What I Did This Summer

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The original idea was to explore symplectic cobordisms between lens spaces. Considering a lens space as the boundary of a four-manifold, it is easiest to consider Stein fillable four-manifolds, which automatically carry a symplectic structure.

Let X be a smooth, compact, connected, oriented 4-manifold which admits a stein structure. Then $M = \partial X$ has a holomorphically fillable induced tight contact structure ξ, and X may be presented as a handlebody with 2-handles attached to a framed link in $\partial(D^4 \cup$ 1-handles), with the framing coefficient of each component K given by $tb(K) - 1$, where $tb(K)$ is the Thurston-Bennequin invariant of the knot K. Conversely, every such manifold is Stein. The resulting surgery on M is called Legendrian surgery, and in the case of lens spaces gives a convenient method for representing cobordisms. As any lens space $L(p, q)$ can be represented by a surgery diagram as a chain of unknots with surgery coefficient ≤ -2, where the coefficients are the coefficients of the continued partial fraction expansion of $-p/q$, the result of attaching more handles is an exercise in combinatorics - introduce a 2-handle, slide over some component of the original link, use twists and other moves to return the diagram to one recognizeable as a lens space, then check the new coefficients to find exactly which lens space it is. For example, by construction such a diagram cannot have any surgery coefficient ≥ -1, so it is impossible to construct a Stein cobordism from a lens space to one with less coefficients, i.e. with a shorter continued fraction expansion.

After realizing this, we began to instead consider the induced contact structures themselves, and to examine relationships between their various representations.

By Giroux, there is a 1-1 correspondence between contact structures and open-book decompositions of any 3-manifold. Also, given a 1-form α for a contact structure ξ, there exists a unique Reeb vector field V ($\iota_V d\alpha = 0$, $\alpha(V) = 1$). V then respects a unique open-book decomposition, i.e. is tranverse to and co-orients the pages, tangent to and orients the binding, and the orientation of the binding agrees with its orientation as the boundary of a page. Given an open-book decomposition, Thurston and Wilkenkemper describe a (actually the unique) contact structure on M as follows: Let λ be a primitive for an area form on a fiber, and let $\lambda_t = t \cdot \lambda + (1-t)h^*\lambda,\ \ t \in [0,1]$, where h is the monodromy. Then $\alpha = dt + \lambda_t$ is a contact 1-form which extends over the binding to give a contact structure on M.

As an open-book decomposition consists of a surface bundle over S^1, it gives rise to another nice representation of cobordisms, in that the monodromy of the bundle is an element of the mapping class group of the surface (the page), which is generated by Dehn twists. Furthermore, a Legendrian surgery on an unknot in the manifold is equivalent to a positive Dehn twist on a page, so it is possible to view a cobordism by simply composing a given monodromy with some number of twists. The problem was actually finding the open book decomposition to begin with. One possible method is described below, but has the disadvantage of not preserving the tight contact structure. Fortunately Akbulut has devised an algorithm for constructing Lefchetz fibrations of 4-manifolds which when restricted to the boundary give open-book decompositions respecting the tight contact structures. It turns out that this method allows one to see clearly the decompositions corresponding to the cobordisms mentioned above in terms of surgery diagrams, but as this realization came towards the end of the summer, and as Giroux has yet to publish the details of his result, it is unclear what any of this might accomplish. A possible further direction would be to use this to understand some of the the relation between virtually overtwisted and universally tight contact structures, specifically whether Legendrian surgery preserves universal tightness.

Algorithm for open book decomposition given a surgery diagram of a 3-fold M:
On any component of the surgery link with coefficient not in $\{-1, 0, +1\}$, blow up the manifold by adding ± 1-framed unknots and sliding the corresponding 2-handles about to reduce the coefficient. Once a link with all coefficients in $\{-1, 0, +1\}$ has been attained, consider an unknot J which passes exactly once through each component of the link. Before the surgery on S^3 has been performed, then, an open book decomposition of S^3 with binding J has pages which intersect a neighbourhood of any component of the surgery link in a meridional disk. Then peforming the surgery while keeping track of the page, excluding a small neighborhood of the core circles of the surgery tori, leads to an open book decomposition of M with binding $h(J) \cup \{$the core circles of the surgery tori$\}$, where h is a homeomorphism from $S^3 \backslash \{$the surgery tori$\}$to the given 3-fold$\backslash\{$the surgery tori$\}$.

ξ tight $\Leftrightarrow$ cannot draw legendrian unknot with $tb \geq 0$
note: gives method for proving whether given knot is knotted