The Birman-Schwinger Prinicple

All the proofs of the general inequalities involve in one way or another the Birman-Schwinger principle. To state, we shall from now consider only negative potentials of the form $V(x) = -U(x)$ with U nonnegative. For $E > 0$ introduce the Birman-Schwinger operator

$$K_E = U^{1/2}(-\Delta + E)^{-1}U^{1/2} \ .$$

This operator has an integral kernel given by

$$U^{1/2}(x)(-\Delta + E)^{-1}(x, y)U^{1/2}(y) \ ,$$

where $(-\Delta + E)^{-1}(x, y)$ is the kernel of the Greens functions of the Laplacian. That this operator exists can be seen from the Riesz representation theorem. For every fixed $g \in H^{-1}(R^n)$ there exists a *unique* $u \in H^1(R^n)$ so that

$$\int \nabla u \cdot \nabla f + \lambda \int uf = \int gf$$

holds for all $f \in H^1(R^n)$. Moreover the connection between u and g is linear. Further it is also bounded from $H^{-1}(R^n)$ to $H^1(R^n)$ since

$$\|u\|^2_{H^1(R^n)} \le \frac{1}{\lambda}\left(\int |\nabla u|^2 + \lambda \int u^2\right) = \int gu \le \|g\|_{H^{-1}(R^n)}\|u\|_{H^1(R^n)} \ .$$

We denote this u by

$$u = (-\Delta + \lambda)^{-1} g \ .$$

This operator has a kernel that can calculated. In one dimension it is given by

$$(-\partial^2 + E)^{-1}(x, y) = \frac{1}{2\sqrt{E}} e^{-\sqrt{E}|x-y|} \ .$$

In three dimension it is given by

$$\frac{1}{4\pi} \frac{e^{-\sqrt{E}|x-y|}}{|x-y|} \ .$$

One can work out the expressions in all the other dimensions. In odd dimensions it is given by elementary function and in even dimension it is given by Bessel functions. It is not difficult to see that for our class of potential, i.e., $U \in L^{n/2} + L^\infty$ with U vanishing at infinity, the Birman Schwinger operator is a bounded operator on $L^2(R^n)$. Recall that by Sobolev's inequality

$$\int U(x)|\psi(x)|^2 dx \le \alpha \int |\nabla \psi(x)|^2 dx + \beta \|\psi\|_2^2$$

for constants α, β. This means that $U^{1/2}$ as a multiplication operator is bounded from $H^1(R^n)$ to $L^2(R^n)$. Now consider

$$B = U^{1/2}(-\Delta + E)^{-1/2}$$

and note that the last factor maps from $L^2(R^n)$ to $H^1(R^n)$ and the first factor maps is back to $L^2(R^n)$ all in a bounded fashion. Thus B^* is also bounded from $L^2(R^n)$ to $L^2(R^n)$ and so is the BB^*, the Birman-Schwinger operator. Thus, the Birman-Schwinger operator is a bounded selfadjoint operator on $L^2(R^n)$. Although the discussion did not cover the case of one and two dimensions this properties hold in theses cases too. They are, in fact even easier to prove.

The following summarizes what we need to know about the Birman-Schwinger kernel.

Theorem: Birman-Schwinger principle *The number $-\lambda < 0$ is an eigenvalue of the quadratic form (3) if and only of* 1 *is an eigenvalue of the Birman-Schwinger operator $K_\lambda(U)$. The eigenvalues of the Birman-Schwinger operator are monotone decreasing functions of E.*

PROOF: The monotonicity of the eigenvalues follows from the min-max theorem since for every $f \in L^2(R^n)$

$$(U^{1/2} f, (-\Delta + E)^{-1} U^{1/2} f)$$

is decreasing as a function of E.

Next, suppose that ϕ is a solution of the Schrödinger equation, i.e.,

$$\int \nabla\phi \cdot \nabla f + \lambda \int \phi f = \int U\phi f$$

for all $f \in H^1(R^n)$. Since $U^{1/2}$ maps $H^1(R^n)$ boundedly to $L^2(R^n)$ we have that $U\phi \in H^{-1}(R^n)$ and hence

$$\phi = (-\Delta + \lambda)^{-1} U\phi \ ,$$

from which we conclude that

$$U^{1/2}\phi = U^{1/2}(-\Delta + \lambda)^{-1} U^{1/2} U^{1/2}\phi$$

which means that 1 is an eigenvalue of $K_\lambda(U)$ with $U^{1/2}\phi$ as eigenfunction. Conversely if ψ satisfies

