# Solutions to Practice Test 1 A

1) a) $\nabla f(\vec{x}) = 4\,(x^3-y,\, y^3-x)$

$\vec{x}_0 = (2^{1/4}, 0)$

$\nabla f(\vec{x}_0) = 4\,(2^{3/4}, -2^{1/4})$.

b) $2^{3/4}x - 2^{1/4}y = 2$.

c) Tangent line parellel to x-axis means that the gradient is perpendicular to the x-axis, i.e.,

$x^3 - y = 0 \qquad f(\vec{x}_0) = 2$

$x^4 + y^4 - 4xy = 2$

Eliminate $y$: $x^{12} - 3x^4 = 2$.

Set $z = x^4$: $z^3 - 3z = 2$.

Solutions: $z_1 = -1$ (double root)

$z_2 = 2$

Since $z = x^4 \geq 0$ the positive root is of interest (the neg. not!)

$x = \pm\, 2^{1/4} \quad y = \pm\, 2^{3/4}$

$\vec{x}_1 = (2^{1/4}, 2^{3/4})$, $\vec{x}_2 = -(2^{1/4}, 2^{3/4})$ are the points.

d) The curve is symmetric under the exchange of $x$ and $y$

$$\Rightarrow \vec{x}_3 = (2^{3/4}, 2^{1/4}), \quad \vec{x}_4 = -(2^{3/4}, 2^{1/4})$$

are the points where the tangent is vertical.

e) Tangent line parallel to the $x=y$ axis, i.e. gradient perp. to the $x=y$ axis.

$$x^3 - y = -(y^3 - x) \quad \text{or} \quad (x^3 - x) = -(y^3 - y) \quad \text{and}$$
I

$$x^4 + y^4 - 4xy = 2$$
II

I can be written as

$$(x^3 + y^3) - (x+y) = (x+y)[x^2 - xy + y^2 - 1] = 0$$

a) $x + y = 0 \Rightarrow x = -y$ From II

$$x^4 + x^4 + 4x^2 = 2 \quad \text{or} \quad x^4 + 2x^2 = 1$$

$$z = x^2 \Rightarrow z^2 + 2z = 1 \quad \text{or} \quad z_1 = \sqrt{2} - 1$$
$$z_2 = -\sqrt{2} - 1$$

only $z_1 > 0$ counts. The solutions are

$$x = \pm\sqrt{\sqrt{2} - 1}$$

$$y = \mp\sqrt{\sqrt{2} - 1}$$

b) $x^2+y^2-xy-1=0$ From this we get

$$(x^2+y^2)^2 = (1+xy)^2$$
$$\|$$
$$x^4+y^4+2x^2y^2 = 1+2xy+x^2y^2 \text{ or}$$

$$x^4+y^4 = 1+2xy-x^2y^2 \quad (*)$$

Inserting this into II yields

$$1+2xy-x^2y^2-4xy = 2 \quad \text{or with } z=xy$$

$$z^2+2z+1=0 \quad \text{or} \quad (z+1)^2=0 \quad \text{i.e. } \underline{z=-1}$$

Thus $xy=-1$ or $x^4+y^4 = 1-2-1=-2$

which is not possible.

Hence $\vec{x}_{1,2} = \pm\sqrt{\sqrt{2}-1}\,(1,-1)$

are the only solutions.

Note that the curve is symmetric under the change $x \to -x$, $y \to -y$. Thus, one tangent line must touch the curve in a point with coordinates $(a,-a)$. Thus

$$2a^4+4a^2=2 \quad \text{or} \quad z^2+2z=1 \quad \text{with } z=a^2$$

which leads to the solution obtained before

Note that our previous calculation shows that there are no others.

To find the tangent line parallel to the $x=-y$ axis, we must have that

$$x^3-y = y^3-x \quad \text{or} \quad (x^3-y^3)+(x-y) = (x-y)\left[x^2+xy+y^2+1\right]=0$$

$x=y$ together with $x^4+y^4-4xy=2$ leads to $z^2-2z^2=1$ with $z=x^2$ from which we obtain

$$x=y=\pm\sqrt{1+\sqrt{2}}.$$

Note that $x^2+y^2+xy+1=0$ has no solution since $x^2+y^2 \geqslant 2|x||y|$ and hence

$$x^2+y^2+xy \geqslant 0.$$

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