ON BAKER'S PATCHWORK CONJECTURE FOR DIAGONAL PADÉ APPROXIMANTS

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Abstract. We prove that for entire functions f of finite order, there is a sequence of integers $\mathcal{S}$ such that as $n \to \infty$ through S,

$$\min \left\{ \left| f - [n/n] \right| (z) , \left| f - [n-1/n-1] \right| (z) \right\}^{1/n} \to 0$$

uniformly for z in compact subsets of the plane. More generally this holds for sequences of Newton-Padé approximants and for functions whose errors of approximation by rational functions of type (n,n) decay sufficiently fast. This establishes George Baker's Patchwork Conjecture for large classes of entire functions.

Padé approximation, Multipoint Padé approximants, spurious poles, Baker Patchwork Conjecture. 41A21, 41A20, 30E10.

1. Introduction[1]

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series. Given a non-negative integer n, the (n,n) Padé approximant is a rational function $[n/n] = P_n/Q_n$, where P_n, Q_n are polynomials of degree $\leq n$ with Q_n not identically 0 and

$$(fQ_n - P_n)(z) = O\left(z^{2n+1}\right).$$

The convergence of Padé approximants is a much studied subject. One of the pitfalls of the method is the phenomenon of spurious poles, namely poles that do not reflect the analytic properties of the function f. For this reason, the most general results, such as the Nuttall-Pommerenke theorem, involve convergence in capacity, rather than uniform convergence. In 1961, Baker, Gammel, and Wills nevertheless conjectured that at least a subsequence of the diagonal Padé sequence converges locally uniformly [3]:

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Baker-Gammel-Wills Conjecture (1961)
Let f be meromorphic in $B_1 = \{z : |z| < 1\}$ and analytic at 0. Then there is a subsequence $\{[n/n]\}_{n \in \mathcal{S}}$ of $\{[n/n]\}_{n \geq 1}$ that converges uniformly to f in compact subsets of B_1 omitting poles of f.

The author showed in 2001 [13] that the conjecture is false, by considering the Rogers-Ramanujan function with a value of q on the unit circle. V.I. Buslaev quickly followed [5] with an analytic counterexample, formed from an algebraic function, and then showed that even the Rogers-Ramanujan function provides an analytic counterexample [6]. One of the unresolved issues is whether the Baker-Gammel-Wills conjecture is valid for entire functions, or perhaps even functions meromorphic in the whole plane. To date, there is still no counterexample. The author proved [12] that the Baker-Gammel-Wills conjecture is true for most entire functions in the sense of category, and subsequently that a more general form involving multipoint Padé approximants [14] also holds in the sense of category.

After his original conjecture was disproved, George Baker [1] noted that in the counterexamples, just two subsequences together provide locally uniform convergence in the unit ball. He went on to conjecture that a patchwork of finitely many subsequences can provide locally uniform convergence for functions meromorphic in the ball [2].

Here is a precise statement:

George Baker's Patchwork Conjecture (2005)
Let the function f be analytic in $\overline{B_1} = \{z : |z| \leq 1\}$ except for a finite number of poles in the interior. There exists a finite number of infinite subsequences $\{\mathcal{S}_k\}_{k=1}^{L}$ of positive integers such that these subsequences can be patched together in such a manner that for any $z \in \overline{B_1}$, for some $1 \leq k \leq L$,

$$\lim_{n \to \infty, n \in \mathcal{S}_k} [n/n](z) = f(z)$$

on the sphere.

Here on the sphere means in the chordal metric - so that at poles of f, the approximants diverge to ∞ in absolute value. In this paper, we shall show Baker's patchwork conjecture is true for entire functions whose errors of rational approximation decay sufficiently rapidly, and in particular for all entire functions of finite order. Moreover, we obtain a sequence of integers $\mathcal{S}$ such that either $[n/n]$ or $[n-1/n-1]$ converges for $n \in \mathcal{S}$, so just two subsequences are enough.

We note that one consequence of the Nuttall-Pommerenke theorem [15], [16], is that for functions f meromorphic in the plane (and more generally with singularities of capacity 0), there is a subsequence $\mathcal{S}$ of integers and a set $\mathcal{E}$ of capacity 0, such that

$$\liminf_{n\to\infty, n\in\mathcal{S}} |f-[n/n]|(z)^{1/n} = 0, z\in\mathbb{C}\backslash\mathcal{E}.$$

Baker's Patchwork Conjecture tries to avoid that unknown set $\mathcal{E}$.

For any compact set $K\subset\mathbb{C}$ and a function f continuous on K, we define the error in best rational approximation of f on K by rational functions of type (n,n),

$$E_{nn}(f;K) = \inf\left\{\left\|f-\frac{P}{Q}\right\|_{L_\infty(K)} : \deg(P), \deg(Q)\le n\right\}.$$

A special case of our results is:

Theorem 1.1
Assume that f is entire and that

$$\lim_{n\to\infty} E_{nn}(f;B_1)^{1/\left(n\sqrt{\log n}\right)} = 0. \tag{1.1}$$

Then there is an infinite sequence of positive integers $\mathcal{S}$ such that uniformly for z in compact subsets of the plane

$$\lim_{n\to\infty, n\in\mathcal{S}} \min\{|f-[n/n]|(z), |f-[n-1/n-1]|(z)\}^{1/n} = 0.$$

Remarks
(a) The condition (1.1) is satisfied by all entire functions of finite order: indeed for those functions

$$\limsup_{n\to\infty} E_{nn}(f;B_1)^{1/(n\log n)} < 1.$$

We believe the result above holds for all entire functions.
(b) Note that this does not imply locally uniform convergence of either $[n/n]$ or $[n-1/n-1]$ for $n\in\mathcal{S}$.
(c) A similar result was proved by Khristoforov for elliptic functions [11].

Our method also allows us to treat Newton-Padé approximation. Let $\{a_j\}_{j=1}^\infty$ be a sequence of not necessarily distinct points in the plane and

$$\omega_n(z) = \prod_{j=1}^n (z-a_j), \ n\ge 1.$$

We say $R_n = P_n/Q_n$ where P_n, Q_n have degree at most n and Q_n is not identically 0, is a *Newton-Padé approximant* to f if

$$\frac{fQ_n - P_n}{\omega_{2n+1}}$$

is analytic at the zeros of ω_{2n+1}. Note that as n increases, we keep earlier interpolation points. Moreover if all $a_j = 0$, then $R_n = [n/n]$. Theorem 1.1 is a special case of :

Theorem 1.2
Let U be a simply connected open set. Let f be analytic in U and such that for some compact set $L \subset U$ of positive logarithmic capacity,

$$\lim_{n\to\infty} E_{nn}(f; L)^{1/\left(n\sqrt{\log n}\right)} = 0.$$

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of not necessarily distinct points lying in a compact subset K of U. Let $\{R_n\}$ be the corresponding Newton-Padé approximants to f. Then there is an infinite sequence of positive integers $\mathcal{S}$ such that uniformly for z in compact subsets of U,

$$\lim_{n\to\infty, n\in\mathcal{S}} \left(\min\{|f - R_n|(z), |f - R_{n-1}|(z)\}\right)^{1/n} = 0.$$

The paper is organized as follows: we present the ideas of proof in Section 2. Section 3 contains some lemmas on polynomials and the Gonchar-Grigorjan inequality. In Section 4, we compare Newton-Padé approximants and best rational approximants on different sets. We prove Theorem 1.2 in Section 5, and Theorem 1.1 in Section 6.

