DISCRETE CIRCULAR BETA ENSEMBLES

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Abstract. Let μ be a measure with support on the unit circle and $n \geq 1, \beta > 0$. The associated circular β ensemble involves a probability distribution of the form

$$\mathcal{P}_\beta^{(n)}(\mu; t_1, t_2, ..., t_n) = C\,|V(t_1, t_2, ..., t_n)|^\beta\, d\mu(t_1) ... d\mu(t_n),$$

where C is a normalization constant, and

$$V(t_1, t_2, ..., t_n) = \prod_{1 \leq i < j \leq n} (t_j - t_i).$$

We explicitly evaluate the $m-$point correlation functions when μ is replaced by a discrete measure on the unit circle, generated by paraorthogonal orthogonal polynomials associated with μ, and use this to investigate universality limits for sequences of such measures. We also consider ratios of products of random characteristic polynomials.

1. Identities

Let μ be a finite positive Borel measure on the unit circle $\Gamma = \{e^{i\theta} : \theta \in [-\pi, \pi]\}$, or equivalently on $[-\pi, \pi]$, with infinitely many points in its support. Let $\beta > 0$ and $n \geq 2$. The β-ensemble with temperature $1/\beta$, associated with the measure μ, involves a probability distribution on Γ^n of the form

$$\begin{aligned} &\mathcal{P}_\beta^{(n)}(\mu; t_1, t_2, ..., t_n) \\ = &\frac{1}{Z_n} |V(t_1, t_2, ..., t_n)|^\beta\, d\mu(t_1) ... d\mu(t_n), \end{aligned} \tag{1.1}$$

where

$$V(t_1, t_2, ...t_n) = \prod_{1 \leq i < j \leq n} (t_j - t_i) = \det\left[t_i^{j-1}\right]_{1 \leq i,j \leq n} \tag{1.2}$$

and

$$Z_n = \int ... \int |V(t_1, t_2, ...t_n)|^\beta\, d\mu(t_1) ... d\mu(t_n). \tag{1.3}$$

These ensembles arise in analysing random unitary ($\beta = 2$), orthogonal ($\beta = 1$), and symplectic matrices ($\beta = 4$) in mathematical physics [1], [7],

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[8], [10], [19]. The case of general β is attracting more and more attention [5], [22] .

One of the most important statistics is the m-point correlation function

$$\begin{aligned} & R_n^{m,\beta}(\mu; u_1, u_2, .., u_m) \\ = & \frac{n!}{(n-m)!} \frac{\int ... \int |V(u_1, u_2, .., u_m, t_{m+1}, ..., t_n)|^\beta d\mu(t_{m+1}) ... d\mu(t_n)}{\int ... \int |V(t_1, t_2, ..., t_n)|^\beta d\mu(t_1) ... d\mu(t_n)}. \end{aligned}$$

(1.4)

In its analysis, it is standard to use orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$, associated with μ, satisfying the orthonormality conditions

$$\int_\Gamma \varphi_n \overline{\varphi_m} d\mu = \delta_{jk}.$$

Throughout we use μ' to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(\mu, z, \zeta) = \sum_{k=0}^{n-1} \varphi_k(z) \overline{\varphi_k(\zeta)}.$$

We note that many researchers use n as the upper index of summation in the sum defining K_n. The nth Christoffel function is

$$\lambda_n(\mu, z) = 1/K_n(\mu, z, z) = 1/\sum_{j=0}^{n-1} |\varphi_k(z)|^2.$$

When it is clear that the measure is μ, we'll omit the μ, just writing $\lambda_n(z)$ and $K_n(z, \zeta)$.

The $*$ operation plays a basic role in orthogonal polynomials on the unit circle. If P is a polynomial of degree n, we define

$$P^*(z) = z^n \overline{P(1/\bar{z})}.$$

It permits us to formulate analogues of the Christoffel-Darboux formula [20, p. 124]:

$$K_n(\mu, z, \zeta) = \frac{\overline{\varphi_n^*(\zeta)}\varphi_n^*(z) - \overline{\varphi_n(\zeta)}\varphi_n(z)}{1 - \bar{\zeta}z} \tag{1.5}$$

$$= \frac{\overline{\varphi_{n-1}^*(\zeta)}\varphi_{n-1}^*(z) - z\bar{\zeta}\overline{\varphi_{n-1}(\zeta)}\varphi_{n-1}(z)}{1 - \bar{\zeta}z}. \tag{1.6}$$

For a given $\zeta \in \Gamma$, and n, let

$$\chi = \chi_n(\zeta) = -\frac{\overline{\varphi_{n-1}^*(\zeta)}}{\bar{\zeta}\overline{\varphi_{n-1}(\zeta)}}, \tag{1.7}$$

so that $|\chi| = 1$, and

$$K_n(\mu, z, \zeta) = \left(-\bar{\zeta}\overline{\varphi_{n-1}(\zeta)}\right) \frac{z\varphi_{n-1}(z) + \chi\varphi_{n-1}^*(z)}{1 - \bar{\zeta}z}. \tag{1.8}$$

It is known that $z\varphi_{n-1}(z) + \chi\varphi_{n-1}^*(z)$ has n simple zeros $\left\{z_{kn}^{(\zeta)}\right\}$ on Γ, one of which is ζ [20, p. 129]. There χ is denoted by β, (and n is replaced by $n+1$) but of course we have already assigned a different role to β. The Gauss quadrature due to Jones, Njastad, and Thron [20, p. 129] asserts that

$$\sum_{k=1}^{n} \lambda_n\left(\mu, z_{kn}^{(\zeta)}\right) P\left(z_{kn}^{(\zeta)}\right) = \int P d\mu \tag{1.9}$$

for every Laurent polynomial P of the form

$$P(z) = \sum_{k=-n+1}^{n-1} c_k z^k. \tag{1.10}$$

We define the discrete measure

$$\mu_{n,\zeta} = \sum_{k=1}^{n} \lambda_n\left(\mu, z_{kn}^{(\zeta)}\right) \delta_{z_{kn}^{(\zeta)}}. \tag{1.11}$$

Thus the Gauss quadrature (1.9) may be expressed as

$$\int P d\mu_{n,\zeta} = \int P d\mu \tag{1.12}$$

for every P of the form (1.10).

Our basic identity is:

Theorem 1.1
Let μ be a measure on the unit circle Γ, with infinitely many points in its support. Let $|\zeta| = 1$ and $n \geq 1$; and $\mu_{n,\zeta}$ be the discrete measure defined by (1.11). For any $m \geq 1$ and $u_1, u_2, .., u_m \in \mathbb{C}$,

$$R_n^{m,\beta}\left(\mu_{n,\zeta}; u_1, u_2, .., u_m\right)$$

$$= \frac{1}{m!} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq n} \left(\prod_{k=1}^{m} \lambda_n\left(\mu, z_{j_k n}^{(\zeta)}\right)\right)^{\beta-1} \times \left|\det \begin{bmatrix} K_n\left(\mu, z_{j_1 n}^{(\zeta)}, u_1\right) & \ldots & K_n\left(\mu, z_{j_1 n}^{(\zeta)}, u_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(\mu, z_{j_m n}^{(\zeta)}, u_1\right) & \ldots & K_n\left(\mu, z_{j_m n}^{(\zeta)}, u_m\right) \end{bmatrix}\right|^{\beta}.$$

(1.13)

Remarks

(a) The point of the theorem is that all the determinants in the last line are $m \times m$, and m is fixed, while typically we want to investigate the behavior as $n \to \infty$. Thus instead of having to deal with the $n-m$ fold integral in the numerator in (1.4), we can analyze fixed size determinants.

