Contemporary Mathematics

Discrete Beta Ensembles based on Gauss Type Quadratures

D. S. Lubinsky

Abstract. Let μ be a measure with support on the real line and $n \geq 1$, $\beta > 0$. In the theory of random matrices, one considers a probability distribution on the eigenvalues $t_1, t_2, \ldots, t_n$ of random matrices, of the form

$$\mathcal{P}_\beta^{(n)}\left(\mu; t_1, t_2, \ldots, t_n\right) = C \left|V\left(t_1, t_2, \ldots, t_n\right)\right|^\beta d\mu\left(t_1\right) \ldots d\mu\left(t_n\right),$$

where C is a normalization constant, and

$$V\left(t_1, t_2, \ldots, t_n\right) = \prod_{1 \leq i < j \leq n} \left(t_j - t_i\right).$$

This is the so-called β ensemble with temperature $1/\beta$. We explicitly evaluate the $m-$point correlation functions when μ is a Gauss quadrature type measure, and use this to investigate universality limits for sequences of such measures.

1. Introduction

Let μ be a finite positive Borel measure on the real line with infinitely many points in the support, and all finite moments. Let $\beta > 0$ and $n \geq 2$. The β-ensemble, with temperature $1/\beta$, associated with the measure μ places a probability distribution on the eigenvalues $t_1, t_2, \ldots, t_n$ of an n by n Hermitian matrix, of the form

$$\begin{aligned} &\mathcal{P}_\beta^{(n)}\left(\mu; t_1, t_2, \ldots, t_n\right) \\ &= \frac{1}{Z_n} \left|V\left(t_1, t_2, \ldots, t_n\right)\right|^\beta d\mu\left(t_1\right) \cdots d\mu\left(t_n\right), \end{aligned} \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 41A10, 41A17, 42C99; Secondary 33C45.

Key words and phrases. Random Matrices.

Research supported by NSF grant DMS1001182 and US-Israel BSF grant 2008399.

where

$$V\left(t_1,t_2,\ldots,t_n\right)=\prod_{1\le i<j\le n}\left(t_j-t_i\right)=\det\left[t_i^{j-1}\right]_{1\le i,j\le n} \tag{1.2}$$

and

$$Z_n=\int\cdots\int\left|V\left(t_1,t_2,\ldots,t_n\right)\right|^{\beta}d\mu\left(t_1\right)\cdots d\mu\left(t_n\right). \tag{1.3}$$

These ensembles arise in scattering theory in mathematical physics. Their analysis has generated interest amongst mathematicians and physicists for decades [**2**], [**3**], [**4**].

One of the important statistics is the m-point correlation function

$$\begin{aligned}&R_n^{m,\beta}\left(\mu;y_1,y_2,\ldots,y_m\right)\\&=\frac{n!}{(n-m)!}\int\cdots\int\mathcal{P}_{\beta}^{(n)}\left(\mu;y_1,y_2,\ldots,y_m,t_{m+1},\ldots,t_n\right)d\mu\left(t_{m+1}\right)\cdots d\mu\left(t_n\right)\\&=\frac{n!}{(n-m)!}\frac{\int\cdots\int\left|V\left(y_1,y_2,\ldots,y_m,t_{m+1},\ldots,t_n\right)\right|^{\beta}d\mu\left(t_{m+1}\right)\cdots d\mu\left(t_n\right)}{\int\cdots\int\left|V\left(t_1,t_2,\ldots,t_n\right)\right|^{\beta}d\mu\left(t_1\right)\cdots d\mu\left(t_n\right)}.\end{aligned} \tag{1.4}$$

It can be used to study local spacing properties of eigenvalues, and local density of eigenvalues. For example, if $m=2$, and $B\subset\mathbb{R}$ is measurable, then

$$\int_B\int_B R_2^{n,\beta}\left(\mu;t_1,t_2\right)d\mu\left(t_1\right)d\mu\left(t_2\right)$$

is the expected number of pairs (t_1,t_2) of eigenvalues, with both $t_1,t_2\in B$.

The best understood case is $\beta=2$ [**2**], where there are close connections to the the theory of orthogonal polynomials associated with the measure μ. The cases $\beta=1$ and $\beta=4$ are also well understood [**3**], [**4**], although the analysis is far more complicated. For Jacobi weights, one can use the Selberg integral to partly analyze general β. For the case where β is the square of an integer, some analysis has been undertaken by Chris Sinclair [**17**]. A recent breakthrough by Borgade, Erdős, and Yau [**1**] gives a new approach to handling β-ensembles for varying weights of the form e^{-nV} with V convex and real analytic.