In the sequel, $C, C_1, C_2, \ldots$ denote constants independent of n, z, and polynomials P of degree $\leq m$ or n. The same symbol does not necessarily denote the same constant in different occurrences. For $R > 0$, we let

$$B_R = \{z : |z| < R\};$$

cap denotes logarithmic capacity [17], [18], while m_2 denotes planar measure, and m_1 denotes one dimensional Hausdorff outer measure. Thus for $S \subset \mathbb{C}$,

$$m_1(S) = \inf\left\{\sum_{j=1}^{\infty} diam\left(B^{(j)}\right) : E \subset \bigcup_j B^{(j)}\right\}$$

where each $B^{(j)}$ is a ball with diameter $diam\left(B^{(j)}\right)$. If Γ is a simple closed curve, we let $\hat{\Gamma}$ denote the compact set consisting of the union of Γ and the simply connected set enclosed by Γ.

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2. Ideas of Proof

Write $R_n = P_n/Q_n$, with some normalization of P_n, Q_n and

$$\Delta_n = fQ_n - P_n. \tag{2.1}$$

Then

$$P_{n+1}Q_n - P_nQ_{n+1} = \Delta_nQ_{n+1} - \Delta_{n+1}Q_n \tag{2.2}$$

vanishes at the zeros of ω_{2n+1}. But then as the left-hand side is a polynomial of degree at most $2n+1$, so for some constant A_n,

$$P_{n+1}Q_n - P_nQ_{n+1} = A_n\omega_{2n+1}. \tag{2.3}$$

Hence also

$$\Delta_nQ_{n+1} - \Delta_{n+1}Q_n = A_n\omega_{2n+1}. \tag{2.4}$$

Now comes the key observation. Suppose that for some ζ_n that is not an interpolation point, and *both* $m = n, n+1$, we have, say,

$$|f - R_m|(\zeta_n) > 1.$$

(If this inequality was initially only known at an interpolation point, then by lower semi-continuity, it would also hold in a neighborhood, so would hold at some ζ_n that is not an interpolation point). Then for $m = n, n+1$,

$$|\Delta_m|(\zeta_n) > |Q_m(\zeta_n)|.$$

Substituting these inequalities into (2.4) gives

$$|A_n||\omega_{2n+1}(\zeta_n)| \leq 2|\Delta_n\Delta_{n+1}|(\zeta_n).$$

If Γ is a simple closed curve enclosing ζ_n, the maximum modulus principle gives

$$|A_n| \leq 2\left\|\frac{\Delta_n\Delta_{n+1}}{\omega_{2n+1}}\right\|_{L_\infty(\Gamma)}.$$

Here the size of ω_{2n+1}^{-1} may be controlled if all interpolation points lie in a compact set at a positive distance to Γ. So we have a bound on $|A_n|$ decaying roughly like the *square* of $\|\Delta_n\|_{L_\infty(\Gamma)}$, whereas it really ought to decay like $\|\Delta_n\|_{L_\infty(\Gamma)}$. It is this simple fact that makes our proofs work.

Next, we choose $m < n$ and write, using (2.3),

$$R_n - R_m = \sum_{j=m}^{n-1} \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}},$$

or equivalently

$$P_n Q_m - P_m Q_n = Q_n Q_m \sum_{j=m}^{n-1} \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}}$$

and

$$\Delta_m Q_n - \Delta_n Q_m = Q_n Q_m \sum_{j=m}^{n-1} \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}}. \tag{2.5}$$

Then also

$$|f - R_n| \leq |f - R_m| + \sum_{j=m}^{n-1} \left| \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}} \right|.$$

We can use Cartan's Lemma, or Polya's estimate on the area of a set where monic polynomials are small, to bound all the $|Q_j|$ below outside a set of not too large measure. More precisely, this incurs a factor of size $n^{o\left(n/\sqrt{\log n}\right)}$ which can be absorbed by the rapid decay of A_j. This leads to estimates for $P_m Q_n - P_n Q_m$ on a set of positive area. Polynomial growth lemmas then provide estimates for $P_m Q_n - P_n Q_m$ on any disk. This in turn allows us to show that

$$|f - R_n| < |f - R_m| + \text{ a small term.}$$

If n is large enough compared to m, and lies in a suitable subsequence of integers, then this contradicts the rate of approximation provided by Newton-Padé approximants - which is essentially the same as best rational approximants. It follows that the ζ_n above does not exist, at least for a subsequence. Of course the rigorous details involve work.

3. Preliminary Lemmas

We start with a simple growth lemma.

Lemma 3.1
Let $\rho \geq 1$. *Let K be a compact set in $\overline{B_\rho}$ of positive capacity.*
(a) Then for $n \geq 1$ and polynomials P of degree $\leq n$,

$$\|P\|_{L_\infty\left(\overline{B_\rho}\right)} \leq \left(\frac{2\rho}{cap\,(K)} \right)^n \|P\|_{L_\infty(K)}.$$

(b) Assume now that K is a compact set in $\overline{B_\rho}$ of positive area. Then for $n \geq 1$ and polynomials P of degree $\leq n$,

$$\|P\|_{L_\infty(\overline{B_\rho})} \leq 3^n \left(\frac{\pi \rho^2}{m_2(K)} \right)^{\frac{n}{2}} \|P\|_{L_\infty(K)} .$$

Proof

(a) Let us assume, as we may, that P is monic of degree n. Let μ be the equilibrium measure for K in the sense of potential theory, and g be the Green function for K with pole at ∞. Thus

$$g(z) = \int \log|z-t| \, d\mu(t) - \log cap(K) .$$

The Bernstein-Walsh inequality asserts that for z in the unbounded component of $\mathbb{C}\backslash K$, [17, p. 156]

$$|P(z)| \leq \|P\|_{L_\infty(K)} e^{ng(z)}.$$

Here for $|z| \leq \rho$, we see that

$$g(z) \leq \log(|z|+\rho) - \log cap(K) \leq \log(2\rho) - \log cap(K) .$$

Then for $|z| = \rho$,

$$|P(z)| \leq \left(\frac{2\rho}{cap(K)} \right)^n \|P\|_{L_\infty(K)} .$$

The maximum modulus principle also shows that this holds for all $|z| \leq \rho$.