(b) Suppose that $u_k = z_{j_k n}^{(\zeta)}$, $1 \le k \le m$, for some distinct $1 \le j_1, j_2, ..., j_m \le n$. Then the above reduces to

$$R_n^{m,\beta}\left(\mu_{n,\zeta}; z_{j_1 n}^{(\zeta)}, z_{j_2 n}^{(\zeta)}, .., z_{j_m n}^{(\zeta)}\right) = \prod_{k=1}^{m} \lambda_n\left(\mu, z_{j_k n}^{(\zeta)}\right)^{-1}.$$

(c) The above is a unit circle analogue of an identity derived in [16] for beta ensembles on the real line, associated with Gauss quadratures.

(d) When $\beta = 2$, this reduces to a familiar identity in random matrix theory:

Corollary 1.2

For $u_1, u_2, ..., u_m \in \mathbb{C}$,

$$\begin{aligned} & R_n^{m,2}\left(\mu_{n,\zeta}; u_1, u_2, ..., u_m\right) \\ = \; & R_n^{m,2}\left(\mu; u_1, u_2, ..., u_m\right) \\ = \; & \det\left[K_n\left(\mu, u_i, u_j\right)\right]_{1 \le i,j \le m}. \end{aligned} \tag{1.14}$$

Another standard quantity in random matrix theory is expected values of products of characteristic polynomials, or their ratios [2], [6], [9], [18]. In our case, these are simple to compute, and are somewhat different from the continuous analogue:

Proposition 1.3

Let $f : \Gamma^n \to \mathbb{C}$ *be a symmetric function of* n *variables. Then*

$$\int f\left(t_1, t_2, ..., t_n\right) d\mathcal{P}_\beta^{(n)}\left(\mu_{n,\zeta}; t_1, t_2, ..., t_n\right) = f\left(z_{1n}^{(\zeta)}, z_{2n}^{(\zeta)}, .., z_{nn}^{(\zeta)}\right). \tag{1.15}$$

In particular, if

$$S\left(\alpha; t_1, t_2, ..., t_n\right) = \prod_{k=1}^{n}\left(\alpha - t_k\right),$$

and $\{\alpha_j\}_{j=1}^m, \{\beta_j\}_{j=1}^m$ *are complex numbers,*

(1.16)
$$\int \prod_{k=1}^{m} \frac{S\left(\alpha_k; t_1, t_2, ..., t_n\right)}{S\left(\beta_k; t_1, t_2, ..., t_n\right)} d\mathcal{P}_\beta^{(n)}\left(\mu; t_1, t_2, ..., t_n\right) = \prod_{k=1}^{m} \frac{K_n\left(\alpha_k, \zeta\right)\left(1 - \alpha_k \bar{\zeta}\right)}{K_n\left(\beta_k, \zeta\right)\left(1 - \beta_k \bar{\zeta}\right)}.$$

We prove Theorem 1.1, Corollary 1.2 and Proposition 1.3 in Section 3.

2. Asymptotics

The formulae of Section 1 permit us to establish *universality limits* as $n \to \infty$. The latter are a major topic in random matrix theory, with many different facets, and we cannot hope to survey this here. See the monographs [1], [3], [7], [8], [10], [19]. For $\beta = 2$, the narrower setting on which we focus is covered in, for example, [12], [13], [14], [16], [21], [25]. As noted above, asymptotic aspects of general β ensembles, are considered, for example, in [5], [16].

We need to assume that the underlying measure μ is *regular* on Γ in the sense of Stahl, Totik, and Ullman [23], that is,

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1.$$

A sufficient condition for regularity is that $\mu' > 0$ a.e. on Γ (equivalently on $[-\pi, \pi]$). See [23] for further background on this concept. We also need the sinc kernel

$$S(t) = \frac{\sin \pi t}{\pi t}. \tag{2.1}$$

We prove:

Theorem 2.1
Let μ be a regular measure on Γ. Let Γ_1 be an open subarc of Γ and Γ_2 be a compact subarc of Γ_1. Assume that μ' is positive and continuous in $\overline{\Gamma_1}$, and moreover, that either

$$\sup_{n\geq 1} n \left\| \lambda_n(\mu, \cdot) \right\|_{L_\infty(\Gamma)} < \infty \tag{2.2}$$

or, for some compact subarc Γ_3 of Γ_1 containing Γ_2 in its interior,

$$\sup_{n\geq 1} \left\| \varphi_n \right\|_{L_\infty(\Gamma_3)} < \infty. \tag{2.3}$$

For $n \geq 1$, let $\zeta_n \in \Gamma_2$ and let μ_{n,ζ_n} be the measure defined by (1.11). Then for $\beta \geq 2$, and real $a_1, a_2, \ldots a_m$,

$$\lim_{n\to\infty} \frac{1}{n^m} R_n^{m,\beta}\left(\mu_{n,\zeta_n}; \zeta_n e^{2\pi i a_1/n}, \ldots, \zeta_n e^{2\pi i a_m/n}\right)$$
$$= \frac{1}{m!} \sum_{j_1, j_2, \ldots, j_m = -\infty}^{\infty} \left| \det\left[S(a_i - j_k)\right]_{1\leq i,k\leq m} \right|^\beta. \tag{2.4}$$

For $1 < \beta < 2$, *the same result holds if we assume (2.3) and the additional condition*

$$\sum_{j=1}^{n} \lambda_n^{-1}\left(z_{jn}^{(\zeta_n)}\right) = o\left(n^{\frac{1}{1-\beta/2}}\right). \tag{2.5}$$

Remarks
(a) Note that if μ is absolutely continuous on Γ, satisfying there $0 < C_1 \leq \mu' < C_2 < \infty$, then both (2.2) and (2.5) hold for all $\beta > 1$. Indeed in this

case, the sum in the left-hand side of (2.5) is $O\left(n^2\right)$. More generally, if $\log\mu' \in L_1(\Gamma)$ and

$$\sup_{e^{i\phi}\in\Gamma_1}\int_{-\pi}^{\pi}\left|\frac{\mu'\left(e^{i\theta}\right)-\mu'\left(e^{i\phi}\right)}{\theta-\phi}\right|^2 d\theta<\infty,$$

then (2.3) holds [11, p. 223, Thm. V.4.4].
(b) In the special case $\beta = 2$, the limit (2.4) reduces to the usual universality limit, and the right-hand side of (2.4) equals $\det\left[S\left(a_i-a_k\right)\right]_{1\le i,k\le m}$.

For ratios of characteristic polynomials, we prove:

Theorem 2.2
Let μ be a regular measure on Γ. Let Γ_1 be an open subarc of Γ and Γ_2 be a compact subarc of Γ_1. Assume that μ' is positive and continuous in Γ_1. For $n \ge 1$, let $\zeta_n \in \Gamma_2$ and let μ_{n,ζ_n} be the measure defined by (1.11). Then for $\beta \ge 2$, and real $\{a_i\}_{i=1}^m$, and real non-integer $\{b_i\}_{i=1}^m$,

$$\lim_{n\to\infty}\int\prod_{k=1}^{m}\frac{S\left(\zeta_n e^{2\pi i a_k/n};t_1,t_2,...,t_n\right)}{S\left(\zeta_n e^{2\pi i b_k/n};t_1,t_2,...,t_n\right)}d\mathcal{P}_\beta^{(n)}\left(\mu;t_1,t_2,...,t_n\right)$$

$$= \prod_{k=1}^{m}\left(e^{i\pi\left(a_k-b_k\right)}\frac{\sin\left(\pi a_k\right)}{\sin\left(\pi b_k\right)}\right). \tag{2.6}$$

We prove Theorems 2.1 and 2.2 in Section 4. Throughout $C, C_1, C_2, ...$ denote positive constants independent of n, x, t, that are different in different occurrences.