In this paper, we show that when we take μ to be a Gauss type quadrature measure, then we can explicitly evaluate the correlation function, and hence analyze universality limits for sequences of such measures, at least for the case $\beta>1$.

Define orthonormal polynomials

$$p_n(x)=\gamma_n x^n+\cdots,\quad \gamma_n>0,$$

$n=0,1,2,\cdots$, satisfying the orthonormality conditions

$$\int p_j p_k d\mu=\delta_{jk}.$$

Throughout we use μ' to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(\mu, x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y).$$

Its normalized cousin is

$$\tilde{K}_n(\mu, x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(\mu, x, y).$$

The nth Christoffel function is

$$\lambda_n(\mu, x) = 1/K_n(\mu, x, x) = 1/\sum_{j=0}^{n-1} p_j^2(x).$$

When it is clear that the measure is μ, we'll omit the μ, just writing $\lambda_n(x)$ and $K_n(x, y)$. Recall that given any real ξ with

$$p_{n-1}(\xi) \neq 0, \tag{1.5}$$

there is a Gauss quadrature including ξ as one of the nodes:

$$\int P \, d\mu = \sum_{j=1}^{n} \lambda_n(\mu, x_{jn}) P(x_{jn}) \tag{1.6}$$

for P of degree $\leq 2n-2$. We shall usually order $\{x_{jn}\}_{j=1}^n = \{x_{jn}(\xi)\}_{j=1}^n$ in increasing order; in Section 3, we shall adopt a different notation, setting $x_{0n} = \xi$. The $\{x_{jn}\}$ are zeros of

$$\psi_n(t, \xi) = p_n(\xi) p_{n-1}(t) - p_{n-1}(\xi) p_n(t).$$

In the special case that $p_n(\xi) = 0$, these are the zeros of p_n, and the precision of the quadrature is actually $2n-1$. Note that when $p_{n-1}(\xi) = 0$, there is still a quadrature like (1.6), but involving $n-1$ points, namely the zeros of p_{n-1}, and exact for polynomials of degree $\leq 2n-3$.

We define the discrete measure μ_n by

$$\int f \, d\mu_n = \sum_{j=1}^{n} \lambda_n(\mu, x_{jn}) f(x_{jn}). \tag{1.7}$$

Equivalently,

$$\mu_n = \sum_{j=1}^{n} \lambda_n(\mu, x_{jn}) \delta_{x_{jn}}, \tag{1.8}$$

where $\delta_{x_{jn}}$ denotes a Dirac delta at x_{jn}. Note that μ_n depends on ξ, but we shall not explicitly display this dependence.

Our basic identity is:

THEOREM 1.1. *Let μ be a measure on the real line with infinitely many points in its support, and all finite power moments. Let $\beta > 0$, $n \geq 1$; let $\xi \in \mathbb{R}$ satisfy* (1.5)*, and μ_n be the discrete measure defined by* (1.8)*. For any real $y_1, y_2, \ldots, y_m$,*

$$\begin{aligned} & R_n^{m,\beta}\left(\mu_n; y_1, y_2, \ldots, y_m\right) \\ & = \frac{1}{m!} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq n} \left(\prod_{k=1}^{m} \lambda_n\left(\mu, x_{j_k n}\right) \right)^{\beta-1} \\ & \times \left| \det \begin{bmatrix} K_n\left(\mu, x_{j_1 n}, y_1\right) & \ldots & K_n\left(\mu, x_{j_1 n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(\mu, x_{j_m n}, y_1\right) & \ldots & K_n\left(\mu, x_{j_m n}, y_m\right) \end{bmatrix} \right|^{\beta}. \end{aligned} \tag{1.9}$$

REMARK. (a) Suppose that $y_k = x_{j_k n}$, $1 \leq k \leq m$, for some distinct $1 \leq j_1, j_2, \ldots, j_m \leq n$. Then the above reduces to

$$R_n^{m,\beta}\left(\mu_n; x_{j_1 n}, x_{j_2 n}, \ldots, x_{j_m n}\right) = \prod_{k=1}^{m} \lambda_n\left(\mu, x_{j_k n}\right)^{-1}.$$

(b) If $m = 1$, we see that

$$\begin{aligned} R_n^{1,\beta}\left(\mu; y\right) &= \sum_{j=1}^{n} \lambda_n\left(\mu, x_{jn}\right)^{\beta-1} \left|K_n\left(\mu, x, x_{jn}\right)\right|^{\beta}. \\ &= \sum_{j=1}^{n} \lambda_n\left(\mu, x_{jn}\right)^{-1} \left|\ell_{jn}\left(x\right)\right|^{\beta}, \end{aligned}$$

where $\{\ell_{jn}\}$ are the fundamental polynomials of Lagrange interpolation for $\{x_{jn}\}$.

