THE DISCRETE FOURIER TRANSFORM

1. Roots of 1

First a little review about complex numbers, namely roots of 1. You know of course that the equation $x^2 = 1$ has two roots, $+1$ and -1. If we consider the equation $x^4 = 1$ and look for solution in the complex domain we find the four roots $1, i, -1, -i$ where you recall that $i^2 = -1$. Thus we can factor

$$x^4 - 1 = (x-1)(x-i)(x+1)(x+i) \ .$$

The roots of the equation $x^3 - 1 = 0$ are given by $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$. and hence

$$x^3 - 1 = (x-1)(x - \frac{-1+i\sqrt{3}}{2})(x + \frac{1+i\sqrt{3}}{2}) \ .$$

Things become much clearer if we jot down these points in the complex plane. Th roots of the equation $x^4 - 1 = 0$ are on the unit circle and are the corners of a square and the roots $x^3 - 1 = 0$ are also on the unit circle and are the corners of an equilateral triangle.

Using a bit of trigonometry we find that the roots of $x^4 - 1 = 0$ can be written as

$$1 = \cos 0 + i \sin 0 = e^{i0} \ , \ i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2} \ ,$$

$$-1 = \cos(\pi) + i\sin(\pi) = e^{i\pi} \ , \ -i = \cos(3\pi/2) + i\sin(3\pi/2) = e^{i3\pi/2} \ .$$

and likewise, the roots of the equation $x^3 - 1 = 0$ can be written as

$$1 = \cos 0 + i \sin 0 = e^{i0} \ , \frac{-1+i\sqrt{3}}{2} = \cos(2\pi/3) + i\sin(2\pi/3) = e^{i2\pi/3} \ ,$$

$$\frac{-1-i\sqrt{3}}{2} = \cos(4\pi/3) + i\sin(4\pi/3) = e^{i4\pi/3} \ .$$

For the general equation $x^n - 1 = 0$ we get the roots

$$1 \ , \ e^{2\pi i \frac{1}{n}} \ , \ e^{2\pi i \frac{2}{n}} \ , \ e^{2\pi i \frac{3}{n}} \ , \ \dots \ , e^{2\pi i \frac{n-1}{n}} \ .$$

To abbreviate the notation we set

$$\omega_n = e^{2\pi i \frac{1}{n}}$$

and can write the set of roots as

$$K' = \{1 \ , \ \omega_n \ , \ \omega_n^2 \ , \ \omega_n^3 \dots \ , \ \omega_n^{n-1}\} \ .$$

There is a close relationship of the roots of unity and cyclic permutations which we explore in the next section.

2. The permutation matrix T

The $n \times n$ matrix T is is the matrix that maps the vector $[x_1, x_2, \ldots, x_n]$ to the vector $[x_2, x_3, \ldots, x_n, x_1]$. The matrix T can be written as

$$T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} .$$

The matrix is orthogonal and hence can be diagonalized, in fact there exists a unitary matrix U such that

$$T = UDU^*$$

where D is diagonal. It is clear that $T^n = I$ and that tells us that the eigenvalues must be roots of 1. Indeed

$$I = T^n = UD^nU^*$$

and hence $D^n = I$ and the diagonal elements of D must be roots of 1. Let ω be one of the eigenvalues and denote the components of the corresponding eigenvector by $[\vec{x}_1, \ldots, x_n]$. The eigenvalue equation then reads as

$$x_2 = \omega x_1 \ , x_3 = \omega x_2, \ldots, x_1 = \omega x_n$$

and we can solve this easily by choosing $x_1 = 1$ so that $x_2 = \omega, x_2 = \omega^2, \ldots, x_n = \omega^{n-1}$. we may write this as

$$\begin{bmatrix} \omega^0 \\ \omega \\ \omega^2 \\ . \\ . \\ \omega^{n-1} \end{bmatrix} .$$

Any root of unity can be expressed through ω_n and hence we can write down n vectors