3. Proof of Theorem 1.1, Corollary 1.2, and Proposition 1.3

We shall fix n, ζ and abbreviate $z_{jn}^{(\zeta)}$ as z_j and $\mu_{n,\zeta}$ as μ_n in this section. We also abbreviate $\lambda_n(\mu, z)$ as $\lambda_n(z)$ and $K_n(\mu, z, u)$ as $K_n(z, u)$. We often use

$$K_n\left(z_j,z_k\right)=0,\ j\ne k. \tag{3.1}$$

Indeed, as $\varphi_n^*(z) = z^n\overline{\varphi_n(z)}$ for $|z| = 1$ and as (1.5) shows that

$$\frac{\varphi_n^*\left(z_j\right)}{\varphi_n\left(z_j\right)}=\frac{\overline{\varphi_n^*\left(\zeta\right)}}{\overline{\varphi_n\left(\zeta\right)}}=\frac{\varphi_n^*\left(z_k\right)}{\varphi_n\left(z_k\right)},$$

so

$$\begin{aligned} K_n\left(z_j,z_k\right) &= \frac{\overline{\varphi_n^*\left(z_k\right)}\varphi_n^*\left(z_j\right)-\overline{\varphi_n\left(z_k\right)}\varphi_n\left(z_j\right)}{1-\overline{z_k}z_j} \\ &= \frac{\overline{\varphi_n^*\left(z_k\right)}z_j^n\overline{\varphi_n\left(z_j\right)}-\overline{\varphi_n\left(z_k\right)}z_j^n\overline{\varphi_n^*\left(z_j\right)}}{1-\overline{z_k}z_j}=0. \end{aligned}$$

We also use the notations

$$\underline{r}=\left(r_1,r_2,...,r_n\right);\ \underline{t}=\left(t_1,t_2,...,t_n\right);\ d\mu^{\times n}\left(\underline{t}\right)=d\mu\left(t_1\right)d\mu\left(t_2\right)...d\mu\left(t_n\right)$$

and

$$D\left(r_1, r_2, ..., r_n\right) = D\left(\underline{r}\right) = \det\left[K_n\left(r_i, r_j\right)\right]_{1\leq i,j\leq n}. \tag{3.2}$$

Lemma 3.1

$$\int ... \int \left|V\left(\underline{t}\right)\right|^{\beta} d\mu_n^{\times n}\left(\underline{t}\right) = \left(\kappa_0 ... \kappa_{n-1}\right)^{-\beta} n! \left(\prod_{k=1}^{n} \lambda_n\left(z_k\right)\right)^{1-\beta/2}. \tag{3.3}$$

Proof
We see by taking linear combinations of columns that

$$\kappa_0\kappa_1...\kappa_n V\left(\underline{t}\right) = \det\left[\varphi_{k-1}\left(t_j\right)\right]_{1\leq j,k\leq n}.$$

Then as the determinant of a matrix equals that of its transpose,

$$\begin{aligned}
& \left(\kappa_0\kappa_1...\kappa_{n-1}\right)^2 \left|V\left(\underline{t}\right)\right|^2 \\
= \; & \det\left[\varphi_{k-1}\left(t_j\right)\right]_{1\leq j,k\leq n} \overline{\det\left[\varphi_{k-1}\left(t_\ell\right)\right]_{1\leq k,\ell\leq n}} \\
= \; & \det\left[\sum_{k=1}^{n} \varphi_{k-1}\left(t_j\right) \overline{\varphi_{k-1}\left(t_\ell\right)}\right]_{1\leq j,\ell\leq n} \\
= \; & \det\left[K_n\left(t_j, t_\ell\right)\right]_{1\leq j,\ell\leq n} = D\left(\underline{t}\right). \qquad (3.4)
\end{aligned}$$

Let $(j_1, ..., j_n)$ be a permutation of $(1, 2, ..., n)$. Then

$$\left[\kappa_0\kappa_1...\kappa_{n-1}\left|V\left(z_{j_1}, ..., z_{j_n}\right)\right|\right]^2 = \det\left[K_n\left(z_{j_i}, z_{j_\ell}\right)\right]_{1\leq j,\ell\leq n} = \prod_{j=1}^{n} K_n\left(z_j, z_j\right),$$

by (3.1). Note that this is independent of the permutation $(j_1, ..., j_n)$. (Alternatively, this follows as $|V|$ is symmetric in its entries). Then by definition of μ_n, and as $V(t_1, ..., t_n)$ vanishes unless $t_1, t_2, ..., t_n$ are distinct,

$$\begin{aligned}
& \left[\kappa_0\kappa_1...\kappa_{n-1}\right]^{\beta} \int ... \int \left|V\left(t_1, t_2, ..., t_n\right)\right|^{\beta} d\mu_n\left(t_1\right)...d\mu_n\left(t_n\right) \\
= \; & \sum_{\substack{j_1=1 \\ j_1, j_2, ... j_n \text{ distinct}}}^{n} \sum_{j_2=1}^{n} ... \sum_{j_n=1}^{n} \left(\prod_{k=1}^{n} \lambda_n\left(z_{j_k}\right)\right) \left[\left(\kappa_0\kappa_1...\kappa_{n-1}\right)^2 \left|V\left(z_{j_1}, ..., z_{j_n}\right)\right|^2\right]^{\beta/2} \\
= \; & \sum_{\substack{j_1=1 \\ j_1, j_2, ... j_n \text{ distinct}}}^{n} \sum_{j_2=1}^{n} ... \sum_{j_n=1}^{n} \left(\prod_{k=1}^{n} \lambda_n\left(z_k\right)\right) \left[\prod_{k=1}^{n} K_n\left(z_k, z_k\right)\right]^{\beta/2} \\
= \; & n! \left(\prod_{k=1}^{n} \lambda_n\left(z_k\right)\right)^{1-\beta/2}.
\end{aligned}$$

■

Lemma 3.2
Let $m \geq 2$ and $y_1, y_2, ..., y_m \in \mathbb{C}$. Let $j_{m+1}, j_{m+2}, ..., j_n$ be distinct indices in $\{1, 2, ..., n\}$. Let $\{j_1, j_2, ..., j_m\} = \{1, 2, ..., n\} \setminus \{j_{m+1}, ..., j_n\}$. Then

$$D\left(y_1...y_m, z_{j_{m+1}}, z_{j_{m+2}}, ..., z_{j_n}\right)$$

$$= \left(\prod_{k=1}^{m} \lambda_n\left(z_{j_k}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)\right)\left|\det\left[K_n\left(z_{j_i}, y_\ell\right)\right]_{1\leq i,\ell\leq m}\right|^2.$$

(3.5)