(c) When $\beta = 2$, this reduces to a familiar identity in random matrix theory:

COROLLARY 1.2.

$$\begin{aligned} R_n^{m,2}\left(\mu_n; y_1, y_2, \ldots, y_m\right) &= R_n^{m,2}\left(\mu; y_1, y_2, \ldots, y_m\right) \\ &= \det\left[K_n\left(\mu, y_i, y_j\right)\right]_{1 \leq i,j \leq m}. \end{aligned} \tag{1.10}$$

The representation in Theorem 1.1 lends itself to asymptotics: let

$$S\left(t\right) = \frac{\sin \pi t}{\pi t} \tag{1.11}$$

denote the sinc kernel. Recall that a compactly supported measure μ is said to be *regular in the sense of Stahl, Totik, and Ullman*, or just *regular*, if the leading coefficients $\{\gamma_n\}$ of its orthonormal polynomials satisfy

$$\lim_{n \to \infty} {\gamma_n}^{1/n} = \frac{1}{\operatorname{cap}\left(\operatorname{supp}\left[\mu\right]\right)}.$$

Here cap(supp $[\mu]$) is the logarithmic capacity of the support of μ. We recall only a very simple criterion for regularity, namely a version of the Erdős-Turán criterion: if the support of μ consists of finitely many intervals, and $\mu' > 0$ a.e. with respect to Lebesgue measure in that support, then μ is regular [**18**, p. 102]. There are many deeper criteria in [**18**].

We also need the density ω_J of the equilibrium measure for a compact set J. Thus $\omega_J(x)\,dx$ is the unique probability measure that minimizes the energy integral

$$\int\int \log \frac{1}{|s-t|} d\nu(s)\, d\nu(t)$$

amongst all probability measures ν with support in J [**13**], [**14**]. In the special case $J = [-1,1]$, $\omega_J(x) = \frac{1}{\pi\sqrt{1-x^2}}$.

Theorem 1.3. *Let μ be a regular measure with compact support J. Let I be a compact subinterval of J such that μ is absolutely continuous in an open interval I_1 containing I. Assume that μ' is positive and continuous in I_1, and moreover, that either*

$$\sup_{n\geq 1} \|p_n\|_{L_\infty(I_1)} < \infty, \tag{1.12}$$

or

$$\sup_{n\geq 1} n\,\|\lambda_n\|_{L_\infty(J)} < \infty. \tag{1.13}$$

Fix $\xi \in I$, and for $n \geq 1$, assume (1.5) *holds. Let μ_n include the point ξ as one of the quadrature points. Then for $\beta \geq 2$ and real $a_1, a_2, \ldots, a_m,$,*

$$\lim_{n\to\infty} \left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R_n^{m,\beta}\left(\mu_n; \xi + \frac{a_1}{n\omega_J(x)}, \ldots, \xi + \frac{a_m}{n\omega_J(x)}\right)$$

$$= \frac{1}{m!} \sum_{j_1,j_2\cdots j_m=-\infty}^{\infty} \left|\det\left[S(a_i - j_k)\right]_{1\leq i,k\leq m}\right|^\beta. \tag{1.14}$$

For $1 < \beta < 2$, the same result holds if we assume (1.12) *and the additonal restriction*

$$\sum_{k=1}^{n} \lambda_n(\mu, x_{kn})^{-1} = O\left(n^{\frac{1}{1-\beta/2}}\right). \tag{1.15}$$

Remarks. (a) We can also write the limit as

$$\lim_{n\to\infty} \frac{1}{K_n(\mu,\xi,\xi)^m} R_n^{m,\beta}\left(\mu_n; \xi + \frac{a_1}{\tilde{K}_n(\mu,\xi,\xi)}, \ldots, \xi + \frac{a_m}{\tilde{K}_n(\mu,\xi,\xi)}\right)$$

$$= \frac{1}{m!} \sum_{j_1,j_2\cdots j_m=-\infty}^{\infty} \left|\det\left[S(a_i - j_k)\right]_{1\leq i,k\leq m}\right|^\beta, \tag{1.16}$$

because, uniformly in compact subsets of I_1,

$$\lim_{n\to\infty} \frac{1}{n} \tilde{K}_n(x,x) = \frac{\omega_J(x)}{\mu'(x)}.$$