(b) Normalize P as follows, with the zeros denoted by v :

$$P(z) = c \prod_{|v| \leq 2\rho} (z-v) \prod_{|v| > 2\rho} \left(1 - \frac{z}{v}\right) .$$

We may assume that $c \neq 0$. Assume there are k terms in the first product and ℓ in the second. Choose ε such that

$$\|P\|_{L_\infty(K)} = \varepsilon^{\deg(P)} = \varepsilon^{k+\ell}.$$

Then for z in K,

$$\varepsilon^{k+\ell} \geq |c| \left| \prod_{|v| \leq 2\rho} (z-v) \right| \left(\frac{1}{2} \right)^\ell$$

so

$$\left| \prod_{|v| \leq 2\rho} (z-v) \right| \leq |c|^{-1} 2^\ell \varepsilon^{k+\ell}.$$

By Polya's lemma [4, p. 320, Thm. 6.6.4], for any $\delta > 0$

$$m_2\left(\left\{z : \left|\prod_{|v|\leq 2\rho}(z-v)\right| \leq \delta^k\right\}\right) \leq \pi\delta^2.$$

so

$$m_2(K) \leq \pi\left[|c|^{-1} 2^\ell \varepsilon^{k+\ell}\right]^{2/k}.$$

So

$$|c| \leq 2^\ell \varepsilon^{k+\ell}\left(\frac{\pi}{m_2(K)}\right)^{\frac{k}{2}}.$$

From our normalization, and choice of ε,

$$\begin{aligned}
\|P\|_{L_\infty\left(\overline{B_\rho}\right)} &\leq |c|(3\rho)^k\left(\frac{3}{2}\right)^\ell \\
&\leq 2^\ell\varepsilon^{k+\ell}\left(\frac{\pi}{m_2(K)}\right)^{\frac{k}{2}}(3\rho)^k\left(\frac{3}{2}\right)^\ell \\
&\leq 3^{k+\ell}\left(\frac{\pi\rho^2}{m_2(K)}\right)^{\frac{k}{2}}\|P\|_{L_\infty(K)} \\
&\leq 3^n\left(\frac{\pi\rho^2}{m_2(K)}\right)^{\frac{n}{2}}\|P\|_{L_\infty(K)}
\end{aligned}$$

as $k \leq n$ and $m_2(K) \leq \pi\rho^2$.■

Lemma 3.2

Let $\rho \geq \frac{1}{2} > \varepsilon > 0$. Let Q be a polynomial of degree $\leq n$, admitting the representation

$$Q(z) = \prod_{|v|\leq 2\rho}(z-v)\prod_{|v|>2\rho}\left(1-\frac{z}{v}\right). \tag{3.1}$$

We then say Q is normalized w.r.t. ρ.

(a) Then for $|z| \leq \rho$,

$$|Q(z)| \leq (3\rho)^n$$

while if k is the number of zeros of Q in $B_{2\rho}$,

$$\frac{1}{|Q(z)|} \leq \frac{2^n}{\varepsilon^k}.$$

for $|z| \leq \rho$, $z \notin \mathcal{E}$, where $m_2(\mathcal{E}) \leq \pi\varepsilon^2$ and $m_1(\mathcal{E}) \leq 4e\varepsilon$.

(b) Let Γ_1 and Γ_2 be simple closed Jordan curves such that Γ_2 encloses

Γ_1. *Then there exists* $c > 0$ *depending only on* Γ_1, Γ_2 *and a simple closed contour* $\Gamma^{(n)}$ *depending on* Q *between* Γ_1 *and* Γ_2 *such that*

$$\frac{1}{|Q(z)|} \leq c^n \text{ on } \Gamma^{(n)}.$$

Remark

It is essential for our applications of this lemma that c in (b) does not depend on n or Q.

Proof

(a) Suppose n_1 is the degree of Q. Let k be the number of zeros in $|z| \leq 2\rho$ and ℓ be the number of zeros outside this disk. We see that for $|z| \leq \rho$,

$$|Q(z)| \leq (3\rho)^k (3/2)^\ell \leq (3\rho)^n .$$

Next for $|z| \leq \rho$,

$$\begin{aligned} \frac{1}{|Q(z)|} &= \frac{1}{\left|\prod\limits_{|v|\leq 2\rho} (z-v)\right| \left|\prod\limits_{|v|>2\rho} \left(1-\frac{z}{v}\right)\right|} \\ &\leq \left(\varepsilon^{-k}\right)\left(2^\ell\right) \leq \frac{2^n}{\varepsilon^k} \end{aligned}$$

outside a set $\mathcal{E}$ with $m_2(\mathcal{E}) \leq \pi\varepsilon^2$ [4, p. 320, Thm. 6.6.4] and $m_1(\mathcal{E}) \leq 4e\varepsilon$ [4, p. 325, Thm. 6.6.7].

(b) Since Γ_2 encloses a simply connected open set, say V, there is a conformal map ϕ of V onto B_1. As Γ_1 is contained inside Γ_2, there exists $r \in (0,1)$ such that $\phi(\Gamma_1) \subset B_r$. Let $r < r' < 1$. Next, as Γ_2 is a Jordan curve, Caratheodory's Theorem [7, p. 93] ensures that ϕ has a continuous one-one extension to the boundary. Thus ϕ is a one-one continuous map of $\overline{V} = \hat{\Gamma}_2$ onto $\overline{B_1}$. Its inverse map $\phi^{[-1]}$ is a continuous map of $\overline{B_1}$ onto $\hat{\Gamma}_2$. Then for small enough $\eta > 0$, $\phi(\{z : dist(z, \Gamma_2) < \eta \text{ and } z \text{ inside } \Gamma_2\})$ is contained in $\{z : r' \leq |z| < 1\}$. Let $A > 0$ be such that

$$\left|\phi'\left(\phi^{[-1]}(z)\right)\right| \leq A \text{ whenever } |z| \leq r'.$$

We then have for any set $F \subset \phi^{[-1]}(B_{r'})$,

$$m_1(\phi(F)) \leq A m_1(F) .$$

We now choose

$$\varepsilon = \frac{1}{8Ae}(r'-r)$$

Then if

$$\mathcal{E}_\varepsilon = \left\{ z : \left| \prod_{|v|\leq 2\rho} (z-v) \right| \leq \varepsilon^k \right\},$$

we have

$$m_1\left(\phi\left(\mathcal{E}_\varepsilon \cap \phi^{[-1]}(B_{r'})\right)\right) \leq Am_1(\mathcal{E}_\varepsilon) \leq A4e\varepsilon = \frac{1}{2}(r'-r).$$

As one dimensional Hausdorff measure does not increase under circular projection, (this follows directly from the definition)

$$m_1\left\{|z| : z \in \phi\left(\mathcal{E}_\varepsilon \cap \phi^{[-1]}(B_{r'})\right)\right\} \leq m_1\left(\phi\left(\mathcal{E}_\varepsilon \cap \phi^{[-1]}(B_{r'})\right)\right) \leq \frac{1}{2}(r'-r).$$

It follows that we can choose $s \in (r, r')$ such that $\phi\left(\mathcal{E}_\varepsilon \cap \phi^{[-1]}(B_{r'})\right)$ does not intersect the circle $\{z : |z| = s\}$. Let

$$\Gamma^{(n)} = \phi^{[-1]}(\{z : |z| = s\}),$$

the image of the circle under the inverse conformal map. Then $\Gamma^{(n)}$ is a Jordan curve between Γ_1 and Γ_2 and does not intersect $\mathcal{E}_\varepsilon$. By (b),

$$\frac{1}{|Q(z)|} \leq \frac{2^n}{\varepsilon^k} \leq \left(\frac{2}{\varepsilon}\right)^n.$$

Here ε depends only on Γ_1 and Γ_2 and not on the particular Q nor its degree n. So we have the result with $c = \frac{2}{\varepsilon}$. ■

We shall need one more form of Lemma 3.2(a), focusing on zeros of Q in a compact set T :