$$\psi = U^{1/2}(-\Delta + \lambda)^{-1} U^{1/2}\psi$$

then if we set

$$\phi = (-\Delta + \lambda)^{-1} U^{1/2}\psi$$

we see that $\phi \in H^1(R^n)$. This means that ϕ satisfies

$$\int \nabla\phi \cdot \nabla f + \lambda \int \phi f = \int U^{1/2}\psi f$$

for all $f \in H^1(R^n)$. But $U^{1/2}\psi \in H^{-1}(R^n)$ and and moreover from the eigenvalue relation we learn that

$$U^{1/2}\psi = U\phi \ .$$

This proves the claim.

The first and most important version of the Lieb–Thirring inequality goes back to the paper by Lieb and Thirring 'Bound for the Kinetic Energy of Fermions Which Proves the Stability of Matter' [LT1].

Theorem: Lieb-Thirring bound *The negative eigenvalues $-\lambda_j$ of the quadratic form in three dimensions*

$$\int |\nabla\psi|^2 dx + \int V(x)|\psi|^2 dx \ , \ \int |\psi|^2 dx = 1$$

satisfy the estimate

$$\sum_j \lambda_j \leq L(3,1) \int [V(x)]_-^{5/2} dx \ ,$$

where $L(3,1) \leq 4/(15\pi)$.

PROOF: Using the Birman Schwinger principle we give a bound on the number of bound states less than $-e$. Start with λ small, so that some of the eigenvalues of K_λ are big. These values decrease as λ grows and every time one of them hits the value 1 the λ-value or rather its negative is and eigenvalue of the Schrödinger problem. If λ arrives at e, the number of these crossings equals the number of eigenvalues of $K_e(U)$ that are greater or equals to 1. In other words the number

$$N_e(U) \ ,$$

the number of eigenvalues of the *Schrödinger problem* that are less than $-e$, is given by the number of eigenvalues greater or equals 1 of the Birman-schwinger operator $K_e(U)$.

The quantity $N_e(U)$ can be used to calculate the sum of the eigenvalues since

$$\sum_j \lambda_j = \int_0^\infty N_e(U) de \ .$$

In general

$$\sum_j (\lambda_j)^\gamma = \gamma \int_0^\infty e^{\gamma-1} N_e(U) de \ . \tag{1}$$

To see this, note that

$$(\lambda_j)^\gamma = \int_0^\infty \chi_{\{(\lambda_j)^\gamma > e\}}(e) de$$

Summing over j and noting that

$$\sum_j \chi_{\{(\lambda_j)^\gamma > e\}}(e) = N_e(U)$$

yields (1) by a change of variables.

The most obvious upper bound would be $\mathrm{Tr} K_e(U)$ since we add up all the eigenvalues not just the one greater or equals 1. This, trace is however infinity. Thus the next step would be to consider the Hilbert-Schmidt norm

$$\mathrm{Tr} K_e(U)^2 \ .$$

This is easily calculated to give

$$\int U(x)[(-\Delta + e)^{-1}(x,y)]^2 U(y) dx dy \ .$$

Unfortunately, this is a bit tricky to estimate in terms of $\int U^{5/2} dx$. Following Lieb and Thirring one splits e into two pieces

$$\int \nabla\phi \cdot \nabla f - \int [U - e/2]\phi f = -\lambda + e/2 \int \phi f$$

and replacing $[U - e/2]$ by $[U - e/2]_+$ lowers the eigenvalues, i.e., increases their magnitude. Moreover we have that

$$N_e(U) \leq N_{e/2}([U - e/2]_+) \tag{1}$$

since there are more eigenvalues in the $[U - e/2]_+$ problem that are below $-e/2$ than there are eigenvalues in the U problem that fall below $-e$. Now we trace all the steps for the Birman-Schwinger principle for this tne problem and obtain the upper bound

$$N_{e/2}([U - e/2]_-) \leq \mathrm{Tr} K_e([U - e/2]_-)^2$$

which yields

$$\int [U - e/2]_-(x)[(-\Delta + e)^{-1}(x - y)]^2 [U - e/2]_-(y) dx dy \ .$$

Using Young's inequality* this is bounded above by

$$\|[U - e/2]_-\|_2^2 \int \frac{1}{(4\pi)^2} \frac{e^{-\sqrt{2}\sqrt{e}|x|}}{|x|^2} dx$$