Proof
Using orthogonality, we see that for any $1 \leq \ell \leq m$, and any $u \in \mathbb{C}$,

$$K_n(u, y_\ell) = \int K_n(u,t) K_n(t, y_\ell)\, d\mu(t).$$

Here, for $|t| = 1$, $K_n(u,t) K_n(t, y_\ell)$ is a Laurent polynomial in t of the form (1.10). The Gauss quadrature formula (1.9) gives

$$K_n(u, y_\ell) = \sum_{i=1}^{m} \lambda_n\left(z_{j_i}\right) K_n\left(u, z_{j_i}\right) K_n\left(z_{j_i}, y_\ell\right) \tag{3.6}$$

since $\{j_1...j_n\}$ is a permutation of $\{1, 2, ...n\}$. Substituting (3.6) with $u \in \left\{y_1, y_2, ..., y_m, z_{j_{m+1}}, ..., z_{j_n}\right\}$ in the first m rows of $D = D\left(y_1...y_m, z_{j_{m+1}}, ..., z_{j_n}\right)$ and then extracting each of the m sums, gives

$$D = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}...\sum_{i_m=1}^{n}\left(\prod_{k=1}^{m} \lambda_n\left(z_{j_{i_k}}\right) K_n\left(y_k, z_{j_{i_k}}\right)\right) \times$$

$$\det\begin{bmatrix} K_n\left(z_{j_{i_1}}, y_1\right) & \dots & K_n\left(z_{j_{i_1}}, y_m\right) & K_n\left(z_{j_{i_1}}, z_{j_{m+1}}\right) & \dots & K_n\left(z_{j_{i_1}}, z_{j_n}\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(z_{j_{i_m}}, y_1\right) & \dots & K_n\left(z_{j_{i_m}}, y_m\right) & K_n\left(z_{j_{i_m}}, z_{j_{m+1}}\right) & \dots & K_n\left(z_{j_{i_m}}, z_{j_n}\right) \\ K_n\left(z_{j_{m+1}}, y_1\right) & \dots & K_n\left(z_{j_{m+1}}, y_m\right) & K_n\left(z_{j_{m+1}}, z_{j_{m+1}}\right) & \dots & K_n\left(z_{j_{m+1}}, z_{j_n}\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(z_{j_n}, y_1\right) & \dots & K_n\left(z_{j_n}, y_m\right) & K_n\left(z_{j_n}, z_{j_{m+1}}\right) & \dots & K_n\left(z_{j_n}, z_{j_n}\right) \end{bmatrix}.$$

We see that this last determinant vanishes unless $\{i_1, i_2, .., i_m\} = \{1, 2, ..., m\}$ (for if not, two rows of the determinant are identical). When $\{i_1, i_2, .., i_m\} =$

$\{1, 2, ..., m\}$, the determinant in the last equation becomes

$$\det\begin{bmatrix} K_n\left(z_{j_{i_1}}, y_1\right) & \dots & K_n\left(z_{j_{i_1}}, y_m\right) & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(z_{j_{i_m}}, y_1\right) & \dots & K_n\left(z_{j_{i_m}}, y_m\right) & 0 & \dots & 0 \\ K_n\left(z_{j_{m+1}}, y_1\right) & \dots & K_n\left(z_{j_{m+1}}, y_m\right) & K_n\left(z_{j_{m+1}}, z_{j_{m+1}}\right) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(z_{j_n}, y_1\right) & \dots & K_n\left(z_{j_n}, y_m\right) & 0 & \dots & K_n\left(z_{j_n}, z_{j_n}\right) \end{bmatrix}$$

$$= \det\begin{bmatrix} K_n\left(z_{j_{i_1}}, y_1\right) & \dots & K_n\left(z_{j_{i_1}}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(z_{j_{i_m}}, y_1\right) & \dots & K_n\left(z_{j_{i_m}}, y_m\right) \end{bmatrix} \prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)$$

$$= \varepsilon_\sigma \det\begin{bmatrix} K_n\left(z_{j_1}, y_1\right) & \dots & K_n\left(z_{j_1}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(z_{j_m}, y_1\right) & \dots & K_n\left(z_{j_m}, y_m\right) \end{bmatrix} \prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right),$$

where ε_σ denotes the sign of the permutation $\sigma = \{i_1, i_2, .., i_m\}$ of $\{1, 2, ..., m\}$, that is $i_j = \sigma(j)$ for each j, $1 \le j \le m$. Then

$$D = \sum_{\substack{i_1=1 \\ \{i_1,i_2,\dots,i_m\}=\{1,2,\dots,m\}}}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_m=1}^{n} \left(\prod_{k=1}^{m} \lambda_n\left(z_{j_{i_k}}\right) K_n\left(y_k, z_{j_{i_k}}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)\right) \times$$

$$\times \varepsilon_\sigma \det\begin{bmatrix} K_n\left(z_{j_1}, y_1\right) & \dots & K_n\left(z_{j_1}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(z_{j_m}, y_1\right) & \dots & K_n\left(z_{j_m}, y_m\right) \end{bmatrix}$$

$$= \left(\prod_{k=1}^{m} \lambda_n\left(z_{j_k}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)\right) \det\begin{bmatrix} K_n\left(z_{j_1}, y_1\right) & \dots & K_n\left(z_{j_1}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(z_{j_m}, y_1\right) & \dots & K_n\left(z_{j_m}, y_m\right) \end{bmatrix}$$

$$\times \sum_\sigma \varepsilon_\sigma \prod_{k=1}^{m} K_n\left(y_k, z_{j_{\sigma(k)}}\right)$$

$$= \left(\prod_{k=1}^{m} \lambda_n\left(z_{j_k}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)\right) \det\left[K_n\left(z_{j_i}, y_\ell\right)\right]_{1\le i,\ell\le m} \det\left[K_n\left(y_\ell, z_{j_i}\right)\right]_{1\le \ell,i\le m}$$

$$= \left(\prod_{k=1}^{m} \lambda_n\left(z_{j_k}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(z_{j_k}, z_{j_k}\right)\right) \left|\det\left[K_n\left(z_{j_i}, y_\ell\right)\right]_{1\le i,\ell\le m}\right|^2.$$

■

Proof of Theorem 1.1

We first deal with the numerator in $R_n^{m,\beta}$ defined by (1.4), but multiplied by $(\kappa_0\kappa_1...\kappa_{n-1})^\beta$. Using the definition (1.11) of μ_n, the identity (3.4), and then Lemma 3.2,

(3.7)

$$\begin{aligned} I \quad : \quad &= (\kappa_0\kappa_1...\kappa_{n-1})^\beta \int ... \int |V(y_1, y_2, ..., y_m, t_{m+1}, ..., t_n)|^\beta \, d\mu_n(t_{m+1}) ... d\mu_n(t_n) \\ &= \sum_{j_{m+1}=1}^{n} ... \sum_{j_n=1}^{n} \left(\prod_{k=m+1}^{n} \lambda_n(z_{j_k}) \right) \\ &\quad \times \left| D\left(y_1, ..., y_m, z_{j_{m+1}}, z_{j_{m+2}}, ..., z_{j_n}\right) \right|^{\beta/2} \\ &= \sum_{\substack{j_{m+1}=1 \\ j_{m+1}...j_n \text{ distinct}}}^{n} ... \sum_{j_n=1}^{n} \left(\prod_{k=m+1}^{n} \lambda_n(z_{j_k}) \right) \\ &\quad \times \left\{ \left(\prod_{k=1}^{m} \lambda_n(z_{j_k}) \right) \left(\prod_{k=m+1}^{n} K_n(z_{j_k}, z_{j_k}) \right) \left| \det\left[K_n(z_{j_i}, y_\ell)\right]_{1\leq i,\ell\leq m} \right|^2 \right\}^{\beta/2}. \end{aligned}$$

Here $\{j_1, j_2, ..., j_m\} = \{1, 2, ..., n\} \setminus \{j_{m+1}, ..., j_n\}$. Because of the symmetry in this last expression, it is the same as it would be if $j_1 < j_2 < ... < j_m$. Moreover, once we have chosen $j_1, ..., j_m$, there are $(n-m)!$ choices for $\{j_{m+1}, ..., j_n\}$ (not ordered in increasing size). Also

$$\prod_{k=m+1}^{n} K_n(z_{j_k}, z_{j_k}) = \prod_{k=m+1}^{n} \lambda_n^{-1}(z_{j_k}) = \left(\prod_{k=1}^{n} \lambda_n^{-1}(z_k) \right) \prod_{k=1}^{m} \lambda_n(z_{j_k}).$$