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} , \vec{v}_1 = \begin{bmatrix} 1 \\ \omega_n \\ \omega_n^2 \\ . \\ . \\ \omega_n^{n-1} \end{bmatrix} , \vec{v}_3 = \begin{bmatrix} 1 \\ (\omega_n^2) \\ (\omega_n^2)^2 \\ . \\ . \\ (\omega_n^2)^{n-1} \end{bmatrix} , \ldots \ , \vec{v}_{n-1} = \begin{bmatrix} 1 \\ (\omega_n^{n-1}) \\ (\omega_n^{n-1})^2 \\ . \\ . \\ (\omega_n^{n-1})^{n-1} \end{bmatrix} .$$

These vectors are pairwise orthogonal. To see this we compute the inner products for $k \neq \ell$

$$\langle v_k, \vec{v}_\ell \rangle = \sum_{j=0}^{n-1} \overline{(\omega_n^k)}^j (\omega_n^\ell)^j = \sum_{j=0}^{n-1} \omega_n^{(\ell-k)j} = \frac{1 - \omega_n^{(\ell-k)n}}{1 - \omega_n^{\ell-k}} = 0$$

since $\omega_n^{(\ell-k)n} = 1$. If $k = \ell$ the inner product is n and if we divide the vectors by $\sqrt{n}$ they are normalized and form an orthonormal basis of $\mathbb{C}^n$.

Thus the unitary matrix U is given by $n^{-1/2}F_n$ where

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & (\omega_n^2)^2 & (\omega_n^3)^2 & \dots & (\omega_n^{n-1})^2 \\ . & . & . & . & \dots & . \\ 1 & \omega_n^{n-1} & (\omega_n^2)^{n-1} & (\omega_n^3)^{n-1} & \dots & (\omega_n^{n-1})^{n-1} \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \dots & \omega_n^{2(n-1)} \\ . & . & . & . & \dots & . \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix}$$

We call this the **Fourier matrix**. Lets work all this out when $n = 4$. The roots are, as we have seen, $1, i, i^2, i^3$ or $1, i, -1, -i$. Then the matrix of eigenvectors is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & (i^2) & (i^2)^2 & (i^2)^3 \\ 1 & (i^3) & (i^3)^2 & (i^3)^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Note that the column vectors of this matrix are orthogonal with respect to the inner product and hence, normalizing these vectors, we get

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, \tag{1}$$

a unitary matrix. One computes easily that $TU = UD$ where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

3. Application to differential equations

Imagine n pearls each having mass m on a circle. We assume that the pearls can slide on this circle and they are coupled to each other via springs with spring constant k. The first pearl interacts with the second and the last the second pearl interacts with the first and the third and so on. The last interacts with the penultimate and the first. We denote the positions of the pearls on the circle by $x_1, \dots, x_n$ and we assume that the ordering is clockwise. The force acting on the first pearl is given by

$$-k(x_1 - x_n) - k(x_1 - x_2)$$

where $-k(x_1 - x_4)$ is the force that pearl n exerts on the first pearl and $-k(x_1 - x_2)$ is the force that the second pearl exerts on the first one. Newton's equation states that

$$m\ddot{x}_1 = -k(x_1 - x_n) - k(x_1 - x_2)$$

or, if we set $\omega^2 = k/m$

$$\ddot{x}_1 = \omega^2(x_n - 2x_1 + x_2)$$

Repeating this argument for all the pearls we find the system of equations:

$$\ddot{x}_1 = \omega^2(x_n - 2x_1 + x_2)\,, \ddot{x}_2 = \omega^2(x_1 - 2x_2 + x_3)\,, \ddot{x}_3 = \omega^2(x_2 - 2x_3 + x_4)\,, \ldots\,, \ddot{x}_n = \omega^2(x_{n-1} - 2x_n + x_1)\,.$$

The problem is to solve this system of equations given initial conditions $x_i(0) = a_i$ and $\dot{x}_i = b_i, i = 1, \ldots, n$. The a_i are the initial position and the b_i the initial velocities.