$$= \frac{1}{4\pi} \int_0^\infty e^{-\sqrt{2}\sqrt{e} r} dr \|[U - e/2]_-\|_2^2$$

$$= \frac{1}{4\sqrt{2}\pi\sqrt{e}} \|[U - e/2]_-\|_2^2 \ .$$

Integrating this expression over the positive semi axis leads to

$$\sum_j \lambda_j = \int_0^\infty N_e(U) de \leq \frac{1}{4\sqrt{2}\pi} \int_0^\infty \frac{1}{\sqrt{e}} \int [U - e/2]_-^2(x) dx de \ .$$

Interchanging the two integrals and a bit of scaling leads to

$$\frac{1}{4\sqrt{2}\pi}\int\int_0^\infty \frac{1}{\sqrt{e}}[U-e/2]_-^2(x)dedx \ .$$

$$=\frac{1}{2\pi}\int_0^1[1-s^2]^2ds\int U(x)^{5/2}dx \ ,$$

$$=\frac{4}{15\pi}\int U(x)^{5/2}dx \ .$$

The semi classical constant is given by

$$\frac{1}{30\pi^2} \ .$$

*Young's inequality states that

$$\int f(x)g(x-y)h(y)dxdy \leq C_{p,q}\|f\|_p\|g\|_q\|h\|_r$$

with $1/p+1/q+1/r=2$.

Let us return to Sobolev's inequality, but this time for systems of orthonormal functions. Recall the definition of the one particle density

$$\rho_\Psi(x)=N\int|\Psi|^2(x,x_2,\ldots,x_N)dx_2\cdots dx_N \ .$$

The following theorem is a classical result of Lieb and Thirring [LT1].

Theorem: Uncertainty priniple for fermions. *Let Ψ be any normalized antisymmetric function in $H^1(R3N)$. Then*

$$T_\Psi=\sum_{j=1}^N\int|\nabla_j\Psi|^2(x_1,\ldots,x_N)dx_1\cdots dx_N\geq\frac{3}{5}(\frac{2}{5})^{2/3}L(3,1)^{-2/3}\int_{R^3}\rho_\Psi^{5/3}(x)dx \ .$$

We have that

$$\frac{3}{5}(\frac{2}{5})^{2/3}L(3,1)^{-2/3}<\frac{3^{5/3}\pi^{2/3}}{5\cdot 2^{2/3}}>1.68 \ .$$

PROOF: For a given Ψ consider the Schrödinger form

$$T_\Phi+V_\Phi \qquad (2)$$

where

$$V(x)=-c\rho_\Psi^{2/3}(x) \ ,$$

and hence

$$V_\Phi = -c\sum_j \int_{R^{3N}} \rho_\Psi^{2/3}(x_j)|\Phi|^2(x_1,\ldots,x_N)dx_1\cdots dx_N \ = -c\int_{R^3} \rho_\Psi^{2/3}(x)\rho_\Phi(x)dx. \quad (3)$$

We want to minimize the energy of (2) over all normalized antisymmetric functions Φ of n variables. Since we are talking about noninteracting fermions we fill the energy levels and find that

$$-\sum_j \lambda_j \le T_\Phi + V_\Phi$$

for all $\Phi \in H^1(R^{3N})$. Further, by the Lieb -Thirring inequality

$$\sum_j \lambda_j \le L(3,1)\int [V(x)]_-^{5/2}dx = L(3,1)c^{5/2}\int \rho_\Psi^{5/3}(x)dx \ .$$

Hence

$$-L(3,1)c^{5/2}\int \rho_\Psi^{5/3}(x)dx \le T_\Phi + V_\Phi$$

for all $\Phi \in H^1(R^{3N})$. In particular the inequality holds for Φ replaced by Ψ and using (3) we get

$$T_\Psi - c\int_{R^3} \rho_\Psi^{5/3}(x)\rho_\Phi(x)dx \ge -L(3,1)c^{5/2}\int \rho_\Psi^{5/3}(x)dx$$

or

$$T_\Psi \ge c\int_{R^3} \rho_\Psi^{5/3}(x)\rho_\Phi(x)dx \ge -L(3,1)c^{5/2}\int \rho_\Psi^{5/3}(x) \ .$$

Maximizing the right side over c yields the result.

References

[LT1] Phys. Rev.Lett. **35**, p.687-689, (1975)