# PRACTICE MIDTERM EXAM

## 1. Topics

Some simple facts from standard real analysis, exterior measure, Lebesgue measure, countable additivity, Caratheodory's criterion, non-measurable sets, measurable functions, Egorov's and Luzin's theorem, the Lebesgue intergal for non-negative functions. Monotone convergence and Fatou's lemma. Thus the test might cover everything up to and including section 4.2.2 in Heil.

## 2. Help for the test

You may prepare a sheet both sides with information and bring it to the exam. Otherwise no help is allowed.

## 3. Practice Test

**Problem 1:** Let $C \subset \mathbb{R}^d$ be compact and $f : C \to \mathbb{R}$ an upper semicontinuous function. Prove that $f$ attains its maximum.
**Solution:** Pick any sequence $x_n$ such that $f(x_n) \to \sup_{x \in C} f(x)$. Such a sequence exists by the definition of the supremum although the supremum might be $+\infty$. Since $C$ is compact, there exists point $x \in C$ and a subsequence $x_{n_k}$ such that $x_{n_k} \to x$ as $k \to \infty$. Clearly $\lim_{k\to\infty} f(x_{n_k}) = \sup_{x\in C} f(x)$. Since $f$ is upper semi continuous, $\lim_{k\to\infty} f(x_{n_k}) \le f(x)$ and hence $f(x) = \sup_{x\in C} f(x)$. Thus $f$ attains its maximum on $C$ and the maximum is finite.

**Problem 2:** Recall that

$$\liminf_{k\to\infty} E_k = \cap_{k=1}^{\infty}(\cup_{j=k}^{\infty} E_k)$$

Suppose that $\sum_{k=1}^{\infty} |E_k|_e < \infty$. Show that $\liminf_{k\to\infty} E_k$ has measure zero.
**Solution:** Pick any $\varepsilon > 0$. By assumption, there exists $m$ such that

$$\sum_{k=m}^{\infty} |E_k|_e < \varepsilon \ .$$

By countable sub-additivity

$$\Big| \cup_{k=m}^{\infty} E_k \Big| \le \sum_{k=m}^{\infty} |E_k|_e < \varepsilon \ .$$

Since

$$\liminf_{k\to\infty} E_k \subset \cup_{k=m}^{\infty} E_k$$

for all $m = 1, 2, \ldots$ we have that

$$\Big| \liminf_{k\to\infty} E_k \Big| < \varepsilon \ .$$

Since $\varepsilon$ is arbitrary this proves the claim.

**Problem 3:** Let $E_j \subset \mathbb{R}^d, j = 1, 2, \dots$ be a sequence of sets, not necessarily measurable. Assume that $E_j \subset E_{j+1}$ for $j = 1, 2, \dots$. Prove that

$$| \cup_{j=1}^{\infty} E_j|_e = \lim_{j \to \infty} |E_j|_e \ .$$

**Solution:** We know that there exists a $G_\delta$ set $G_k$ such that $E_k \subset G_k$ and

$$|E_k|_e = |G_k| \ .$$

We know that the $G_k$ are measurable, but they are not necessarily nested. Consider $A_m = \cap_{k=m}^{\infty} G_k$ this set is also measurable and we have that $A_m \subset A_{m+1}$. Thus by continuity

$$\lim_{m \to \infty} |A_m| = \left| \cup_{m=1}^{\infty} A_m \right| \ ,$$

Since $E_k \subset E_n$ for all $n \geq k$ we have that $E_k \subset G_n$ for all $n \geq k$ and hence $E_k \subset A_k$ all $k$. Moreover, $|G_k| = |E_k|_e \leq |A_k| \leq |G_k|$ and hence $|E_k|_e = |A_k|$. Thus

$$\lim_{m \to \infty} |E_m|_e = \left| \cup_{m=1}^{\infty} A_m \right| \geq \left| \cup_{m=1}^{\infty} E_m \right|$$

since $\cup_{m=1}^{\infty} E_m \subset \cup_{m=1}^{\infty} A_m$. Because $E_k \subset \cup_{m=1}^{\infty} E_m$ we have that

$$|E_k|_e \leq \left| \cup_{m=1}^{\infty} E_m \right|$$

for all $k$ which proves the claim.