Lemma 3.3
Let $\rho \geq \frac{1}{2} > \varepsilon > 0$. Let $\frac{1}{2} > \eta > 0$. Let Q be a polynomial of degree $\leq n$, admitting the representation (3.1). Let T be a compact subset of B_ρ with boundary ∂T. Assume that Q has $\leq M$ zeros in T. Then there is a set $\mathcal{E}$ with $m_2(\mathcal{E}) \leq \pi\varepsilon^2$ such that for $z \in T \backslash \mathcal{E}$ with dist$(z, \partial T) \geq \eta$, we have

$$\frac{1}{|Q(z)|} \leq \frac{1}{\eta^n}\frac{1}{\varepsilon^M}. \tag{3.2}$$

Proof
We further split (3.1) as

$$Q(z) = \prod_{v\in T}(z-v) \prod_{\substack{|v|\leq 2\rho,\\ v\notin T}} (z-v) \prod_{|v|>2\rho}\left(1-\frac{z}{v}\right).$$

Assume there are $m \leq M$ terms in the first product, k in the second and ℓ in the third. As above for $z \in T \subset B_\rho$, the third term has absolute value bounded below by $\left(\frac{1}{2}\right)^\ell$. Next, for $z \in T$, with $\operatorname{dist}(z, \partial T) \geq \eta$

$$\left| \prod_{\substack{|v| \leq 2\rho, \\ v \notin T}} (z - v) \right| \geq \eta^k.$$

Finally, as above, there is a set $\mathcal{E}$ with $m_2(\mathcal{E}) \leq \pi\varepsilon^2$ such that for $z \in \mathbb{C} \backslash \mathcal{E}$,

$$\left| \prod_{v \in T} (z - v) \right| \geq \varepsilon^m.$$

Combining the 3 lower bounds and using that $m \leq M$, $k + \ell \leq n$, while $\varepsilon, \eta \leq \frac{1}{2}$, gives the result. ■

We shall make substantial use of a result of Gonchar and Grigorjan. If f is meromorphic inside a simply connected domain D, then we can form the sum R_f of the principal parts of f in D, so that it has the form

$$R_f(z) = \sum_j \sum_{k \geq 1} c_{jk} (z - b_j)^{-k}$$

where $\{b_j\}$ are the poles of f in D. The analytic part of f in D is then

$$\mathcal{A}f = f - R_f.$$

The following result is a weaker form of the remarkable results of Gonchar and Grigorjan, see [9], [10]:

Lemma 3.4

Let D be a bounded simply connected domain with boundary Γ. Let f be meromorphic in D with poles of total multiplicity at most n, and analytic on Γ. Then

$$\|\mathcal{A}f\|_{L_\infty(\Gamma)} \leq 7n^2 \|f\|_{L_\infty(\Gamma)}.$$

Proof

This follows directly from Theorem 1 in [9, p. 571]. ■

4. Comparing Δ_n and E_{nn} on different sets

We begin by comparing errors of Newton-Padé approximation and best rational approximation. Throughout, as in Section 1, the $\{a_j\}$ are the interpolation points, while R_n is the (n, n) Newton-Padé approximant.

Lemma 4.1
Let Γ_1, Γ_2 be rectifiable Jordan curves such that Γ_1 lies in the open set enclosed by Γ_2. Let K be a compact set containing all interpolation points $\{a_j\}$ and assume that K lies in the interior of the set enclosed by Γ_1. Let f be analytic in an open set U that contains Γ_2.
(a) There exists $C_1 > 0$ independent of n, f and for large enough n, a Jordan curve $\Gamma^{(n)}$ between Γ_1 and Γ_2 (depending on n, f) such that

$$\|f - R_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq C_1^n E_{nn}\left(f; \hat{\Gamma}_2\right).$$

(b) If inside Γ_1, the total multiplicity of poles of R_n is at least N_n, then

$$E_{n-N_n, n-N_n}\left(f; \hat{\Gamma}_1\right) \leq C_2^n E_{nn}\left(f; \hat{\Gamma}_2\right).$$

Here C_2 is independent of n, f.
Proof
(a) Let $R_n^* = P_n^*/Q_n^*$ be a best approximant to f in the L_∞ norm on the compact set $\hat{\Gamma}_2$ enclosed by Γ_2. Write $R_n = P_n/Q_n$. Then we have for z inside Γ_2,

$$\left(\frac{Q_n^*\left(fQ_n - P_n\right)}{\omega_{2n+1}}\right)(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{Q_n(t)\left(fQ_n^* - P_n^*\right)(t)}{\omega_{2n+1}(t)(t-z)}dt.$$

Recall this follows from the fact that $\left(Q_n^* P_n - P_n^* Q_n\right)(t) / \left((t-z)\,\omega_{2n+1}(t)\right) = O\left(t^{-2}\right)$ at ∞ and is analytic outside Γ_2. From this we obtain for z inside Γ_2,
(4.1)

$$|f - R_n|(z) \leq \frac{length\left(\Gamma_2\right)}{2\pi dist\left(\Gamma_2, z\right)} E_{nn}\left(f; \hat{\Gamma}_2\right) \max_{t\in\Gamma_2} \frac{|Q_n Q_n^*|(t)}{|Q_n Q_n^*|(z)} \frac{\max_{t\in\Gamma_2}|\omega_{2n+1}(t)|}{\min_{t\in\Gamma_2}|\omega_{2n+1}(t)|}.$$

Here as all zeros of ω_{2n+1} lie inside a compact set inside Γ_1, so for some constant $C_3 > 0$ depending on Γ_2, Γ_1, but not on n,

$$\frac{\max_{t\in\Gamma_2}|\omega_{2n+1}(t)|}{\min_{t\in\Gamma_2}|\omega_{2n+1}(t)|} \leq C_3^n. \tag{4.2}$$

We assume that QQ_n^* is normalized as in Lemma 3.2 with ρ so large that $\hat{\Gamma}_2 \subset B_\rho$. Then

$$\max_{t\in\Gamma_2} |Q_n Q_n^*|(t) \leq (3\rho)^{2n}.$$

Next, by Lemma 3.2(b), there is a constant c depending only on Γ_1 and Γ_2 and for large enough n, a simple closed contour $\Gamma^{(n)}$ between Γ_1 and Γ_2 such that

$$\frac{1}{|Q_n Q_n^*|(z)} \leq c^{2n},\ z \in \Gamma^{(n)}.$$

Substituting this and (4.2) into (4.1) gives the desired estimate.
(b) Let $\mathcal{A}R_n$ denote the analytic part of R_n inside $\Gamma^{(n)}$. As R_n has $\geq N_n$ poles inside Γ_1 and hence inside $\Gamma^{(n)}$, so $\mathcal{A}R_n$ is a rational function of type $(n-N_n, n-N_n)$. Also inside $\Gamma^{(n)}$, $\mathcal{A}(f-R_n) = f - \mathcal{A}R_n$, so

$$\begin{aligned} E_{n-N_n,n-N_n}\left(f;\hat{\Gamma}_1\right) &\leq E_{n-N_n,n-N_n}\left(f;\hat{\Gamma}^{(n)}\right) \\ &\leq \|f-\mathcal{A}R_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \\ &= \|\mathcal{A}(f-R_n)\|_{L_\infty\left(\Gamma^{(n)}\right)} \\ &\leq 7n^2\|f-R_n\|_{L_\infty\left(\Gamma^{(n)}\right)}, \end{aligned}$$

by the Gonchar-Grigorjan Lemma (Lemma 3.4). This and the estimate in (a) give the result. ■

Next, we compare errors of best rational approximation on different sets:

Lemma 4.2
Let f be analytic in an open connected set U. Assume that for some compact set $L \subset U$ of positive capacity,

$$\lim_{n\to\infty} E_{nn}(f;L)^{1/(n\phi(n))} = 0, \tag{4.3}$$

where $\phi:[1,\infty)\to[1,\infty)$ is a non-decreasing function. Let T and S be compact subsets of U of positive logarithmic capacity. Let $\eta>0$. Then for large enough n,

$$E_{nn}(f;S) \leq E_{nn}(f;T)^{1-\frac{\eta}{\phi(n)}}.$$

Proof
Choose $R_n^* = P_n^*/Q_n^*$ of type (n,n) such that

$$\|f-R_n^*\|_{L_\infty(T)} = E_{nn}(f;T).$$

Choose a Jordan curve Γ_2 in U that encloses both S and T. We also choose a second Jordan curve Γ_1 inside Γ_2 that encloses both S and T. We initially assume that instead of (4.3),

$$\lim_{n\to\infty} E_{nn}\left(f;\hat{\Gamma}_2\right)^{1/(n\phi(n))} = 0. \tag{4.4}$$

For large enough n, choose the smallest integer $k=k(n)\geq n$ such that

$$E_{kk}\left(f;\hat{\Gamma}_2\right) \leq E_{nn}(f;T). \tag{4.5}$$

Then either $k=n$ or

$$E_{k-1,k-1}\left(f;\hat{\Gamma}_2\right) > E_{nn}(f;T). \tag{4.6}$$

Choose $R_k^{\#} = P_k^{\#}/Q_k^{\#}$ such that

$$\left\| f - R_k^{\#} \right\|_{L_\infty(\hat{\Gamma}_2)} = E_{kk}\left(f; \hat{\Gamma}_2 \right). \tag{4.7}$$

Then from (4.5),

$$\left\| R_k^{\#} - R_n^* \right\|_{L_\infty(T)} \leq 2E_{nn}(f;T).$$

We may normalize the numerators and denominators in R_n^* and $R_k^{\#}$ as we please. It is convenient to normalize them as in Lemma 3.2 so that

$$\begin{aligned} \|Q_n^*\|_{L_\infty(B_\rho)} &\leq (3\rho)^n ; \\ \left\| Q_k^{\#} \right\|_{L_\infty(B_\rho)} &\leq (3\rho)^k . \end{aligned}$$

Here ρ is chosen so large that B_ρ contains Γ_2. Then

$$\left\| P_k^{\#} Q_n^* - P_n^* Q_k^{\#} \right\|_{L_\infty(T)} \leq 2(3\rho)^{k+n} E_{nn}(f;T).$$

By Lemma 3.1(a), there exists a constant $A > 0$ depending only on T and ρ such that

$$\left\| P_k^{\#} Q_n^* - P_n^* Q_k^{\#} \right\|_{L_\infty(B_\rho)} \leq 2(3A\rho)^{k+n} E_{nn}(f;T).$$

Also by Lemma 3.2(b), we can choose a constant c depending only on Γ_1,Γ_2, and not on n, as well as a contour $\Gamma^{(n)}$ between Γ_1 and Γ_2 that does depend on n, such that on $\Gamma^{(n)}$,

$$\frac{1}{\left| Q_n^* Q_k^{\#} \right|(z)} \leq c^{n+k}. \tag{4.8}$$

Then

$$\left\| R_k^{\#} - R_n^* \right\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq 2(3cA\rho)^{k+n} E_{nn}(f;T).$$

Hence using (4.7) and (4.5),

$$\|f - R_n^*\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq E_{nn}(f;T)\left\{ 1 + 2(3cA\rho)^{k+n} \right\}.$$

From the Gonchar-Grigorjan Lemma, if $\mathcal{A}(f - R_n^*) = f - \mathcal{A}R_n^*$ is the analytic part of $f - R_n^*$ inside $\Gamma^{(n)}$, we have

$$\|f - \mathcal{A}R_n^*\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq \left(7n^2\right) E_{nn}(f;T)\left\{ 1 + 2(3cA\rho)^{k+n} \right\}.$$

As $\mathcal{A}R_n^*$ is also a rational function of type (n,n), while $\Gamma^{(n)}$ encloses Γ_1, we obtain

$$E_{nn}\left(f; \hat{\Gamma}_1 \right) \leq \left(7n^2\right) E_{nn}(f;T)\left\{ 1 + 2(3cA\rho)^{k+n} \right\}. \tag{4.9}$$

Now if $\eta > 0$, and as $T \subset \hat{\Gamma}_2$, (4.4) shows that

$$\lim_{n\to\infty} \left(E_{nn} (f; T)^{\eta/\phi(n)} \right)^{1/n} = 0,$$

so if $k = n$, then for n large enough,

$$\left(7n^2\right) \left\{ 1 + 2 (3cA\rho)^{k+n} \right\} \leq E_{nn} (f; T)^{-\eta/\phi(n)}.$$

If on the other hand $k > n$, then given any constant C, for large enough k, from (4.3) and (4.6),

$$\begin{aligned} C^k &\leq E_{k-1,k-1} \left(f; \hat{\Gamma}_2 \right)^{-\eta/\phi(k-1)} \\ &\leq E_{nn} (f; T)^{-\eta/\phi(k-1)} \leq E_{nn} (f; T)^{-\eta/\phi(n)}. \end{aligned}$$

It follows from (4.9) that

$$E_{nn} \left(f; \hat{\Gamma}_1 \right) \leq E_{nn} (f; T)^{1-\frac{\eta}{\phi(n)}}.$$

Since Γ_1 encloses S, so

$$E_{nn} (f; S) \leq E_{nn} (f; T)^{1-\frac{\eta}{\phi(n)}}.$$

So we have the result. We still need to deal with the assumption (4.4), which is different from our original (4.3) involving L. To do this, we use a basic result about the Gonchar-Walsh class. Now (4.3) implies that for one compact subset L of positive capacity,

$$\lim_{n\to\infty} E_{nn}^{\frac{1}{n}} (f; L) = 0.$$

It then follows that for every compact subset L of U, we have this last relation. See for example [8, p. 153, Theorem 1]. It also follows from the method of proof above. Then choosing $\hat{\phi}(x) = 1$, $x \in [1, \infty)$, we have

$$\lim_{n\to\infty} E_{nn}^{1/\left(n\hat{\phi}(n)\right)} (f; \Gamma_1) = 0.$$

Our proof above applied to $S = \hat{\Gamma}_2$; $T = L; \eta = \frac{1}{2}$ and $\hat{\phi}$ rather than ϕ shows that for large enough n,

$$E_{nn} \left(f; \hat{\Gamma}_2 \right) \leq E_{nn} (f; L)^{1-\frac{1/2}{\hat{\phi}(n)}} = E_{nn} (f; L)^{1/2}.$$

Then (4.4) follows from (4.3), so our modified hypothesis is satisfied. ■

Now we compare the linearized form of the error in Newton-Padé approximation on different sets:

Lemma 4.3
Let K be a compact set containing all interpolation points $\{a_j\}$. Let f be analytic in an open set U that contains K and assume (4.3) holds. Let $\eta > 0$. Let S and T be compact subsets of U such that S has non-empty interior while T has positive capacity. Let $R_n = P_n/Q_n$ denote the Newton-Padé approximant R_n to f at the zeros of ω_{2n+1} and $\Delta_n = fQ_n - P_n$, where for some $\rho > 0$, Q_n is normalized as in (3.1). Let $\eta > 0$. Then
(a) For n large enough,

$$E_{nn}(f;T)^{1+\frac{\eta}{\phi(n)}} \leq \|\Delta_n\|_{L_\infty(S)} \leq E_{nn}(f;T)^{1-\frac{\eta}{\phi(n)}}. \tag{4.10}$$

