Stability of matter

One of the fundamental attributes of matter is its **extensivity**, that is, its size as well as its energy content is proportional to the number of particles. This is intimately related to the fact that one can combine material systems. While it is an everyday experience that when we pour two separate liters of water together we still have two liters, this is a nontrivial fact from a theoretical point of view. Classical mechanics is unable to explain this most obvious of all the facts. It is indeed curious if one thinks of water as a huge number of molecules, themselves made up of electrons and nuclei interacting with each other via electrostatic forces. If the two liters are poured together the number of molecules that potentially might interact with each other is doubled. This means that the terms in the Coulomb interaction has quadrupled, that is, there are four times as many interaction terms as there were before pouring that stuff together. Indeed the Coulomb energy in each of the separate containers consists of the electron-nuclear attraction ($N \times K$ terms) plus the electron-electron repulsion ($N^2/2$ terms). Thus in total $2 \times N \times K$ attraction terms and N^2 repulsion terms. If we pour them together we have $4 \times N \times K$ attraction terms and $2 \times N^2$ repulsion terms. Thus the electrostatic energy is not just additive, it grows with the square of the numbers of particles and not linearly.

Let us assume for the moment that the energy content of matter consisting of N particles is proportional to $-N^2$. The minus sign indicates that we have to spend energy in order to separate the particles. Two separate containers have total energy content of $-2N^2$. Next we pour the two containers together and get a system with $2N$ particles and hence and energy content of $-4N^2$. Thus by pouring the two containers together we have liberated an additional energy $2N^2$ which is enormous since N is a huge number around 10^{26} for a liter of water. In such a situation, the ground state of the 'universe' would consist of a huge lump of charged particles sticking together. i.e., a world that looks very different from what we see. To summarize, 'stability of matter' means that the energy content of a lump of matter must be proportional to the number of particles. Thus, there must be a mechanism that beats somehow the quadratic dependence on the number of particles of the Coulomb energy. As we shall see the uncertainty principle for fermions will be that mechanism.

We call a physical quantity extensive if it is proportional to the number of particles involved. While there are other extensive quantities, such as the volume of the system, the free energy, the entropy etc. we shall concentrate on the ground state energy of the system and see later that the extensivity of all the other important quantities follow from this.

We are now ready to define the ground state energy of a quantum system consisting of K nuclei and N electrons. The electrons are **fermions** which will be key in our investigation while the nuclei maybe fermions as well as bosons. Since their masses are by at least three orders of magnitudes heavier than the one of the electron we shall fix the nuclei at arbitrary positions $R_1, \dots, R_K$. Let us add that the electrons carry an additional degree of freedom, called the spin. There are electrons with spin up and spin down, i.e., two kinds. It is often convenient to leave the number of spin states as a variable q. Thus, when filling up energy levels, we can fill q electrons into the ground state and then q electrons into the first excited and so on. We set

$$\Psi(x_1,\sigma_1;\ldots;x_N,\sigma_N)$$

and the normalization condition is now given by

$$\sum_{\sigma_1,\ldots,\sigma_N=1}^{q}\int|\Psi(x_1,\sigma_1;\ldots;x_N,\sigma_N)|^2dx_1\cdots dx_N=1$$

We shall adopt the notation

$$\int dz=\sum_{\sigma}\int dx\ .$$

The one particle density will be given by

$$\rho_\Psi(x)=\sum_{\sigma_1,\ldots,\sigma_N=1}^{q}\int|\Psi(x,\sigma_1;\ldots;x_N,\sigma_N)|^2dx_2\cdots dx_N\ .$$

For such a functions Ψ we have the kinetic energy

$$T_\Psi=\sum_{\sigma_1,\ldots,\sigma_N=1}^{q}\sum_{j}\int|\nabla_j\Psi(x_1,\sigma_1;\ldots;x_N,\sigma_N)|^2dx_1\cdots dx_N$$

and the potential energy

$$V_\Psi=\int V_{\mathrm{C}}(x_1,\ldots,x_N;R_1,\ldots,R_K)|\Psi(x,\sigma_1;\ldots;x_N,\sigma_N)|^2dz_1\cdots dz_N\ ,$$

the variable $R_1,dots,R_K$ we keeps fixed. Here

$$V_{\mathrm{C}}(x_1,\ldots,x_N;R_1,\ldots,R_K)=-\sum_{k=1,j=1}^{K,N}\frac{Z_k}{|x_j-R_k|}+\sum_{i<j}^{N}\frac{1}{|x_i-x_j|}+\sum_{k<l}\frac{Z_kZ_l}{|R_k-R_l|}$$

the first term being the attraction between the nuclei and elctrons (the charge number of nucleus k is Z_k), the second term is the repulsion between the electrons and the third terms is the repulsion between the nuclei. The latter is jsut a function of the nuclear positions and does not take part in a dynamical fashion but nevertheless it will be an important term.

The ground state energy is defined by

$$E_0(N,K,R_1,\cdots R_K,q):=$$

$$\inf\{T_\Psi+V_\Psi:\int|\Psi|^2dz_1\cdots dz_N=1,\Psi(z_1,\ldots,z_N)\text{ is antisymmetric in the particles labels}\}$$

Now we are ready to define the notion of stability:

We call

$$\inf_{R_1,\ldots,R_K} E_0(N, K, R_1, \cdots R_K, q) > -\infty$$

stability of the first kind and

$$\inf_{R_1,\ldots,R_K} E_0(N, K, R_1, \cdots R_K, q) > -C(Z_1, \ldots, Z_K, q)(N + K)$$

stability of the second kind.

While stability of the first kind was shown in the early sixties by Kato, stability of the second kind is much more difficult an it was proved by Freeman Dyson and Andrew Lenard around 1968. We shall present another proof due to Lieb and Thirring from the mid seventies which is much more conceptual and yields much better constants. It is one of the classical works that started the industry which we now call Quantum Coulomb systems.