(b) If the support of μ is the interval $[-1,1]$ and μ satisfies the Szego condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty,$$

while in some open subinterval I_2 of $(-1,1)$, μ is absolutely continuous, μ' is bounded above and below by positive constants, and μ' satisfies the condition

$$\int \left| \frac{\mu'(t) - \mu'(\theta)}{t-\theta} \right|^2 dt < \infty$$

uniformly in I_1, then (1.12) holds (cf. [**5**, p. 223, Thm. V.4.4]). In particular, this holds for Jacobi and generalized Jacobi weights. The bound (1.12) is also known for exponential weights that violate Szegő's condition [**7**].
(c) The global condition (1.13) is satisfied if for example the support is $[-1,1]$ and $\mu'(x) \leq C/\sqrt{1-x^2}$ for a.e. $x \in (-1,1)$. In fact, as we show in Section 3, one can replace (1.12) and (1.13) by the more implicit condition (which they both imply)

$$\sup_{t \in J, x \in I_2} \lambda_n(t) |K_n(x,t)| \leq C, \quad n \geq 1. \tag{1.17}$$

Here I_2 is a compact subinterval of I_1 that contains I in its interior.
(d) (1.15) places severe restrictions on the measure μ, especially near the endpoints of the support. But some such restriction may well be necessary. It seems that universality is most universal for the "natural" case $\beta = 2$.
(e) When $\beta = 2$, the last right-hand side reduces to a familiar universality limit:

Corollary 1.4.

$$\lim_{n\to\infty} \left(\frac{\mu'(x)}{n\omega_J(x)} \right)^m R_n^{m,2} \left(\mu_n; \xi + \frac{a_1}{n\omega_J(x)}, \ldots, \xi + \frac{a_m}{n\omega_J(x)} \right)$$
$$= \det \left[S(a_i - a_j) \right]_{1 \leq 1, j \leq m}.$$

Of course, this last limit has been established under much more general conditions elsewhere, using special techniques available for $\beta = 2$ [**9**], [**10**], [**16**], [**21**]. For $\beta = 4$, the form of the universality limit differs from the standard one for $\beta = 4$ as the determinant of a 2 by 2 matrix involving S and its derivatives and integrals [**3**, p. 142]. It remains to be seen if (1.14) coincides with that form.

We prove Theorem 1.1 and Corollary 1.2 in Section 2, and Theorem 1.3 and Corollary 1.4 in Section 3. Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, that are different in different occurrences.

2. Proof of Theorem 1.1 and Corollary 1.2

We shall often use

$$K_n(\mu, x_{jn}, x_{kn}) = 0, \quad j \neq k. \tag{2.1}$$

We also use the notation

$$\underline{\mathrm{r}}_n = (r_1, r_2, \dots, r_n) \text{ and } \underline{\mathrm{s}}_n = (s_1, s_2, \dots, s_n)$$

and

$$\begin{aligned} & D\left((r_1, r_2, \dots, r_n), (s_1, s_2, \dots, s_n)\right) \\ & = D\left(\underline{\mathrm{r}}_n, \underline{\mathrm{s}}_n\right) = \det\left[K_n\left(r_i, s_j\right)\right]_{1 \le i,j \le n} \\ & = \det \begin{bmatrix} K_n(r_1, s_1) & K_n(r_1, s_2) & \dots & K_n(r_1, s_n) \\ K_n(r_2, s_1) & K_n(r_2, s_2) & \dots & K_n(r_2, s_n) \\ \vdots & \vdots & \ddots & \vdots \\ K_n(r_n, s_1) & K_n(r_n, s_2) & \dots & K_n(r_n, s_n) \end{bmatrix}. \end{aligned} \tag{2.2}$$

Lemma 2.1.

$$\begin{aligned} & \int \cdots \int \left|V(t_1, t_2, \dots, t_n)\right|^\beta d\mu_n(t_1) \cdots d\mu_n(t_n) \\ & = (\gamma_0 \cdots \gamma_{n-1})^{-\beta} n! \left(\prod_{k=1}^n \lambda_n(\mu, x_{kn})\right)^{1-\beta/2}. \end{aligned} \tag{2.3}$$

Proof. We see by taking linear combinations of columns that

$$\gamma_0 \gamma_1 \cdots \gamma_{n-1} V(t_1, \dots, t_n) = \det\left[p_{k-1}(t_j)\right]_{1 \le j,k \le n}.$$