(b) Assume now T has non-empty interior. For n large enough,

$$\|\Delta_n\|^{1+\frac{\eta}{\phi(n)}}_{L_\infty(T)} \leq \|\Delta_n\|_{L_\infty(S)} \leq \|\Delta_n\|^{1-\frac{\eta}{\phi(n)}}_{L_\infty(T)}. \tag{4.11}$$

Proof
(a) Let Γ_1, Γ_2 be as in Lemma 4.1. We can assume that Γ_1 encloses K, S, T. Then for some contour $\Gamma^{(n)}$ between Γ_1, Γ_2, we have from Lemma 4.1(a),

$$\|\Delta_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq \|Q_n\|_{L_\infty\left(\Gamma^{(n)}\right)} C_1^n E_{nn}\left(f;\hat{\Gamma}_2\right) \leq C_3^n E_{nn}\left(f;\hat{\Gamma}_2\right)$$

in view of our normalization of Q_n. Then using the fact that for large enough n,

$$C_3^n \leq E_{nn}\left(f;\hat{\Gamma}_2\right)^{-\frac{\eta}{2\phi(n)}} \tag{4.12}$$

we obtain from the maximum modulus principle

$$\|\Delta_n\|_{L_\infty(S)} \leq E_{nn}\left(f;\hat{\Gamma}_2\right)^{1-\frac{\eta}{2\phi(n)}}.$$

In view of Lemma 4.2, we then obtain for large enough n,

$$\|\Delta_n\|_{L_\infty(S)} \leq E_{nn}(f;T)^{\left(1-\frac{\eta}{2\phi(n)}\right)^2} \leq E_{nn}(f;T)^{1-\frac{\eta}{\phi(n)}}.$$

So we have the upper bound in (4.10).

For the lower bound, since S has non-empty interior, we may simply assume that S is a ball of radius $r > 0$. By Lemma 3.2(b), we can choose a circle $\Gamma^{(n)}$, concentric with the ball S, and of radius between $r/2$ and r such that

$$\frac{1}{|Q_n(z)|} \leq c^n \text{ on } \Gamma^{(n)}$$

with c depending only on r. Then

$$\|f - R_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq \|\Delta_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \left\|\frac{1}{Q_n}\right\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq \|\Delta_n\|_{L_\infty(S)} c^n.$$

By the Gonchar-Grigorjan Lemma, irrespective of if there are poles of R_n inside $\Gamma^{(n)}$ or not,

$$\begin{aligned} E_{nn}\left(f; \hat{\Gamma}^{(n)}\right) &\leq \|f - \mathcal{A}R_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \\ &\leq 7n^2 \|f - R_n\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq 7n^2 c^n \|\Delta_n\|_{L_\infty(S)}. \end{aligned}$$

If S_1 is the ball concentric with S but of radius $r/2$, (so that S_1 lies in $\Gamma^{(n)}$) also then

$$\begin{aligned} E_{nn}(f; S_1) &\leq 7n^2 c^n \|\Delta_n\|_{L_\infty(S)} \\ &\leq E_{nn}(f; S_1)^{-\frac{\eta}{\phi(n)}} \|\Delta_n\|_{L_\infty(S)} \end{aligned}$$

for n large enough. Here we are using that S_1 does not depend on n. So we have the lower bound in (4.10) when T is replaced by S_1. We can replace it by T using Lemma 4.2 and modifying the value of η.
(b) This follows from (a) and Lemma 4.2. ■

5. Proof of Theorem 1.2

Throughout this section, we assume the hypotheses of Theorem 1.2 - in particular that f is analytic in U. Define

$$\phi(x) = \sqrt{1 + \log x}, x \geq 1.$$

For a given n, we let

$$M_n = \left[\frac{n}{\phi(n)}\right],$$

where $[x]$ denotes the largest integer $\leq x$. We also let L be as in the hypothesis of Theorem 1.2, and

$$\varepsilon_n = E_{nn}(f; L)^{1/(n\phi(n))}.$$

Choose an infinite sequence of integers $\mathcal{S}$ such that for $n \in \mathcal{S}$,

$$\varepsilon_n \leq \varepsilon_k,\ 1 \leq k \leq n. \tag{5.1}$$

Of course as $\{\varepsilon_n\}$ has limit 0, such an infinite sequence exists. We first make a simple observation:

Lemma 5.1

(a) Let $\delta \in (0,1)$. Let $T \subset U$ have positive capacity. For all large enough $n \in \mathcal{S}$, we have

$$E_{n-M_n,n-M_n}(f;T) > E_{nn}(f;T)^{1-\frac{\delta}{\phi(n)}}. \tag{5.2}$$

(b) Let Γ_1 be a Jordan curve inside U that encloses all the interpolation points $\{a_j\}$. For large enough $n \in \mathcal{S}$, R_n has $< M_n$ poles in Γ_1, counting multiplicity.

Proof

(a) From (5.1),

$$E_{nn}(f;L) \leq E_{n-M_n,n-M_n}(f;L)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}}.$$

Let η be so small that $\delta + 2\eta < 1$. Using Lemma 4.2 twice, we obtain for a given $\eta > 0$ and large enough n,

$$\begin{aligned} E_{nn}(f;T) &\leq E_{nn}(f;L)^{1-\frac{\eta}{\phi(n)}} \\ &\leq E_{n-M_n,n-M_n}(f;T)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\left(1-\frac{\eta}{\phi(n)}\right)\left(1-\frac{\eta}{\phi(n-M_n)}\right)}. \end{aligned}$$

If (5.2) is false, we then obtain

$$E_{nn}(f;T) \leq E_{nn}(f;T)^{\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\left(1-\frac{\eta}{\phi(n)}\right)\left(1-\frac{\eta}{\phi(n-M_n)}\right)\left(1-\frac{\delta}{\phi(n)}\right)}. \tag{5.3}$$

Here the exponent is, using that ϕ is increasing, and that $\frac{M_n}{n} = \frac{1}{\phi(n)} + O\left(\frac{1}{n}\right)$, while $\phi(n-M_n) = \phi(n)(1+o(1))$,

$$\begin{aligned} &\frac{n\phi(n)}{(n-M_n)\phi(n-M_n)}\left(1-\frac{\eta}{\phi(n)}\right)\left(1-\frac{\eta}{\phi(n-M_n)}\right)\left(1-\frac{\delta}{\phi(n)}\right) \\ \geq\ & \frac{1}{1-\left(\frac{1}{\phi(n)}+O\left(\frac{1}{n}\right)\right)}\left(1-\frac{\delta+2\eta}{\phi(n)}+o\left(\frac{1}{\phi(n)}\right)\right) \\ =\ & 1+\frac{1-\delta-2\eta}{\phi(n)}+o\left(\frac{1}{\phi(n)}\right). \end{aligned}$$

As $1 > \delta + 2\eta$, we obtain that the exponent in the right-hand side of (5.3) exceeds 1 for large enough n, leading to a contradiction. So we must have (5.2).