**Problem 4:** Define the inner Lebesgue measure of a set $A \subset \mathbb{R}^d$ to be

$$|A|_i = \sup\{|F| : F \text{ is closed } F \subset A\}$$

Prove that if $A$ is Lebesgue measurable then $|A|_e = |A|_i$. Moreover, show that if $|A|_e < \infty$ and $|A|_e = |A|_i$, then $A$ is Lebesgue measurable.

**Solution:** Let $A$ be a measurable set. For any closed set $F \subset A$ we have that $|F| \leq |A|$ and hence $|A|_i \leq |A|$. For any $\varepsilon > 0$ there exists a closed set $F \subset A$ such that $|A \setminus F| < \varepsilon$. Hence

$$|F| \geq |A| - \varepsilon$$

and hence $|A|_i \geq |A| - \varepsilon$ and thus $|A|_i \geq |A|$.

Assume now that $A$ is set with $|A|_e < \infty$ and $|A|_e = |A|_i$. Pick any $\varepsilon > 0$. By assumption there exists $F \subset A$ closed such that

$$|F| \geq |A|_e - \varepsilon/2 \ .$$

Moreover, there exists an open set $A \subset U$ such that $|U| \leq |A|_e + \varepsilon/2$. Now

$$U \setminus A \subset U \setminus F$$

and by monotonicity

$$|U \setminus A|_e \leq |U \setminus F| \ ,$$

and

$$|U \setminus F| = |U| - |F| \leq |A|_i + \varepsilon/2 - |A|_i + \varepsilon/2 = \varepsilon \ .$$

Hence

$$|U \setminus A|_e < \varepsilon$$

and $A$ is measurable.

**Problem 5, (5 points):** In Egorov's theorem we had to assume that $|E| < \infty$. Give an example of a sequence of functions on the whole real line which converges but where Egorov's theorem fails.

**Solution:** Take the sequence $f_n(x) = \max |x|, n$ on the real line. The point wise limit of this sequence is $f(x) = |x|$. If $A$ is any set the uniform convergence on this set means that

$$\sup_{x \in A} |f_n(x) - f(x)| \to 0$$

as $n \to \infty$. If $\mathbb{R} \setminus A$ has finite measure, there exists a sequence of points $x_j \in A$ with $x_j \to \infty$ as $j \to \infty$. But then

$$\sup_A |f_n(x) - f(x)| = \infty \ ,$$

and the convergence is not uniform.

**Problem 6, (5 points):** Prove that $f : E \to [-\infty, \infty]$ is measurable if and only if

$$\{f > r\}$$

is measurable for every $r$ rational.

**Solution:** If $f$ is measurable then $\{f > a\}$ is measurable for all $a$ and hence for rational $a$. Pick any $a$ real. There exists a sequence of rational number $r_n < a$ with $r_n \to a$ as $n \to \infty$. Then

$$\cap_{n=1}^{\infty} \{f > r_n\} = \{f \geq a\} \ .$$

and hence $\{f \geq a\}$ is measurable. Since $a$ is arbitrary, $f$ is measurable.

**Problem 7, (5 points):** Assume Fatou's lemma and deduce the monotone convergence theorem from it.

**Solution:** Assume that $f_n$ converges monotonically to $f$. Then we have that

$$\int_E f_n \leq \int_E f$$

for all $n = 1, 2, \ldots$. Since the numbers $\int_E f_n$ is an increasing sequence we find that

$$\limsup_{n \to \infty} \int f_n \leq \int_E f \ .$$

Fatou's lemma, however says, that

$$\liminf_{n \to \infty} \int_E f_n \geq \int_E \liminf_{n \to \infty} f_n = \int_E f$$

which proves the monotone convergence theorem.