Then as the determinant of a matrix equals that of its transpose,

$$\begin{aligned} (\gamma_0 \gamma_1 \cdots \gamma_{n-1})^2 V(t_1, \dots, t_n)^2 & = \det\left[p_{k-1}(t_j)\right]_{1 \le j,k \le n} \det\left[p_{k-1}(t_\ell)\right]_{1 \le k,\ell \le n} \\ & = \det\left[\sum_{k=1}^n p_{k-1}(t_j)\, p_{k-1}(t_\ell)\right]_{1 \le j,\ell \le n} \\ & = \det\left[K_n(t_j, t_\ell)\right]_{1 \le j,\ell \le n}. \end{aligned} \tag{2.4}$$

Let $(j_1, \dots, j_n)$ be a permutation of $(1, 2, \dots, n)$. Then

$$\begin{aligned} \left[\gamma_0 \gamma_1 \cdots \gamma_{n-1} V(x_{j_1 n}, \dots, x_{j_n n})\right]^2 & = \det\left[K_n(x_{j_i n}, x_{j_\ell n})\right]_{1 \le j,\ell \le n} \\ & = \prod\nolimits_{j=1}^n K_n(x_{jn}, x_{jn}), \end{aligned}$$

by (2.1). Note that this is independent of the permutation $(j_1, \dots, j_n)$. Then by definition of μ_n, and as $V(t_1, \dots, t_n)$ vanishes unless all its entries are distinct,

$$\begin{aligned} & [\gamma_0 \gamma_1 \cdots \gamma_{n-1}]^\beta \int \cdots \int \left|V(t_1, t_2, \dots, t_n)\right|^\beta d\mu_n(t_1) \cdots d\mu_n(t_n) \\ & = \underset{j_1, j_2, \dots, j_n \text{ distinct}}{\sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_n=1}^n} \left(\prod_{k=1}^n \lambda_n(x_{j_k n})\right) \left[(\gamma_0 \gamma_1 \cdots \gamma_{n-1})^2 \left(V(x_{j_1 n}, \dots, x_{j_n n})\right)^2\right]^{\beta/2} \end{aligned}$$

$$= \sum_{\substack{j_1=1 \\ j_1, j_2, \ldots, j_n \text{ distinct}}}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} \left(\prod_{k=1}^{n} \lambda_n \left(x_{kn} \right) \right) \left[\prod_{k=1}^{n} K_n \left(x_{kn}, x_{kn} \right) \right]^{\beta/2}$$

$$= n! \left(\prod_{k=1}^{n} \lambda_n \left(x_{kn} \right) \right)^{1-\beta/2}.$$

□

Recall that we use the abbreviations $\lambda_n(x)$ for $\lambda_n(\mu, x)$, and $K_n(x, y)$ for $K_n(\mu, x, y)$. We shall do this fairly consistently in the proof of Lemma 2.2 and Theorem 1.1.

LEMMA 2.2. *Let $m \geq 2$ and $y_1, y_2, \ldots, y_m \in \mathbb{R}$. Let $j_{m+1}, j_{m+2}, \ldots, j_n$ be distinct indices in $\{1, 2, \ldots, n\}$. Let $\{j_1, j_2, \ldots, j_m\} = \{1, 2, \ldots, n\} \setminus \{j_{m+1}, \ldots, j_n\}$. Then*

$$D\left(\left(y_1 \cdots y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \ldots, x_{j_n n} \right), \left(y_1 \cdots y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \ldots, x_{j_n n} \right) \right)$$
$$= \left(\prod_{k=1}^{m} \lambda_n \left(x_{j_k n} \right) \right) \left(\prod_{k=m+1}^{n} K_n \left(x_{j_k n}, x_{j_k n} \right) \right)$$
$$\times \left(\det \begin{bmatrix} K_n \left(x_{j_1 n}, y_1 \right) & \ldots & K_n \left(x_{j_1 n}, y_m \right) \\ \vdots & \ddots & \vdots \\ K_n \left(x_{j_m n}, y_1 \right) & \ldots & K_n \left(x_{j_m n}, y_m \right) \end{bmatrix} \right)^2. \tag{2.5}$$

PROOF. We use the reproducing kernel and Gauss quadrature in the form

$$K_n \left(y_k, u \right) = \sum_{i=1}^{n} \lambda_n \left(x_{j_i n} \right) K_n \left(y_k, x_{j_i n} \right) K_n \left(x_{j_i n}, u \right). \tag{2.6}$$