(b) Suppose R_n has at least M_n poles inside Γ_1. Choose a Jordan curve Γ_2 inside U enclosing Γ_1. By Lemma 4.1(b),

$$E_{n-M_n,n-M_n}\left(f;\hat{\Gamma}_1\right) \leq C_2^n E_{nn}\left(f;\hat{\Gamma}_2\right).$$

Then for n large enough, Lemma 4.2 shows that

$$E_{n-M_n,n-M_n}\left(f;\hat{\Gamma}_1\right) \leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{2\phi(n)}},$$

contradicting (a). ■

Remark
This shows that R_n has $< \frac{n}{\phi(n)}$ poles in any compact set when $n \in \mathcal{S}$, which is of independent interest.

Proof of Theorem 1.2
We prove that for every compact subset T of U,

$$\liminf_{n\to\infty} \left(\sup_{z\in T} \left(\min \left\{ \left| f - R_n \right|(z), \left| f - R_{n-1} \right|(z) \right\} \right) \right)^{1/n} = 0. \tag{5.4}$$

Once we have this, we can choose a sequence $\{T_m\}$ of compact sets increasing to U and for each m, an integer $n_m > n_{m-1}$ such that

$$\sup_{z\in T_m} \left(\min \left\{ \left| f - R_{n_m} \right|(z), \left| f - R_{n_m-1} \right|(z) \right\} \right)^{1/n} < \frac{1}{m}.$$

Then the result follows.

Now suppose that (5.4) fails. Then there exists a compact subset T of U and $A > 0$ such that

$$\liminf_{n\to\infty} \left(\sup_{z\in T} \left(\min \left\{ \left| f - R_n \right|(z), \left| f - R_{n-1} \right|(z) \right\} \right) \right)^{1/n} > A > 0.$$

It is not initially ruled out that the left-hand side is ∞. We may assume (by increasing the size of T) that T is the closure of a simply connected set and moreover T contains all the interpolation points at a positive distance to the boundary of T. We may assume that the boundary of T is a Jordan curve Γ_1 and choose another Jordan curve Γ_2 inside U enclosing Γ_1. We choose a small $\eta > 0$ and let T_1 denote the subset of T of points whose distance to the boundary Γ_1 of T is at least η. For large enough n, there exists $\zeta_n \in T$ such that for $\ell = n, n+1$,

$$\left| f - R_\ell \right|(\zeta_n) > A^n. \tag{5.5}$$

By lower semi-continuity, this also holds in a neighborhood of ζ_n, so we may assume that ζ_n is neither an interpolation point nor a pole of R_ℓ. Assume that $R_\ell = P_\ell / Q_\ell$ where Q_ℓ is normalized as in Lemma 3.2 - we choose ρ there so large that B_ρ contains Γ_2. We then have for $\ell = n, n+1$,

$$\left| Q_\ell(\zeta_n) \right| < \frac{1}{A^n} \left| \Delta_\ell(\zeta_n) \right|.$$

Then from (2.4),

$$(5.6)\qquad \begin{aligned} |A_n| &= |\omega_{2n+1}(\zeta_n)|^{-1} |\Delta_n Q_{n+1} - \Delta_{n+1} Q_n| (\zeta_n) \\ &\le \frac{2}{A^n} \frac{|\Delta_n \Delta_{n+1}|(\zeta_n)}{|\omega_{2n+1}(\zeta_n)|} \le \frac{2}{A^n} \left\| \frac{\Delta_n \Delta_{n+1}}{\omega_{2n+1}} \right\|_{L_\infty(\Gamma_1)}, \end{aligned}$$

by the maximum-modulus principle (recall that Δ_n/ω_{2n+1} is analytic inside Γ_1). Recall too that $\phi(n) = \sqrt{1+\log n}$ and $M_n = \left[\frac{n}{\phi(n)}\right]$. Let $\mathcal{S}$ be as in (5.1) and $n \in \mathcal{S}$. Let

$$m = m(n) = n - M_n.$$

We now consider two subcases.

Case I: For infinitely many $n \in \mathcal{S}$, R_k has $< M_n$ poles in T for $n \ge k \ge n - M_n$

As in Section 2,

$$(5.7)\qquad P_n Q_m - P_m Q_n = Q_n Q_m \sum_{j=m}^{n-1} \frac{A_j \omega_{2j+1}}{Q_j Q_{j+1}}.$$

Now by Lemma 3.3, with $\varepsilon = n^{-1/2}$

$$(5.8)\qquad \frac{1}{|Q_j(z)|} \le \frac{1}{\eta^j n^{-M_n/2}}$$

in $T_1 \backslash \mathcal{E}_j$ where $m_2(\mathcal{E}_j) \le \pi n^{-1}$. Recall that T_1 is the subset of T of points at least a distance η away from ∂T. Let

$$\mathcal{E}_{m,n} = \bigcup_{j=m}^{n} \mathcal{E}_j,$$

so that

$$m_2(\mathcal{E}_{m,n}) \le (n-m+1)\pi n^{-1} \le \pi \left(\frac{n}{\phi(n)} + 1 \right) \pi n^{-1} = o(1).$$

It follows from (5.6-5.8) and Lemma 3.2(a) that in $T_1 \backslash \mathcal{E}_{m,n}$,

$$|P_m Q_n - P_n Q_m| \le \frac{2}{A^n} (3\rho)^{n+m} \sum_{j=m}^{n-1} \eta^{-n} n^{M_n} \frac{\|\omega_{2j+1}\|_{L_\infty(\Gamma_1)}}{\min_{t\in\Gamma_1} |\omega_{2j+1}(t)|} \|\Delta_j \Delta_{j+1}\|_{L_\infty(\Gamma_1)}.$$

Here as all zeros of ω_{2j+1} lie in some compact subset of T independent of j, so for some C_1 independent of j,

$$\frac{\|\omega_{2j+1}\|_{L_\infty(\Gamma_1)}}{\min_{t\in\Gamma_1} |\omega_{2j+1}(t)|} \le C_1^j.$$

As $m_2(\mathcal{E}_{m,n}) = o(1)$, so by Lemma 3.1(b) and this last inequality,

$$\|P_m Q_n - P_n Q_m\|_{L_\infty(\Gamma_2)} \leq C_2^n n^{M_n} \sum_{j=m}^{n-1} \|\Delta_j \Delta_{j+1}\|_{L_\infty(\Gamma_1)}$$

where C_2 depends only on T, η and not on m, n. The same estimate then holds for $\Delta_m Q_n - \Delta_n Q_m$ and hence if $\Gamma^{(n)}$ is any simple closed curve inside Γ_2,

$$\min_{t\in\Gamma^{(n)}} |Q_n(t)| \, \|\Delta_m\|_{L_\infty\left(\Gamma^{(n)}\right)} \leq \|\Delta_n\|_{L_\infty(\Gamma_2)} (3\rho)^m + C_2^n n^{M_n} \sum_{j=m}^{n-1} \|\Delta_j \Delta_{j+1}\|_{L_\infty(\Gamma_1)}.$$

Here by Lemma 3.2(b), we may choose a curve $\Gamma^{(n)}$ lying between Γ_1 and Γ_2 and a constant c independent of n such that $\min_{t\in\Gamma^{(n)}} |Q_n(t)| \geq c^n$. Thus the maximum modulus principle gives

$$c^n \|\Delta_m\|_{L_\infty(\Gamma_1)} \leq \|\Delta_n\|_{L_\infty(\Gamma_2)} (3\rho)^m + C_2^n n^{M_n} \sum_{j=m}^{n-1} \|\Delta_j \Delta_{j+1}\|_{L_\infty(\Gamma_1)}.$$

