The cheese theorem

First some notation: Let Ω be some domain in R^n. If $h > 0$ then we denote by the *inner rim* of Ω the set

$$\Omega_h = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le h\}$$

and for $h < 0$ we define the inner rim

$$\Omega_h = \{x \in \Omega^{\mathrm{c}} : \mathrm{dist}(x, \partial\Omega) \le -h\} \ ,$$

where Ω^{c} denotes the complement of Ω.

Lemma 1: *Let Ω be a domain in R^n and assume that it is covered by a collection of closed cubes of side length a whose interior are disjoint. Denote by N the nuber of cubes that are inside Ω. Denote by $\Delta\Omega$ the set Ω with all these cubes removed, then*

$$|\Delta\Omega| = |\Omega| - Na^n \le |\Omega_{\sqrt{n}d}| \ .$$

PROOF: Remove all the cubes that are inside Ω. The cubes left over intersect the boundary and hence the interior of these cubes is not farther away than $\sqrt{n}a$ from the boundary. Hence, the union of the intersection of these cubes with Ω is contained in the inner rim of size $\sqrt{n}a$ from which the above estimate follows.

From this we deduce the bound

$$N \le \frac{1}{a^n}(|\Omega| - |\Omega_{\sqrt{n}a}|) \ . \tag{1}$$

A further elementary lemma concerning the geometry of balls is the following one.

Lemma 2 *Let $B(r)$ be a ball of radius r. Pick b so that*

$$r \ge 2b\sqrt{n} \ge 0 \ .$$

Then

$$|B_{2b\sqrt{n}}| \le |B_{-2b\sqrt{n}}| \le \gamma_n \omega_n r^{n-1} b$$

where

$$\gamma_n = 2\sqrt{n}[2^n - 1]$$

and ω_n is the volume of the unit ball in R^n.

PROOF: The first inequality is obvious. We therefore have that

$$\omega_n[r^n - (r - 2b\sqrt{n})^n] \le \omega_n[(r + 2b\sqrt{n})^n - r^n] = \omega r^n[(1+\varepsilon)^n - 1]$$

where

$$0 \le \varepsilon = \frac{2b\sqrt{n}}{r} \le 1 \ .$$

Now, by the binomial formula

$$[(1+\varepsilon)^n - 1] = \varepsilon \sum_{k=1}^{n} \binom{n}{k} \varepsilon^{k-1} \leq \varepsilon \sum_{k=1}^{n} \binom{n}{k} = \varepsilon[2^n - 1] \ .$$

The main goal of this section is the proof of the following theorem of Lieb and Lebowitz.

Cheese theorem *Let p be a positive integer and for all $j \geq 1$ define the radii*

$$r_j = \frac{1}{(1+p)^j}$$

and the integers

$$m_j = p^{j-1}(1+p)^{j(n-1)} \ .$$

Then, if

$$1 + p \geq \gamma_n \omega_n + \frac{2^n}{\omega_n}$$

it is possible to pack

$$\cup_{j=1}^{\infty}(m_j \text{ balls of radius } r_j)$$

in the unit n-dimensional ball.

PROOF: Cover the unit ball with closed cubes of size $2r_1$ and put balls of radius r_1 into each of the cubes that sit inside the unit ball. Removing these balls wee get a remaining set Ω^1 which we cover by closed cubes of size $2r_2$. Again, we fill those that are inside Ω^1 with balls of radius r_2 and so on. Having done this j-times we have to show that the remaining uncovered set Ω^j can be packed with balls of radius r_{j+1}. Let is start with $j = 0$, i.e., with the ball itself. We have that

$$|\Omega^1| = \omega_n(1 - m_1 r_1^n) == \omega_n(1 - (1+p)^{(n-1)}(1+p)^{-n}) = \omega_n \frac{p}{1+p} \ .$$