So

$$\begin{aligned} I &= (n-m)! \left\{ \prod_{k=1}^{n} \lambda_n(z_k) \right\}^{1-\beta/2} \\ &\quad \times \sum_{1\leq j_1<j_2<...j_m\leq n} \left(\prod_{k=1}^{m} \lambda_n(z_{j_k}) \right)^{\beta-1} \left| \det\left[K_n(z_{j_i}, y_\ell)\right]_{1\leq i,\ell\leq m} \right|^\beta \\ &= \frac{(n-m)!}{m!} \left\{ \prod_{k=1}^{n} \lambda_n(z_k) \right\}^{1-\beta/2} \\ &\quad \times \sum_{1\leq j_1,j_2...j_m\leq n} \left(\prod_{k=1}^{m} \lambda_n(z_{j_k}) \right)^{\beta-1} \left| \det\left[K_n(z_{j_i}, y_\ell)\right]_{1\leq i,\ell\leq m} \right|^\beta. \end{aligned}$$

Then (1.4), (3.3), and our definition (3.7) of I give

$$\begin{aligned} & R_n^{m,\beta}(\mu_n; y_1, y_2, .., y_m) \\ = & \frac{n!}{(n-m)!} \frac{\int ... \int |V(y_1, y_2, ..., y_m, t_{m+1}, ..., t_n)|^\beta d\mu_n(t_{m+1}) ... d\mu_n(t_n)}{\int ... \int |V(t_1, t_2, ..., t_n)|^\beta d\mu_n(t_1) ... d\mu_n(t_n)} \\ = & \frac{n!}{(n-m)!} \frac{I}{(\kappa_0 ... \kappa_{n-1})^\beta \int ... \int |V(t_1, t_2, ..., t_n)|^\beta d\mu_n(t_1) ... d\mu_n(t_n)} \\ = & \frac{1}{m!} \sum_{1 \le j_1, j_2 ... j_m \le n} \left(\prod_{k=1}^m \lambda_n(z_{j_k}) \right)^{\beta - 1} \left| \det [K_n(z_{j_i}, y_\ell)]_{1 \le i, \ell \le m} \right|^\beta . \end{aligned}$$

■

Proof of Corollary 1.2
For $\beta = 2$,

$$\begin{aligned} & |V(y_1, y_2, ..., y_m, t_{m+1}, ..., t_n)|^2 \\ = & V(y_1, y_2, ..., y_m, t_{m+1}, ..., t_n) \overline{V(y_1, y_2, ..., y_m, t_{m+1}, ..., t_n)} \end{aligned}$$

is a Laurent polynomial of form (1.10) in $t_{m+1}, t_{m+2}, ..., t_n \in \Gamma$. Similarly for $|V(t_1, ..., t_n)|^2$. Then the Gauss quadrature formula (1.9) gives the first equality in (1.14). Next for $\beta = 2$, (1.13) becomes

$$\begin{aligned} & \frac{1}{m!} \sum_{1 \le j_1, j_2 ... j_m \le n} \prod_{k=1}^m \lambda_n(z_{j_k}) \left| \det [K_n(z_{j_i}, y_j)]_{1 \le i, j \le m} \right|^2 \\ = & \frac{1}{m!} \int ... \int |\det [K_n(t_i, y_j)]|^2 d\mu(t_1) d\mu(t_2) ... d\mu(t_m), \end{aligned}$$

by repeated application of (1.9). It is well known that this integral equals $\det [K_n(y_i, y_j)]_{1 \le i, j \le m}$, but we provide the details. Let σ, η denote permutations of $(1, 2, ..., m)$ with signs $\varepsilon_\sigma, \varepsilon_\eta$. We continue the above as

$$\begin{aligned} = & \frac{1}{m!} \sum_\sigma \sum_\eta \varepsilon_\sigma \varepsilon_\eta \int ... \int \left(\prod_{i=1}^m K_n(t_i, y_{\sigma(i)}) \right) \left(\overline{\prod_{i=1}^m K_n(t_i, y_{\eta(i)})} \right) d\mu(t_1) ... d\mu(t_m) \\ = & \frac{1}{m!} \sum_\sigma \sum_\eta \varepsilon_\sigma \varepsilon_\eta \prod_{i=1}^m K_n(y_{\eta(i)}, y_{\sigma(i)}) \\ = & \frac{1}{m!} \sum_\sigma \sum_\eta \varepsilon_{\sigma \circ \eta^{-1}} \prod_{j=1}^m K_n(y_j, y_{\sigma \circ \eta^{-1}(j)}) \\ = & \frac{1}{m!} \sum_\sigma \det [K_n(y_i, y_j)]_{1 \le i, j \le m} = \det [K_n(y_i, y_j)]_{1 \le i, j \le m} . \end{aligned}$$

In the above, we used $\varepsilon_\sigma \varepsilon_\eta = \varepsilon_{\sigma \circ \eta^{-1}}$, and that $\sigma \circ \eta^{-1}$ runs through all permutations of $(1, 2, ...m)$ as η does. ■

Proof of Proposition 1.3

$$\begin{aligned}
&\int f(t_1,t_2,...,t_n)\,d\mathcal{P}_\beta^{(n)}(\mu;t_1,t_2,...,t_n) \\
=\;& \frac{1}{Z_n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n}\left(\prod_{k=1}^{n}\lambda_n(z_{i_k})\right) f(z_{i_1},z_{i_2},..,z_{i_n})\left|V(z_{i_1},z_{i_2},..,z_{i_n})\right|^\beta \\
=\;& \frac{f(z_1,z_2,..,z_n)}{Z_n}\sum_{\substack{i_1=1\ i_2=1\ \cdots\ i_n=1 \\ \{i_1,i_2...i_n\}=\{1,2,...,n\}}}^{n}\left(\prod_{k=1}^{n}\lambda_n(z_{i_k})\right) f(z_{i_1},z_{i_2},..,z_{i_n})\left|V(z_{i_1},z_{i_2},..,z_{i_n})\right|^\beta \\
=\;& f(z_1,z_2,..,z_n),
\end{aligned}$$

by the symmetry of f, and as $V=0$ unless all its arguments are distinct. Thus we have (1.15). Since

$$K_n(z,\zeta)=\text{Constant}\times\frac{\prod_{j=1}^{n}(z-z_j)}{1-z\bar{\zeta}},$$

(1.16) also follows. ■

4. Proof of Theorem 2.1 and 2.2

In this section, we assume μ is as in Theorem 2.1, except that we don't assume (2.2) or (2.3). For $n\geq 1$, let $\zeta^{(n)}\in\Gamma_2$, and let

$$z_{0n}^{(\zeta_n)}=\zeta_n=e^{i\theta_{0n}} \tag{4.1}$$

and write $z_{kn}^{(\zeta_n)}=e^{i\theta_{jn}}$ for other k. It is important to note that in the sequel, k no longer runs from 1 to n, so there is a notational change from (1.9) and (1.11). Rather, as we center our indexing around $z_{0n}^{(\zeta_n)}=\zeta_n$, the index k may now take both positive and negative values, and

$$-\pi\leq...<\theta_{-1,n}<\theta_{0n}<\theta_{1n}<\theta_{2n}<...\leq\pi \tag{4.2}$$

Of course there are only n distinct θ_{kn}, so the sequence terminates on both sides. Throughout Γ_1,Γ_2 and Γ_3 are as in Theorem 2.1. We also continue to abbreviate $\lambda_n(\mu,z)$ as $\lambda_n(z)$, etc. We begin with

Lemma 4.1
(a) Uniformly for a,b in compact subsets of $\mathbb{C}$, we have

$$\lim_{n\to\infty}\frac{K_n\left(\zeta_n\exp\left(\frac{2\pi ia}{n}\right),\zeta_n\exp\left(\frac{2\pi i\bar{b}}{n}\right)\right)}{K_n(\zeta_n,\zeta_n)}=e^{i\pi(a-b)}S(a-b). \tag{4.3}$$