Substituting (2.6) with $u \in \left\{ y_1, y_2, \ldots, y_m, x_{j_{m+1}n}, \ldots, x_{j_n n} \right\}$ in the first m rows of

$$D = D\left(\left(y_1 \cdots y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \ldots, x_{j_n n} \right), \left(y_1 \cdots y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \ldots, x_{j_n n} \right) \right)$$

and then extracting each of the m sums, gives

$$D = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \left(\prod_{k=1}^{m} \lambda_n \left(x_{j_{i_k} n} \right) K_n \left(y_k, x_{j_{i_k} n} \right) \right)$$

$$\times \det \begin{bmatrix} K_n\left(x_{j_{i_1}n}, y_1\right) & \dots & K_n\left(x_{j_{i_1}n}, y_m\right) & K_n\left(x_{j_{i_1}n}, x_{j_{m+1}n}\right) & \dots & K_n\left(x_{j_{i_1}n}, x_{j_nn}\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(x_{j_{i_m}n}, y_1\right) & \dots & K_n\left(x_{j_{i_m}n}, y_m\right) & K_n\left(x_{j_{i_m}n}, x_{j_{m+1}n}\right) & \dots & K_n\left(x_{j_{i_m}n}, x_{j_nn}\right) \\ K_n\left(x_{j_{m+1}n}, y_1\right) & \dots & K_n\left(x_{j_{m+1}n}, y_m\right) & K_n\left(x_{j_{m+1}n}, x_{j_{m+1}n}\right) & \dots & K_n\left(x_{j_{m+1}n}, x_{j_nn}\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(x_{j_nn}, y_1\right) & \dots & K_n\left(x_{j_nn}, y_m\right) & K_n\left(x_{j_nn}, x_{j_{m+1}n}\right) & \dots & K_n\left(x_{j_nn}, x_{j_nn}\right) \end{bmatrix}.$$

We see that this determinant vanishes unless $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, m\}$ (for if not, two rows of the determinant are identical). When $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, m\}$, the determinant in the last equation becomes

$$\det \begin{bmatrix} K_n\left(x_{j_{i_1}n}, y_1\right) & \dots & K_n\left(x_{j_{i_1}n}, y_m\right) & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(x_{j_{i_m}n}, y_1\right) & \dots & K_n\left(x_{j_{i_m}n}, y_m\right) & 0 & \dots & 0 \\ K_n\left(x_{j_{m+1}n}, y_1\right) & \dots & K_n\left(x_{j_{m+1}n}, y_m\right) & K_n\left(x_{j_{m+1}n}, x_{j_{m+1}n}\right) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K_n\left(x_{j_nn}, y_1\right) & \dots & K_n\left(x_{j_nn}, y_m\right) & 0 & \dots & K_n\left(x_{j_nn}, x_{j_nn}\right) \end{bmatrix}$$

$$= \det \begin{bmatrix} K_n\left(x_{j_{i_1}n}, y_1\right) & \dots & K_n\left(x_{j_{i_1}n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_{i_m}n}, y_1\right) & \dots & K_n\left(x_{j_{i_m}n}, y_m\right) \end{bmatrix} \prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right)$$

$$= \varepsilon_\sigma \det \begin{bmatrix} K_n\left(x_{j_1n}, y_1\right) & \dots & K_n\left(x_{j_1n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_mn}, y_1\right) & \dots & K_n\left(x_{j_mn}, y_m\right) \end{bmatrix} \prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right),$$

where ε_σ denotes the sign of the permutation $\sigma = \{i_1, i_2, \dots, i_m\}$ of $\{1, 2, \dots, m\}$, that is $i_j = \sigma(j)$ for each j, $1 \le j \le m$. Then

$$\begin{aligned} D = & \left(\prod_{k=1}^{m} \lambda_n\left(x_{j_kn}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right)\right) \\ & \det \begin{bmatrix} K_n\left(x_{j_1n}, y_1\right) & \dots & K_n\left(x_{j_1n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_mn}, y_1\right) & \dots & K_n\left(x_{j_mn}, y_m\right) \end{bmatrix} \\ & \times \sum_{\sigma} \varepsilon_\sigma \prod_{k=1}^{m} K_n\left(y_k, x_{j_{\sigma(k)}n}\right) \\ = & \left(\prod_{k=1}^{m} \lambda_n\left(x_{j_kn}\right)\right)\left(\prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right)\right) \end{aligned}$$

$$\times \left(\det \begin{bmatrix} K_n\left(x_{j_1n}, y_1\right) & \dots & K_n\left(x_{j_1n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_mn}, y_1\right) & \dots & K_n\left(x_{j_mn}, y_m\right) \end{bmatrix} \right)^2 .$$