Now, we look at the inner rim

$$\Omega^1_{2\sqrt{n}r_2}$$

which consists of all points that are inside Ω^1 but are not more than a distance $2\sqrt{n}r_2$ away from the boundary of Ω^1. Each of these points is either in the outer rim

$$B^1_{-2\sqrt{n}r_2}$$

of the balls that have been removed or in the inner rim

$$B_{2\sqrt{n}r_2}$$

of the unit ball. Hence

$$|\Omega^1_{2\sqrt{n}r_2}| \le \gamma_n\omega_n r_2(1+n_1 r_1^{n-1}) = 2\gamma_n\omega_n\frac{1}{(1+p)^2}$$

since

$$2\sqrt{n}r_2 < r_1 \ ,$$

because of Lemma 2. Next, we know from Lemma 1 that the number of cubes that have size r_2 and that sit inside Ω_1 is not less than

$$\frac{1}{(2r_2)^n}(|\Omega^1| - |\Omega^1_{2\sqrt{n}r_2}|) \le \frac{1}{2^n}(1+p)^{2n}(\omega_n\frac{p}{1+p} - 2\gamma_n\omega_n\frac{1}{(1+p)^2}) \ .$$

Thus, if it is true that

$$m_2 = p(1+p)^{2(n-1)} \le \frac{1}{2^n}(1+p)^{2n}(\omega_n\frac{p}{1+p} - 2\gamma_n\omega_n\frac{1}{(1+p)^2})$$

we have completed the first step. But this says that p has to satisfy

$$1 \le \frac{\omega_n}{2^n}(1+p-\frac{2\gamma_n}{p}) \ .$$

Since $p > 1$ this is implied by

$$1 \le \frac{\omega_n}{2^n}(1+p-2\gamma_n)$$

which is precisely our condition. Thus we have done the first inductive step.

Suppose that we have arrived at Ω^j which is what remains of the unit ball after removing the m_1 balls of radius r_1, the m_2 balls of radious r_2 etc. and at the end removing the m_j balls of radius r_j. Its volume is

$$|\Omega^j| = \omega_n(1 - \sum_{k=1}^{j} m_k r_k^n = \omega_n(\frac{p}{1+p})^j \ .$$

Next we consider the inner rim

$$\Omega^j_{2\sqrt{n}r_{j+1}}$$

which is the collection of all points in Ω^j that have at most distance $2\sqrt{n}r_{j+1}$ to the boundary of Ω^j. Thus, each of these points is either in the outer rim of some of the balls that have been removed or in the inner rim of the unit ball. Thus by Lemma 2

$$|\Omega^j_{2\sqrt{n}r_{j+1}}| \le \gamma_n\omega_n r_{j+1}(1+\sum_{k=1}^{j} m_k r_k^{n-1} = \gamma_n\omega_n\frac{(p^j+p-2)}{(p-1)(1+p)^{j+1}} \ .$$

Again, by Lemma 1 we now that we can pack Ω^j at least

$$\frac{1}{(2r_{j+1})^n}(|\Omega^j| - |\Omega^j_{2\sqrt{n}r_{j+1}}|)$$

cubes of size $2r_{j+1}$. Since

$$(|\Omega^j| - |\Omega^j_{2\sqrt{n}r_{j+1}}|) \geq \omega_n[(\frac{p}{1+p})^j - \gamma_n \frac{(p^j+p-2)}{(p-1)(1+p)^{j+1}}]$$

this amounts to show that

$$m_{j+1} = p^j(1+p)^{(j+1)(n-1)} \leq \frac{\omega_n}{2^n}(1+p)^{(j+1)n}[(\frac{p}{1+p})^j - \gamma_n \frac{(p^j+p-2)}{(p-1)(1+p)^{j+1}}]$$

or

$$1 \leq \frac{\omega_n}{2^n}[(1+p) - \gamma_n \frac{(1+p^{-j}(p-2))}{(p-1)}]$$

which holds if

$$1 \leq \frac{\omega_n}{2^n}[(1+p) - \gamma_n]$$

which is again our condition. Note that we have used that $p > 1$. Thus, we can continue with our packing indefinitely.

Corollary *The packing is asymptotically complete and rapid.*

Since

$$\sum_{j=1}^{N} m_j r_j^n = \sum_{j=1}^{N} p^{j-1}(1+p)^{j(n-1)} \frac{1}{(1+p)^{jn}} = \frac{1}{1+p}\sum_{j=0}^{N-1}(\frac{p}{1+p})^j = 1 - \delta^N$$

where

$$\delta = \frac{p}{1+p} < 1\ .$$

As $N \to \infty$ this converges to 1.

Further the set that is not covered up and including the N-th packing has volume δ^N and hence the convergence of the packing is exponential.