ is a partition of $\mathbf{R}$.

We can estimate the left hand side of (6) as below, in which the terms ϵ_P are phase factors of modulus 1 which make up for the absolute value signs in (6).

$$\left\| \sum\nolimits_{P \in T} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P} 1_{E_{2P}} \right\|_1 \leq$$

$$\sum\nolimits_{J \in \mathcal{J}} \sum\nolimits_{P \in T: |I_P| \leq |J|} \| \langle f, \phi_{1P}\rangle \phi_{1P} 1_{E_{2P}} \|_{L^1(J)} \tag{10}$$

$$+ \sum\nolimits_{J \in \mathcal{J}} \left\| \sum\nolimits_{P \in T: |I_P| > |J|} \epsilon_P \langle f, \phi_{1P}\rangle \phi_{1P} 1_{E_{2P}} \right\|_{L^1(J)} \tag{11}$$

To estimate (10), we calculate for each $J \in \mathcal{J}$ and $P \in T$:

$$\| \langle f, \phi_{1P} \rangle \, \phi_{1P} 1_{E_{2P}} \|_{L^1(J)} \leq C\varepsilon\mu |I_P| \, \|w_P\|_{L^\infty(J)} \quad .$$

Here, we have used the notation $\varepsilon = \text{energy}(T)$ and $\mu = \text{mass}(T)$. Fix an integer k with $2^k \leq |J|$ and consider $P \in T$ with $|I_P| = 2^k$. These intervals are disjoint and contained in I_T, hence

$$\sum_{P \in T : |I_P| = 2^k} |I_P| \, \|w_P\|_{L^\infty(J)} \leq C \sum_{P \in T : |I_P| = 2^k} |I_P| (\text{dist}(I_P, J) |I_P|^{-1})^{-\nu}$$

If $J \subset 3I_T$, we estimate this sum as

$$C|I_P|(2^k/|J|)^2 \leq C2^k \quad .$$

If $J \not\subset 3I_T$, then $\text{dist}(J, I_T) \geq c|J|$, so that the sum is no more than

$$C|I_T|(2^k/|J|)^2 (\text{dist}(I_T, J)/|I_T|)^{-2} \quad .$$

These estimates, summed over $2^k \leq |J|$ and $J \in \mathcal{J}$, yield no more than $C|I_T|$, completing the estimate of (10).

We consider (11). We can assume that the summation runs only over those $J \in \mathcal{J}$ for which there exists a $P \in T$ with $|J| < |I_P|$. Then we have $J \subset 3I_T$ and $|J| < |I_T|$ for all J occuring in the sum.

Let us fix an interval $J \in \mathcal{J}$ and observe that

$$G_J = J \cap \bigcup_{P \in T : |I_P| > |J|} E_{2P}$$

has measure at most $C\mu|J|$. Indeed, let J' be the dyadic interval which contains J and $|J'| = 2|J| \leq |I_T|$. By maximality of J, $3J'$ contains an interval I_P for some $P \in T$. Let P' be the tile with $|I_{P'}| = |J'|$ and $P < P' < I_T \times \omega_T$. Then $G_J \subset J' \cap E_{P'}$. And since $\text{mass}(\{P\}) \leq \mu$, our claim follows.

Let T_2 be the 2-tree of all $P \in T$ such that $\omega_{2T} \subset \omega_{2P}$, and let $T_1 = T \setminus T_2$. Define, for $j = 1, 2$,

$$F_{jJ} = \sum_{P \in T_j : |I_P| > |J|} \epsilon_P \, \langle f, \phi_{1P} \rangle \, \phi_{1P} 1_{E_{2P}}$$

For P in the 1–tree T_1, the intervals ω_{2P} are disjoint. So the sets E_{2P} are disjoint. Hence

$$\|F_{1J}\|_{L^1(J)} \leq C\varepsilon\mu |G_J| \leq C\varepsilon\mu|J| \quad .$$

This estimate is summed over the disjoint $J \subset 3I_T$.

To complete the estimate of (11), we estimate $F_{2J}(x)$. Fix x and assume that $F_{2J}(x)$ is not zero. Since the intervals ω_{2P} with $P \in T_2$ are all nested, there is a largest (smallest) interval ω_+ (ω_-) of the form ω_P with $P \in T_2$, $x \in E_{2p}$ and $|I_P| > J$. Then $x \in E_{2P}$ for some $P \in T$ with $|I_P| > |J|$ iff $|\omega_-| \leq |\omega_P| \leq |\omega_+|$.

Hence we can write $F_{2J}(x)$ as

$$\sum_{P \in T_2 : |\omega_-| < |\omega_P| \leq |\omega_+|} \epsilon_P \, \langle f, \phi_{1P} \rangle \, \phi_{1P}(x)$$

$$= \sum_{P \in T_2} \epsilon_P \, \langle f, \phi_{1P} \rangle \left(\phi_{1P} * \left(M_{c(\omega_+)} D^1_{0.1|\omega_+|^{-1}} \phi - M_{c(\omega_-)} D^1_{0.1|\omega_-|^{-1}} \phi \right) \right)(x) \quad .$$

The last equality is easily seen from the geometry of the supports of the functions $\widehat{\phi}_{1P}$. Therefore we can estimate $|F_{2J}(x)|$ by

$$C \sup_{J \subset I} \frac{1}{|I|} \int_I \left| \sum_{P \in T_2} \epsilon_P \langle f, \phi_{1P} \rangle \, \phi_{1P}(z) \right| dz \quad ,$$

which is constant on J.

But $F_{2J} 1_J$ is supported on the set G_J of measure $\leq C\mu|J|$, hence

$$\begin{aligned} C \sum_{J \in \mathcal{J} : J \subset 3I_T} & \mu|J| \sup_{J \subset I} \frac{1}{|I|} \int_I \left| \sum_{P \in T_2} \epsilon_P \langle f, \phi_{1P} \rangle \, \phi_{1P}(z) \right| dz \\ & \leq C\mu \left\| M \left(\sum_{P \in T_2} \epsilon_P \langle f, \phi_{1P} \rangle \, \phi_{1P} \right) \right\|_{L^1(3I_T)} \\ & \leq C\mu |I_T|^{\frac{1}{2}} \left\| \sum_{P \in T_2} \epsilon_P \langle f, \phi_{1P} \rangle \, \phi_{1P} \right\|_2 \quad . \end{aligned}$$

Here M denotes the Hardy Littlewood maximal function and we have used the maximal theorem. By direct calculation the L^2 norm in the last expression is bounded by

$$C \left(\sum_{P \in T_2} |\langle f, \phi_{1P} \rangle|^2 \right)^{\frac{1}{2}} \leq C |I_T|^{\frac{1}{2}} \varepsilon \quad .$$

This completes the desired estimate for the second summand in (11) and thereby finishes the proof of (6).

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