(b) Uniformly for $\zeta\in\overline{\Gamma_1}$,

$$\lim_{n\to\infty}n\lambda_n(\zeta)=2\pi\mu'(\zeta). \tag{4.4}$$

Moreover, there exist $C_1, C_2 > 0$ *such that for* $n \geq 1$ *and all* $\zeta \in \overline{\Gamma_1}$,

$$C_1 \leq n\lambda_n(\zeta) \leq C_2. \tag{4.5}$$

(c) There exists $C_3 > 0$ *such that for all* n, j *with* $e^{i\theta_{jn}}, e^{i\theta_{j-1,n}} \in \Gamma_2$,

$$\theta_{jn} - \theta_{j-1,n} \geq C_3/n. \tag{4.6}$$

(d) For each integer j,

$$\lim_{n\to\infty} (\theta_{jn} - \theta_{0n}) \frac{n}{2\pi} = j. \tag{4.7}$$

Proof

(a) This is proved in [12, p. 559, Theorem 6.3], in the form

$$\lim_{n\to\infty} \frac{K_n\left(e^{i\theta}\left(1+\frac{2\pi ia}{n}\right), e^{i\theta}\left(1+\frac{2\pi i\bar{b}}{n}\right)\right)}{K_n\left(e^{i\theta}, e^{i\theta}\right)} = e^{i\pi(a-b)} S(a-b),$$

uniformly for a, b in compact subsets of the plane and $e^{i\theta} \in \overline{\Gamma_1}$. Since

$$e^{2\pi ia} = 1 + \frac{2\pi ia}{n} + O\left(\frac{1}{n^2}\right),$$

the uniformity of the convergence in a, b, gives the result.

(b) See for example, Theorem 3.1 in [12, Theorem 3.1, p. 549]. Much more general asymptotics are known [17].

(c) We need the fundamental polynomial ℓ_{kn} of Lagrange interpolation that satisfies

$$\ell_{kn}(z_{jn}) = \delta_{jk}.$$

One well known representation of ℓ_{kn}, which follows from (3.1) above, is

$$\ell_{kn}(z) = K_n(z, z_{kn}) / K_n(z_{kn}, z_{kn}).$$

Assume that $z_{jn}, z_{j-1,n} \in \Gamma_2$. Then

$$\begin{aligned} 1 &= \ell_{jn}(z_{jn}) - \ell_{jn}(z_{j-1,n}) \\ &= \int_{\theta_{j-1,n}}^{\theta_{j,n}} \ell'_{jn}\left(e^{it}\right) ie^{it} dt \\ &\leq Cn \sup_{\zeta\in\overline{\Gamma_1}} |\ell_{jn}(\zeta)| (\theta_{jn} - \theta_{j-1,n}), \end{aligned} \tag{4.8}$$

by Videnskii's inequality for the derivative of a polynomial on an arc of the circle - see, for example [4, p. 243]. Here for $\zeta \in \overline{\Gamma_1}$, our bounds on the Christoffel function, and Cauchy-Schwarz give

$$\begin{aligned} |\ell_{jn}(\zeta)| &= |K_n(\zeta, z_{jn})| / K_n(z_{jn}, z_{jn}) \\ &\leq K_n(\zeta,\zeta)^{1/2} K_n(z_{jn}, z_{jn})^{1/2} / K_n(z_{jn}, z_{jn}) \leq C, \end{aligned}$$

by (4.5). Then (4.6) follows from (4.8).

(d) The functions

$$f_n(a) = \frac{K_n\left(\zeta_n \exp\left(\frac{2\pi ia}{n}\right), \zeta_n\right)}{K_n(\zeta_n, \zeta_n)}, n \geq 1,$$

are entire and satisfy, uniformly for a in compact subsets of the plane,

$$\lim_{n\to\infty} f_n(a) = e^{i\pi a} S(a).$$

Hurwitz' Theorem on zeros of uniformly convergent sequences of analytic functions, shows that the only zeros of f_n are zeros $a_{j,n}$, with

$$\lim_{n\to\infty} a_{j,n} = j,\ j = \pm 1, \pm 2, \ldots .$$

As the only zeros of $K_n(\cdot, \zeta_n)$ are $z_{jn} = e^{i\theta_{jn}} = \zeta_n \exp\left(\frac{2\pi i a_{j,n}}{n}\right)$, we deduce that

$$\theta_{jn} - \theta_{0n} = \frac{2\pi a_{j,n}}{n} = \frac{2\pi j}{n}(1 + o(1)).$$

■

We now analyze the main part of the sum in (1.13). Recall that we have changed the range of the indices of summation j_i, which now takes both positive and negative values, with $j_i = 0$ corresponding to $z_{0n} = \zeta_n = e^{i\theta_{0n}}$. In particular in (1.13), instead of $1 \leq j_1, j_2, \ldots, j_m \leq n$, the j_i now take positive and negative values.

Lemma 4.2
Assume that for $1 \leq k \leq m$,

$$y_k = y_k(n) = \zeta_n \exp\left(\frac{2\pi i a_{n,k}}{n}\right), \tag{4.9}$$

where for $1 \leq k \leq m, a_{n,k} \in \mathbb{R}$, *and*

$$\lim_{n\to\infty} a_{n,k} = a_k,$$

and $a_1, a_2, \ldots a_m$ *are fixed. Then for each fixed positive integer* L,

$$\lim_{n\to\infty} \sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} \frac{\left(\prod_{k=1}^{m} \lambda_n(z_{j_k n})\right)^{\beta-1}}{K_n(\zeta_n,\zeta_n)^m} \left|\det \begin{bmatrix} K_n(z_{j_1 n}, y_1) & \ldots & K_n(z_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(z_{j_m n}, y_1) & \ldots & K_n(z_{j_m n}, y_m) \end{bmatrix}\right|^\beta$$

$$= \sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} \left|\det\left(S(j_i - a_k)\right)\right|^\beta . \tag{4.10}$$

Proof
Note that for each fixed j, Lemma 4.1(b), (d), and the uniform continuity of μ' give

$$\frac{K_n(z_{jn}, z_{jn})}{K_n(\zeta_n, \zeta_n)} = 1 + o(1). \tag{4.11}$$

Moreover,
(4.12)

$$\frac{K_n(z_{jn}, y_k)}{K_n(\zeta_n, \zeta_n)} = \frac{K_n\left(\zeta_n e^{\frac{2\pi i j(1+o(1))}{n}}, \zeta_n e^{\frac{2\pi i a_k(1+o(1))}{n}}\right)}{K_n(\zeta_n, \zeta_n)} = e^{\pi i(j-a_k)} S(j-a_k) + o(1),$$

because of the uniform convergence in Lemma 4.1(a). Hence, for each $m-$tuple of integers $j_1, j_2, ..., j_m$,

$$\frac{1}{K_n(\zeta_n, \zeta_n)^m}\left|\det\begin{bmatrix} K_n(z_{j_1 n}, y_1) & \dots & K_n(z_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(z_{j_m n}, y_1) & \dots & K_n(z_{j_m n}, y_m)\end{bmatrix}\right|$$

$$= \left|\det\left[e^{\pi i(j-a_k)} S(j_i - a_k)\right]_{1\leq i,k\leq m}\right| + o(1)$$

$$(4.13) \quad = \left|\det\left[S(j_i - a_k)\right]_{1\leq i,k\leq m}\right| + o(1).$$