□

Proof of Theorem 1.1. We first deal with the numerator in $R_n^{m,\beta}$ defined by (1.4). Using the definition (1.8) of μ_n, the identity (2.4), and then Lemma 2.2,

$$I = (\gamma_0\gamma_1 \cdots \gamma_{n-1})^\beta \int \cdots \int \left|V\left(y_1, y_2, \dots, y_m, t_{m+1}, \dots, t_n\right)\right|^\beta d\mu_n\left(t_{m+1}\right) \cdots d\mu_n\left(t_n\right)$$

$$= \sum_{j_{m+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \left(\prod_{k=m+1}^{n} \lambda_n\left(x_{j_kn}\right) \right)$$

$$\times \Big| D\Big(\left(y_1, \dots, y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \dots, x_{j_nn}\right),$$

$$\left(y_1, \dots, y_m, x_{j_{m+1}n}, x_{j_{m+2}n}, \dots, x_{j_nn}\right)\Big)\Big|^{\beta/2}$$

$$= \sum_{\substack{j_{m+1}=1 \\ j_{m+1}\cdots j_n \text{ distinct}}}^{n} \cdots \sum_{j_n=1}^{n} \left(\prod_{k=m+1}^{n} \lambda_n\left(x_{j_kn}\right) \right)$$

$$\times \left\{ \left(\prod_{k=1}^{m} \lambda_n\left(x_{j_kn}\right) \right) \left(\prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right) \right) \times \left(\det \begin{bmatrix} K_n\left(x_{j_1n}, y_1\right) & \dots & K_n\left(x_{j_1n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_mn}, y_1\right) & \dots & K_n\left(x_{j_mn}, y_m\right) \end{bmatrix} \right)^2 \right\}^{\beta/2} \tag{2.7}$$

Here $\{j_1, j_2, \dots, j_m\} = \{1, 2, \dots, n\} \setminus \{j_{m+1}, \dots, j_n\}$. Because of the symmetry in this last expression, it is the same as it would be if $j_1 < j_2 < \cdots < j_m$. Moreover, once we have chosen $j_1, \dots, j_m$, there are $(n-m)!$ choices for $\{j_{m+1}, \dots, j_n\}$ (not necessarily in increasing size). Also

$$\prod_{k=m+1}^{n} K_n\left(x_{j_kn}, x_{j_kn}\right) = \prod_{k=m+1}^{n} \lambda_n^{-1}\left(x_{j_kn}\right)$$
$$= \left(\prod_{k=1}^{n} \lambda_n^{-1}\left(x_{kn}\right) \right) \prod_{k=1}^{m} \lambda_n\left(x_{j_kn}\right).$$

So

$$I = (n-m)! \left\{ \prod_{k=1}^{n} \lambda_n\left(x_{kn}\right) \right\}^{1-\beta/2} \sum_{1 \le j_1 < j_2 < \cdots j_m \le n} \left(\prod_{k=1}^{m} \lambda_n\left(x_{j_kn}\right) \right)^{\beta-1}$$

$$\times \left| \det \begin{bmatrix} K_n(x_{j_1 n}, y_1) & \ldots & K_n(x_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(x_{j_m n}, y_1) & \ldots & K_n(x_{j_m n}, y_m) \end{bmatrix} \right|^\beta$$

$$= \frac{(n-m)!}{m!} \left\{ \prod_{k=1}^{n} \lambda_n(x_{kn}) \right\}^{1-\beta/2} \sum_{1 \le j_1, j_2 \cdots j_m \le n} \left(\prod_{k=1}^{m} \lambda_n(x_{j_k n}) \right)^{\beta-1}$$

$$\times \left| \det \begin{bmatrix} K_n(x_{j_1 n}, y_1) & \ldots & K_n(x_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(x_{j_m n}, y_1) & \ldots & K_n(x_{j_m n}, y_m) \end{bmatrix} \right|^\beta .$$

Then (1.4), Lemma 2.1, and our definition (2.7) of I give

$$R_n^{m,\beta}(\mu_n; y_1, y_2, \ldots, y_m)$$

$$= \frac{n!}{(n-m)!} \frac{\int \cdots \int |V(y_1, y_2, \ldots, y_m, t_{m+1}, \ldots, t_n)|^\beta d\mu_n(t_{m+1}) \cdots d\mu_n(t_n)}{\int \cdots \int |V(t_1, t_2, \ldots, t_n)|^\beta d\mu_n(t_1) \cdots d\mu_n(t_n)}$$

$$= \frac{n!}{(n-m)!} \frac{I}{(\gamma_0 \cdots \gamma_{n-1})^\beta \int \cdots \int |V(t_1, t_2, \ldots, t_n)|^\beta d\mu_n(t_1) \cdots d\mu_n(t_n)}$$

$$= \frac{1}{m!} \sum_{1 \le j_1, j_2 \cdots j_m \le n} \left(\prod_{k=1}^{m} \lambda_n(x_{j_k n}) \right)^{\beta-1}$$

$$\times \left| \det \begin{bmatrix} K_n(x_{j_1 n}, y_1) & \ldots & K_n(x_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(x_{j_m n}, y_1) & \ldots & K_n(x_{j_m n}, y_m) \end{bmatrix} \right|^\beta .$$