This is a formidable problem that, nevertheless has an elegant solution. The first step is to write this system of equations in matrix form. Set

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

and note that the system can be written as

$$\frac{d^2\vec{X}}{dt^2} = \omega^2(T - 2I + T^{-1})\vec{X}$$

As an example, take $n = 4$. Then we get the equations

$$\frac{d^2x_1}{dt^2}_i = \omega^2(x_4 - 2x_1 + x_2)\ ,$$
$$\frac{d^2x_2}{dt^2}_i = \omega^2(x_1 - 2x_2 + x_3)\ ,$$
$$\frac{d^2x_3}{dt^2}_i = \omega^2(x_2 - 2x_3 + x_4)\ ,$$
$$\frac{d^2x_4}{dt^2}_i = \omega^2(x_3 - 2x_4 + x_1)\ .$$

We assume that we are given the initial conditions $\vec{x}(0) = \vec{x}_0$ and $\frac{d}{dt}\vec{x}(0) = \vec{v}_0$. We have diagonalized T. We get the eigenvalues $0, -2\omega^2, -4\omega^2, -2\omega^2$ with the eigenvectors as the column vectors in (1). Thus, we have

$$T = UDU^*$$

and one easily computes that

$$U^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\ .$$

To solve the system of differential equations

$$\frac{d^2\vec{X}}{dt^2} = \omega^2(T - 2I + T^{-1})\vec{X}$$

and hence

$$\frac{d^2\vec{X}}{dt^2} = \omega^2 U(D - 2I + D^{-1})U^*\vec{X}$$

or if we set $\vec{Y} = U^*\vec{X}$

$$\frac{d^2\vec{Y}}{dt^2} = \omega^2(D - 2I + D^{-1})\vec{Y}$$

which we have to solve with the initial conditions $\vec{Y}(0) = U^*\vec{A} =: \alpha$ and $\frac{d\vec{Y}}{dt}(0) = U^*\vec{B} =: \beta$. In the case where $n = 4$ we find that

$$D - 2I + D^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} .$$

Writing it out in detail we have

$$\ddot{y}_1 = 0 \ , \ \ddot{y}_2 = -2\omega^2 y_2 \ , \ \ddot{y}_3 = -4\omega^2 \ , \ \ddot{y}_4 = -2\omega^2 \ .$$

and hence

$$y_1(t) = \alpha_1 + \beta_1 t \ , \ y_2(t) = \cos(\sqrt{2}\omega t)\alpha_2 + \frac{\sin(\sqrt{2}\omega t)}{\sqrt{2}\omega}\beta_2 \ ,$$

$$y_3(t) = \cos(2\omega t)\alpha_3 + \frac{\sin(2\omega t)}{2\omega}\beta_3 \ , \ y_4(t) = \cos(\sqrt{2}\omega t)\alpha_4 + \frac{\sin(\sqrt{2}\omega t)}{\sqrt{2}\omega}\beta_4$$

and recalling that the $\vec{\alpha}$ and $\vec{\beta}$ vector are given

$$\vec{\alpha} = U^*A \ , \ \vec{\beta} = U^*\vec{B} \ .$$

Note that the vectors are complex numbers. Hence we have to multiply the initial conditions $\vec{A}$ and $\vec{B}$ have to be multiplied by U^*. Then we compute $y_1(t), y_2(t), y_3(t), y_4(t)$ which yields $\vec{Y}(t)$. Then we have to compute

$$\vec{X}(t) = U\vec{Y}(t) \ .$$

As an example, consider the initial condition $\vec{B} = \vec{0}$ and $\vec{A} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$U^*\vec{A} = \frac{1}{2}\begin{bmatrix} 0 \\ 1+i \\ 2 \\ 1-i \end{bmatrix}$$

and of course $U^*\vec{B} = \vec{0}$. The $\vec{Y}(t)$ vector is then given by

$$y_1(t) = 0 \ , \ y_2(t) = \cos(\sqrt{2}\omega t)\frac{1+i}{2} \ ,$$

$$y_3(t) = 2\cos(2\omega t) \ , \ y_4(t) = \cos(\sqrt{2}\omega t)\frac{1-i}{2}$$

and then

$$\vec{X}(t) = \frac{1}{2}\cos(\sqrt{2}\omega t)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \cos(2\omega t)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} .$$

Note that the multiplications by the Fourier matrix is rather tedious, because the matrix is a full matrix. Naively, it would take us n^2 operations to compute $F_n\vec{x}$. We shall se in the next section that the situation is much better.

4. Fast Fourier transform

The Fourier matrix is not sparse and hence to compute $F_n\vec{x}$ it takes about n^2 operations. We shall show by arranging the computation in a clever way, that it takes much fewer steps. To describe the result we set $n = 2^k$.