Then using (4.11),

$$\sum_{|j_1|,|j_2|,...,|j_m|\leq L} \frac{\left(\prod_{k=1}^{m} \lambda_n(z_{j_k n})\right)^{\beta-1}}{K_n(\zeta_n, \zeta_n)^m}\left|\det\begin{bmatrix} K_n(z_{j_1 n}, y_1) & \dots & K_n(z_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(z_{j_m n}, y_1) & \dots & K_n(z_{j_m n}, y_m)\end{bmatrix}\right|^\beta$$

$$= (1+o(1)) \sum_{|j_1|,|j_2|,...,|j_m|\leq L} K_n(\zeta_n, \zeta_n)^{-m\beta}\left|\det\begin{bmatrix} K_n(z_{j_1 n}, y_1) & \dots & K_n(z_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(z_{j_m n}, y_1) & \dots & K_n(z_{j_m n}, y_m)\end{bmatrix}\right|^\beta,$$

and the lemma follows from (4.13). ■

Now we estimate the tail. We assume (4.9) throughout. First we deal with the (known) case $\beta = 2$:

Lemma 4.3
As $L \to \infty$,
(4.14)

$$T_{L,2} = \limsup_{n\to\infty} \sum_{\substack{(j_1,j_2,...,j_m): \\ \max_i |j_i| > L}} \frac{\prod_{k=1}^{m} \lambda_n(z_{j_k n})}{K_n(\zeta_n, \zeta_n)^m}\left|\det\left[K_n(z_{j_i n}, y_k)\right]_{1\leq i,k\leq m}\right|^2 \to 0.$$

Proof
Recall that from Theorem 1.1 and Corollary 1.2,

$$\begin{aligned} & \frac{1}{m!}\sum_{j_1\dots j_m}\frac{\prod_{k=1}^{m}\lambda_n\left(z_{j_k n}\right)}{K_n\left(\xi,\xi\right)^m}\left|\det\left[K_n\left(z_{j_i n},y_k\right)\right]_{1\le i,k\le m}\right|^2 \\ = & \det\left[\frac{K_n\left(y_i,y_j\right)}{K_n\left(\zeta_n,\zeta_n\right)}\right]_{1\le i,j\le m} \\ \to & \det\left[S\left(a_i-a_j\right)\right]_{1\le i,j\le m}, \end{aligned}$$

as $n\to\infty$, by the limit (4.3). Moreover, from the proof of Corollary 1.4 in [16, p. 162],

$$\frac{1}{m!}\sum_{j_1\dots j_m=-\infty}^{\infty}\left|\det\left[S\left(a_i-a_{j_k}\right)\right]_{1\le i,k\le m}\right|^2=\det\left[S\left(a_i-a_j\right)\right]_{1\le i,j\le m}.$$

Thus

$$\begin{aligned} & \lim_{n\to\infty}\sum_{j_1\dots j_m}\frac{\prod_{k=1}^{m}\lambda_n\left(z_{j_k n}\right)}{K_n\left(\xi,\xi\right)^m}\left|\det\left[K_n\left(z_{j_i n},y_k\right)\right]_{1\le i,k\le m}\right|^2 \\ = & \sum_{j_1\dots j_m=-\infty}^{\infty}\left|\det\left[S\left(a_i-a_{j_k}\right)\right]_{1\le i,k\le m}\right|^2. \end{aligned}$$

Now we can apply (4.10), and use the convergence of series in the last right-hand side. ■

Next we handle the case $\beta>2$:

Lemma 4.4
Let $\beta>2$. Assume all the hypotheses of Theorem 1.3, except (2.2) and (2.3). Instead of those, assume

$$\sup_{\zeta\in\Gamma,u\in\Gamma_3}\lambda_n\left(\zeta\right)\left|K_n\left(\zeta,u\right)\right|\le C,\ n\ge 1. \tag{4.15}$$

Then as $L\to\infty$,
(4.16)

$$T_{L,\beta}=\limsup_{n\to\infty}\sum_{\substack{(j_1,j_2,\dots,j_m):\\ \max_i|j_i|>L}}\frac{\prod_{k=1}^{m}\lambda_n\left(z_{j_k n}\right)^{\beta-1}}{K_n\left(\zeta_n,\zeta_n\right)^m}\left|\det\left[K_n\left(z_{j_i n},y_k\right)\right]_{1\le i,k\le m}\right|^{\beta}\to 0.$$

In particular, (4.15) holds when (2.2) or (2.3) holds.
Proof

We see that
(4.17)

$$T_{L,\beta} \leq T_{L,2} \left\{ \limsup_{n\to\infty} \max_{\substack{(j_1,j_2,...,j_m): \\ \max_i |j_i|>L}} \left[\prod_{k=1}^{m} \lambda_n (z_{j_k n}) \right] \left| \det \left[K_n (z_{j_i n}, y_k) \right]_{1\leq i,k\leq m} \right| \right\}^{\beta-2},$$

where by Lemma 4.3, $T_{L,2} \to 0$ as $L \to \infty$. Next, if σ denotes a permutation of $\{1, 2, ..., m\}$, we see that

$$\begin{aligned} & \left[\prod_{i=1}^{m} \lambda_n (z_{j_i n}) \right] \left| \det \left[K_n (z_{j_i n}, y_k) \right]_{1\leq i,k\leq m} \right| \\ \leq \; & \sum_{\sigma} \prod_{i=1}^{m} \lambda_n (z_{j_i n}) \left| K_n \left(z_{j_i n}, y_{\sigma(i)} \right) \right| \\ \leq \; & m! \left(\sup_{\zeta\in\Gamma, u\in\Gamma_3} \lambda_n (\zeta) \left| K_n (\zeta, u) \right| \right)^m \leq C, \end{aligned}$$

by our hypothesis (4.15). Combined with (4.17), this gives the result. We turn to proving (4.15) under (2.2) or (2.3). Suppose first (2.2) holds. Then for $\zeta \in \Gamma, u \in \Gamma_3$,

$$\lambda_n (\zeta) \left| K_n (\zeta, u) \right| \leq \lambda_n (\zeta) K_n (\zeta, \zeta)^{1/2} K_n (u, u)^{1/2} \leq C,$$

by (4.5). So (4.15) holds in this case. Next, suppose (2.3) holds. We still have (4.5) for $\zeta \in \Gamma_1, u \in \Gamma_3$, so this last argument gives the requisite bound in this case. Now suppose $\zeta \in \Gamma \backslash \Gamma_1, u \in \Gamma_3$, so that $|\zeta - u| \geq C$. From the Christoffel-Darboux formula (1.6),

$$|K_n (\zeta, u)| \leq 2 \frac{\left|\varphi_{n-1} (\zeta)\right| \left|\varphi_{n-1} (u)\right|}{|\zeta - u|}. \tag{4.18}$$

Then

$$\begin{aligned} \lambda_n (\zeta) \left| K_n (\zeta, u) \right| & \leq C \lambda_n (\zeta) \left|\varphi_{n-1} (\zeta)\right| \left|\varphi_{n-1} (u)\right| \\ & \leq C \lambda_n^{1/2} (\zeta) \left|\varphi_{n-1} (u)\right| \leq C \lambda_1^{1/2} (\zeta) \leq C, \end{aligned}$$

by (2.3). So we still have (4.15). ■

The case $\beta < 2$ is more difficult:

Lemma 4.5
Assume all the hypotheses of Theorem 2.2, including (2.3) and (2.5). Let $1 < \beta < 2$. *Then as* $L \to \infty$, *(4.16) holds.*
Proof