(5.9)

Next for $m \leq j < n$, Lemma 4.3(a) and mononotonicity of errors give

$$\begin{aligned} \|\Delta_j\|_{L_\infty(T)} &\leq E_{jj}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{8\phi(j)}} \\ &\leq E_{mm}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{8\phi(j)}} \leq E_{mm}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{8\phi(m)}}. \end{aligned}$$

Then this and an application of Lemma 4.3 to (5.9) gives

$$E_{mm}\left(f;\hat{\Gamma}_1\right)^{1+\frac{1}{4\phi(m)}} \leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}} + n^{M_n+1} E_{mm}\left(f;\hat{\Gamma}_1\right)^{2-\frac{1}{4\phi(n)}}$$

$$\begin{aligned} \Rightarrow\quad 1 &\leq \frac{E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}}}{E_{mm}\left(f;\hat{\Gamma}_1\right)^{1+\frac{1}{4\phi(m)}}} + n^{M_n+1} E_{mm}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{2\phi(m)}} \\ &=: \tau_1 + \tau_2. \end{aligned}$$

(5.10)

Recall that $m = n - M_n$. By Lemma 5.1, with $\delta = \frac{3}{4}$,

$$\tau_1 \leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{\left(1-\frac{1}{4\phi(n)}\right)-\left(1+\frac{1}{4\phi(m)}\right)\left(1-\frac{3}{4\phi(n)}\right)}.$$

Here the exponent is

$$\begin{aligned}&\left(1-\frac{1}{4\phi(n)}\right)-\left(1+\frac{1}{4\phi(m)}\right)\left(1-\frac{3}{4\phi(n)}\right)\\=\ &\frac{1}{4\phi(n)}(1+o(1))\end{aligned}$$

as $\phi(m)/\phi(n)=1+o(1)$, so

$$\tau_1\leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{\frac{1}{4\phi(n)}(1+o(1))}\rightarrow 0 \text{ as } n\rightarrow\infty. \tag{5.11}$$

Next, our hypothesis (1.1) shows that for some sequence $\{\xi_m\}$ with limit ∞,

$$\begin{aligned}&\tau_2\leq\exp\left((M_n+1)\log n-\xi_m m\phi(m)\left(1-\frac{1}{2\phi(m)}\right)\right)\\=\ &\exp\left(n\sqrt{\log n}\left[1+o(1)-\xi_m(1+o(1))\right]\right)\\\rightarrow\ &0 \text{ as } n\rightarrow\infty.\end{aligned}$$

This and (5.11) show that (5.10) is impossible for large enough n, and we have a contradiction. Thus if Case I holds for infinitely many $n\in\mathcal{S}$, (5.4) is true.

Case II: For large enough $n\in\mathcal{S}$, R_k has $\geq M_n$ poles inside T for at least one k with $n\geq k\geq n-M_n$

In this case, we choose ℓ to be the largest such k, so that R_j has $<M_n$ poles for $n\geq j>\ell$ but R_ℓ has $\geq M_n$ poles in T. Recall from Lemma 5.1(b) that R_n cannot have that many poles so necessarily $\ell<n$. We then proceed much as above, but using

$$P_nQ_\ell-P_\ell Q_n=Q_nQ_m\sum_{j=\ell}^{n-1}\frac{A_j\omega_{2j+1}}{Q_jQ_{j+1}}. \tag{5.12}$$

For $j>\ell$, we let $\mathcal{E}_j$ be as above so that (5.8) holds. For $j=\ell$, we instead let $\mathcal{E}_\ell$ be the set on which

$$\frac{1}{|Q_\ell(z)|}\leq\frac{2^\ell}{(\log n)^\ell}$$

so that $m_2(\mathcal{E}_\ell)\leq\pi\left(\frac{1}{\log n}\right)^2$. We now define

$$\mathcal{E}_{\ell,n}=\bigcup_{j=\ell}^{n}\mathcal{E}_j,$$

so that

$$m_2\left(\mathcal{E}_{\ell,n}\right) \leq \pi\left(\frac{1}{\log n}\right)^2 + o\left(1\right).$$

This is still much smaller than $m_2(T)$, so proceeding as above, we obtain for some contour $\Gamma^{(n)}$ between Γ_1 and Γ_2,

$$\begin{aligned} & \min_{t\in\Gamma^{(n)}} \left|Q_n\right|(t)\left\|\Delta_\ell\right\|_{L_\infty(\Gamma_n)} \\ \leq & \ \left\|\Delta_n\right\|_{L_\infty(\Gamma_2)}(3\rho)^m + C_2^n\left\{\begin{array}{c} 2^n(\log n)^n n^{M_n/2}\left\|\Delta_\ell\Delta_{\ell+1}\right\|_{L_\infty(\Gamma_1)} \\ +n^{M_n}\sum_{j=\ell+1}^{n-1}\left\|\Delta_j\Delta_{j+1}\right\|_{L_\infty(\Gamma_1)} \end{array}\right\} \end{aligned}$$

and hence as above,

$$E_{\ell\ell}\left(f;\hat{\Gamma}_1\right)^{1+\frac{1}{4\phi(\ell)}} \leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}} + n^{M_n+1}E_{\ell\ell}\left(f;\hat{\Gamma}_1\right)^{2-\frac{1}{4\phi(\ell)}}$$

$$\begin{aligned} \Rightarrow \quad & 1 \leq \frac{E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}}}{E_{\ell\ell}\left(f;\hat{\Gamma}_1\right)^{1+\frac{1}{4\phi(\ell)}}} + n^{M_n+1}E_{\ell\ell}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{2\phi(\ell)}} \\ = \quad & :\tau_1+\tau_2. \end{aligned}$$

Exactly as before,

$$\tau_2 \to 0 \text{ as } n \to \infty.$$

To deal with the first term, we now use our assumption in Case II that R_ℓ has $\geq M_n$ poles. Using Lemmas 4.1(b) and 4.2, we obtain

$$E_{\ell-M_n,\ell-M_n}\left(f;\hat{\Gamma}_1\right) \leq E_{\ell\ell}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(\ell)}},$$

so

$$\begin{aligned} & \tau_1 \leq E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}} / E_{\ell-M_n,\ell-M_n}\left(f;\hat{\Gamma}_1\right)^{\left(1+\frac{1}{4\phi(\ell)}\right)\left(1-\frac{1}{4\phi(\ell)}\right)^{-1}} \\ \leq & \ E_{nn}\left(f;\hat{\Gamma}_1\right)^{1-\frac{1}{4\phi(n)}} / E_{n-M_n,n-M_n}^{1+\frac{1+o(1)}{2\phi(\ell)}}\left(f;\hat{\Gamma}_1\right) \to 0 \text{ as } n\to\infty, \end{aligned}$$

by Lemma 5.1(a). Again we have a contradiction. Theorem 1.2 is proven. ■

6. Proof of Theorem 1.1

It follows from Theorem 1.2 that given a ball B_ρ and $\varepsilon > 0$, we can find an integer n such that

$$\sup_{z\in B_\rho}\left(\inf\left\{\left|f-[n/n](z)\right|, \left|f-[n-1/n-1]\right|(z)\right\}\right)^{1/n} \leq \varepsilon.$$

We can then choose a growing sequence of $\rho' s$ and $n' s$ and a decreasing sequence of $\varepsilon' s$ to deduce the result. ■

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