Theorem 4.1. *Let $\omega_n = e^{\frac{2\pi i}{n}}$ with $n = 2^k$. One can calculate $F_n\vec{x}$ for any vector $\vec{x} \in \mathbb{C}^n$ in $4 \cdot 2^k k = 4n\log_2(n)$ operations.*

Consider the case where $n = 2m$. Given an arbitrary n vector $\vec{x}$. Write it in the form $\vec{y} = \begin{bmatrix} \vec{y}_0 \\ \vec{y}_1 \end{bmatrix}$ where $\vec{y}_0$ contains only the entries of $\vec{x}$ with even index and likewise $\vec{y}_1$ the entries with odd index, i.e., we write $\vec{y}_0 = [x_0, x_2, \dots, x_{2(m-1)}]$ and $\vec{y}_1 = [x_1, x_3, \dots, x_{2m-1}]$. Then we can write the vector

$$\begin{bmatrix} \vec{y}_0 \\ \vec{y}_1 \end{bmatrix} = P\vec{x}$$

where P is the permutation matrix that maps the indices $(0, 2, \dots, 2(m-1))$ to $(0, 1, \dots, m-1)$ and the indices $(1, 2, \dots 2m-1)$ to the indices $(m, \dots, 2m-1)$. The point now is that

$$\begin{aligned}
[F_{2m}\vec{x}]_j = \sum_{\ell=0}^{2m-1} \omega_{2m}^{j\ell} x_\ell &= \sum_{\ell=0}^{m-1} \omega_{2m}^{j2\ell} x_{2\ell} + \sum_{\ell=0}^{m-1} \omega_{2m}^{j(2\ell+1)} x_{2\ell+1} \\
&= \sum_{\ell=0}^{m-1} \omega_m^{j\ell} x_{2\ell} + \omega_{2m}^{j} \sum_{\ell=0}^{m-1} \omega_m^{j\ell} x_{2\ell+1}
\end{aligned}$$

using that

$$\omega_{2m}^{2j\ell} = e^{\frac{2\pi i 2l\ell}{2m}} = e^{\frac{2\pi i l\ell}{m}} = \omega_m^{j\ell} \ .$$

We can rewrite this using the vectors $\vec{y}_0$ and $\vec{y}_1$ (which are m vector) as

$$[F_{2m}\vec{x}]_j = [F_m\vec{y}_0]_j + \omega_{2m}^j [F_m\vec{y}_1]_j \ .$$

Note that for $m \le j \le 2m-1$

$$\omega_{2m}^j = \omega_{2m}^{j-m} e^{\frac{2\pi i}{2m} m} = -\omega_{2m}^{j-m} \ .$$

Thus, if we denote by D_m the diagonal $m \times m$ matrix that has ω_{2m}^j on the diagonal we may write the Fourier multiplication as

$$F_{2m} = \begin{bmatrix} I & D_m \\ I & -D_m \end{bmatrix} \begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix} P \ .$$

Computing $P\vec{x}$ does not use any operations, we just group the even indexed and odd indexed elements together. This is achieved by a suitable input routine. Let $c(m)$ be the smallest number of steps to compute $F_m\vec{y}$. That gives us $2c(m)$ steps to compute

$$\begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix} P\vec{x}$$

To compute $D_m F_m$ that takes another m steps, because D_m is diagonal. Hence we need $2c(m) + m$ steps to compute $F_{2m}\vec{x}$. In other words, if $c(m)$ denotes the smallest number of steps to compute $F_m\vec{y}$ then

$$c(2m) \leq 2c(m) + m \ .$$

Suppose we pick $n = 2^k$ then $c(2^k) \leq 2c(2^{k-1}) + 2^{k-1}$ This leads to a recursion which can be solved with the result that

$$c(2^k) \leq 2^{k-1}a_0 + (k-1)2^{k-1} \ .$$

Here a_0 is the number of steps to compute the Fourier transform for a two vector which takes two steps. Thus, if we stick n back in, we get that the multiplication of the Fourier matrix F_n with an arbitrary vector takes

$$n(1 + \frac{1}{2}\log_2 n)$$

step. If we choose $n = 2^{20}$ which about a million by million matrix, it takes about $10^6(a_0 + 10)$ which should be compared with the naive computation which would give 10^{12}.