Each term in $T_{L,\beta}$ has the form

$$\begin{aligned} & \frac{\prod_{k=1}^{m} \lambda_n \left(z_{j_k n}\right)^{\beta-1}}{K_n \left(\zeta_n, \zeta_n\right)^m} \left|\det \left[K_n \left(z_{j_i n}, y_k\right)\right]_{1\leq i,k\leq m}\right|^{\beta} \\ \leq \; & \frac{C}{n^m} \sum_{\sigma} \prod_{k=1}^{m} \left(\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|K_n \left(z_{j_k n}, y_{\sigma(k)}\right)\right|^{\beta}\right), \end{aligned} \tag{4.19}$$

Here the sum is over all permutations σ of $(1,2,...,m)$. If first $z_{j_k n} \in \Gamma_2$, then by the estimate (4.5) for λ_n, and by (4.18),

$$\begin{aligned} & \frac{1}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|K_n \left(z_{j_k n}, y_{\sigma(k)}\right)\right|^{\beta} \\ \leq \; & \frac{C}{n^{\beta}} \frac{\left|\varphi_n \left(z_{j_k n}\right) \varphi_n \left(y_{\sigma(k)}\right)\right|^{\beta}}{\left|z_{j_k n} - y_{\sigma(k)}\right|^{\beta}} \\ \leq \; & \frac{C}{\left(n \left|z_{j_k n} - y_{\sigma(k)}\right|\right)^{\beta}}, \end{aligned}$$

by our bound (2.3) on φ_n. Here, recalling (4.9),

$$\begin{aligned} \left|z_{j_k n} - y_{\sigma(k)}\right| &= \left|z_{j_k n} - \xi(1 + O\left(\frac{a_{n,\sigma(k)}}{n}\right))\right| \\ &\geq C_1 \frac{|j_k|}{n} - C_2 \frac{\max_i |a_i|}{n}, \end{aligned}$$

by (4.6). It follows that there exists $B > 0$ depending only on $\max_i |a_i|$ such that for $|j_k| \geq B$,

$$\left|z_{j_k n} - y_{\sigma(k)}\right| \geq C_3 \frac{|j_k|}{n}.$$

In particular, B is independent of L and n. Then for $|j_k| \geq B$, and $z_{j_k n} \in \Gamma_2$,

$$\frac{1}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|K_n \left(z_{j_k n}, y_{\sigma(k)}\right)\right|^{\beta} \leq \frac{C}{\left(1 + |j_k|\right)^{\beta}}. \tag{4.20}$$

Now if $|j_k| \leq B$, we can just use our bound (4.5) on λ_n and Cauchy-Schwarz to deduce that

$$\frac{1}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|K_n \left(z_{j_k n}, y_{\sigma(k)}\right)\right|^{\beta} \leq C \frac{1}{n^{\beta}} n^{\beta} \leq \frac{C}{\left(1 + |j_k|\right)^{\beta}}.$$

Thus again (4.20) holds, so we have (4.20) for all j_k with $z_{j_k n} \in \Gamma_2$. Next if $z_{j_k n} \notin \Gamma_2$, then $\left|z_{j_k n} - y_{\sigma(k)}\right| \geq C$, so

$$\begin{aligned} & \frac{1}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|K_n \left(z_{j_k n}, y_{\sigma(k)}\right)\right|^{\beta} \\ \leq \; & \frac{C}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_k n}\right) \varphi_n \left(y_{\sigma(k)}\right)\right|^{\beta} \\ \leq \; & \frac{C}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_k n}\right)\right|^{\beta}, \end{aligned}$$

by (2.3). Note that there is no dependence on σ in the bound in this last inequality nor in (4.20). Then

$$T_{L,\beta} \leq C \limsup_{n\to\infty} \sum_{\substack{(j_1,j_2,\dots,j_m): \\ \max_i |j_i|>L}} \left(\prod_{z_{j_k n}\in\Gamma_2} (1+|j_k|)^{-\beta} \right) \prod_{z_{j_k n}\notin\Gamma_2} \left(\frac{1}{n}\lambda_n \left(z_{j_k n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_k n}\right)\right|^{\beta} \right).$$

We can bound this above by a sum of m terms, such that in the kth term, the index j_k exceeds L in absolute value, while all remaining indices may assume any integer value. As each such term is identical, we may assume that j_1 is the index with $|j_1| \geq L$, and deduce that

$$\begin{aligned} T_{L,\beta} &\leq C \limsup_{n\to\infty} \left(\sum_{|j_1|\geq L} (1+|j_1|)^{-\beta} + \sum_{z_{j_1 n}\notin\Gamma_2} \frac{1}{n}\lambda_n \left(z_{j_1 n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_1 n}\right)\right|^{\beta} \right) \\ &\quad \times \left(\sum_{j=-\infty}^{\infty} (1+|j|)^{-\beta} + \sum_{z_{j_1 n}\notin\Gamma_2} \frac{1}{n}\lambda_n \left(z_{j_1 n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_1 n}\right)\right|^{\beta} \right)^{m-1}. \end{aligned}$$

Here by Hölder's inequality with parameters $p = \frac{2}{\beta}$ and $q = \left(1-\frac{\beta}{2}\right)^{-1}$,

$$\begin{aligned} &\sum_{z_{j_1 n}\notin\Gamma_2} \frac{1}{n}\lambda_n \left(z_{j_1 n}\right)^{\beta-1} \left|\varphi_n \left(z_{j_1 n}\right)\right|^{\beta} \\ &= \frac{1}{n}\sum_{j_1} \left(\lambda_n \left(z_{j_1 n}\right) \left|\varphi_n \left(z_{j_1 n}\right)\right|^2\right)^{\beta/2} \lambda_n \left(x_{j_1 n}\right)^{\beta/2-1} \\ &\leq \frac{C}{n} \left(\sum_{j_1} \lambda_n \left(z_{j_1 n}\right) \left|\varphi_n \left(z_{j_1 n}\right)\right|^2 \right)^{\beta/2} \left(\sum_{j_1} \lambda_n \left(z_{j_1 n}\right)^{-1} \right)^{1-\beta/2} \\ &\leq \frac{C}{n} \left(\sum_{j_1} \lambda_n \left(z_{j_1 n}\right)^{-1} \right)^{1-\beta/2} = o(1), \end{aligned}$$

by our hypothesis (2.5). Thus

$$T_{L,\beta} \leq CL^{1-\beta},$$

and the lemma follows. ■.

Proof of Theorem 2.1

This follows directly from Lemmas 4.2 and 4.4 for $\beta > 2$, and from Lemmas 4.2 and 4.5 for $\beta < 2$: we can choose L so large that the tail in Lemma 4.4 or 4.5 is as small as we please. ■

Proof of Theorem 2.2

By Lemma 4.1(a), for $1 \leq k \leq m$, as $n \to \infty$,

$$\begin{aligned} & \frac{K_n\left(\zeta_n e^{2\pi i a_k/n}, \zeta_n\right)\left(1-\left(\zeta_n e^{2\pi i a_k/n}\right)\bar{\zeta}_n\right)}{K_n\left(\zeta_n e^{2\pi i b_k/n}, \zeta_n\right)\left(1-\left(\zeta_n e^{2\pi i b_k/n}\right)\bar{\zeta}_n\right)} \\ = & \frac{e^{i\pi a_k} S\left(a_k\right)}{e^{i\pi b_k} S\left(b_k\right)}\left(1+o\left(1\right)\right)\frac{e^{i\pi a_k/n}}{e^{i\pi b_k/n}}\frac{\sin\left(\pi a_k/n\right)}{\sin\left(\pi b_k/n\right)} \\ = & \frac{e^{i\pi a_k}\sin\left(\pi a_k\right)}{e^{i\pi b_k}\sin\left(\pi b_k\right)}\left(1+o\left(1\right)\right). \end{aligned}$$

Now apply (1.16) of Proposition 1.3. ■

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