# TEST 2, MATH 3406, APRIL 2, 2020

**Print Name:**

**This test is to be taken without calculators and notes of any sort**. The allowed time is 75 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given. Likewise, write legibly!

**I abide by the Georgia Tech honor code. Signature:**

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**Problem 1 (15):** Compute the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

Is it diagonalizable? Write the possible eigenvectors as row vectors. No partial credit will be given because you can check the answer.

**Solution:** Characteristic polynomial: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ Alg. mult. is 2. Eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ geom. mult. is 1

**Problem 2 (10):** For which values of $a$ is the matrix

$$\begin{bmatrix} a & 1-a \\ a-1 & 2-a \end{bmatrix}$$

diagonalizable? Circle the correct option: 2, none, 1 and -1, 1 only, all

**Solution:** Characteristic polynomial is $\lambda^2 - 2\lambda + a(2-a) - (a-1)(1-a) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ alg mult is 2 Eigenvector equation is $ax + (1-a)y = x$ so that $x = y$ if $a \neq 1$. So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the only eigenvector in this case and hence the geom. multiplicity is 1. If $a = 1$ the the matrix is the identity matrix and the geom. mult is 2 and hence diagonal.

**Problem 3 (10):** The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ span the unit cube which has volume 1. The linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\vec{x} = A\vec{x}$ where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

maps the unit cube to a parallelepiped. Which one of the following numbers is the volume of the parallelepiped: $4, -3, 3, 2$?

**Solution:** The parallelepiped is given by the three column vectors of $A$. Hence the volume is $|\text{det}A| = 3$

**Problem 4 (20):** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps the vector $\vec{e}_1$ to the vector $\vec{e}_1 + 2\vec{e}_2$ and the vector $\vec{e}_2$ to $2\vec{e}_1 - \vec{e}_2$.

a) Write the matrix associated with this transformation.

b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the $x = y$ axis.

Write the matrix for the map $T \circ S$ as well as the matrix associated with the map $S \circ T$ .

**Solution:** Matrix for $T$:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Matrix for $S \circ T$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Matrix for $T \circ S$ is

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

**Problem 5 (12):** Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

You do not have to compute the eigenvectors. There is no partial credit, because you can check your answer.

**Solution:** $-1, 3, 2, 5$

**Problem 6 (15):** Construct a matrix $A$ with eigenvalues 1 and 2, such that the eigenvalue 1 has algebraic multiplicity 3, geometric multiplicity 1, and 2 has algebraic multiplicity 2, geometric 1.

**Solution:**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

**Problem 7 (18):** Suppose $S$ is a $k$-dimensional subspace of $\mathbb{R}^5$, $k \leq 5$. Let $P$ be the matrix of the orthogonal projection to $S$.
a) (5) What are the eigenvalues and eigenvectors of $P$?
b) (5) What are the algebraic and geometric multiplicities of the eigenvalues of $P$?
c)(5) Is $P$ diagonalisable?

**Solution:** All the vectors in $S$ are eigenvectors with eigenvalue 1 and those perpedicular to $S$ are the eigenvectors with eigenvalue 0. The algebraic multiplicity of 1 = geoemetric multiplicity of 1 equals $k$. The alg mult of 0 is the geom. mult. of 0 equals $5 - k$. It is diagonalizable.

**Problem EC* (10):** Consider the non-zero rank one matrix $\vec{v}\vec{u}^T$, $\vec{v}, \vec{u} \in \mathbb{R}^3$ and assume that $\vec{u} \cdot \vec{v} \neq 0$. What are the eigenvectors and eigenvalues? Is it diagonalizable? Suppose that $\vec{u} \cdot \vec{v} = 0$ is it again diagonalizable?

**Solution:** Assume that $\vec{u} \cdot \vec{v} \neq 0$: $\vec{v}$ is an eigenvector, since $\vec{v}\vec{u}^T\vec{v} = (\vec{u} \cdot \vec{v})\vec{v}$. Hence the eigenvalue is $\vec{u} \cdot \vec{v}$. Any vector perpendicular to $\vec{u}$ gets mapped to the zero vector. Hence the matrix is diagonalizable.

Suppose now that $\vec{u} \cdot \vec{v} = 0$: Any vector perpendicular to $\vec{u}$ is an eigenvector with eigenvalue 0. Any vector that is not perpendicular to $\vec{u}$ cannot be an eigenvector since the matrix maps this vector on to $\vec{v}$ which is perpendicular. Hence this matrix is not diagonalizable.