□

Proof of Corollary 1.2. For $\beta = 2$, $|V(y_1, y_2, \ldots, y_m, t_{m+1}, \ldots, t_n)|^2$ is a polynomial of degree $\le 2n-2$ in $t_{m+1}, t_{m+2}, \ldots, t_n$. Similarly for $|V(t_1, \ldots, t_n)|^2$. Then the Gauss quadrature formula gives the first equality in (1.10). Next for $\beta = 2$, the right-hand side of (1.9) becomes

$$\frac{1}{m!} \sum_{1 \le j_1, j_2 \cdots j_m \le n} \prod_{k=1}^{m} \lambda_n(\mu, x_{j_k n})$$

$$\times \left| \det \begin{bmatrix} K_n(\mu, x_{j_1 n}, y_1) & \ldots & K_n(\mu, x_{j_1 n}, y_m) \\ \vdots & \ddots & \vdots \\ K_n(\mu, x_{j_m n}, y_1) & \ldots & K_n(\mu, x_{j_m n}, y_m) \end{bmatrix} \right|^2$$

$$= \frac{1}{m!} \int \cdots \int \det [K_n(\mu, t_i, y_j)]^2 \, d\mu(t_1) \, d\mu(t_2) \cdots d\mu(t_m) .$$

By the equality part of Theorem 1.1 in [**11**], this last expression equals $\det [K_n(y_i, y_j)]_{1 \le i,j \le m}$. □

3. Proof of Theorem 1.3 and Corollary 1.4

We begin with

Lemma 3.1. *Assume that μ satisfies the hypotheses of Theorem 1.3. Let I_2 be a compact subinterval of I_1. Then*
(a) Uniformly for $\xi \in I_2$, and uniformly for a, b in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n\left(\mu, \xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\mu, \xi, \xi)} = S(a-b), \tag{3.1}$$

(b) Uniformly for $x \in I_2$,

$$\lim_{n\to\infty} n\lambda_n(\mu, x) = \pi\mu'(x)/\omega_J(x). \tag{3.2}$$

Moreover, there exist $C_1, C_2 > 0$ such that for $n \geq 1$ and all $x \in I_2$,

$$C_1 \leq n\lambda_n(\mu, x) \leq C_2. \tag{3.3}$$

(c) There exists $C_3, C_4 > 0$ such that for all n, j with $x_{jn}, x_{j-1,n} \in I_2$,

$$C_4/n \geq x_{jn} - x_{j-1,n} \geq C_3/n. \tag{3.4}$$

(d) Fix $\xi \in I_1$ and $\{x_{jn}\} = \{x_{jn}(\xi)\}$. Order them in the following way:

$$\cdots < x_{-1,n} < x_{0n} = \xi < x_{1n} < x_{2n} < \cdots \tag{3.5}$$

Then for each integer j,

$$\lim_{n\to\infty} (x_{jn} - \xi)\tilde{K}_n(\xi, \xi) = j. \tag{3.6}$$

Proof. (a) This follows from results of Totik [**21**, Theorem 2.2].
(b) The first part (3.2) also follows from the result of Totik [**21**, Theorem 2.2]. The second part follows from the extremal property of Christoffel functions, and comparison with, e.g. the Christoffel function for the Legendre weight - see [**12**, p. 116].
(c) We need the fundamental polynomial ℓ_{kn} of Lagrange interpolation that satisfies

$$\ell_{kn}(x_{jn}) = \delta_{jk}.$$

One well known representation of ℓ_{kn}, which follows from the Christoffel-Darboux formula, is

$$\ell_{kn}(x) = K_n(x_{kn}, x)/K_n(x_{kn}, x_{kn}). \tag{3.7}$$

Let I_3 be a compact subinterval of I_1 that contains I_2 in its interior. Then

$$\begin{aligned} 1 &= \ell_{jn}(x_{jn}) - \ell_{jn}(x_{j-1,n}) \\ &= \ell'_{jn}(\xi)(x_{jn} - x_{j-1,n}) \\ &\leq Cn \sup_{t\in I_3} |\ell_{jn}(t)| (x_{jn} - x_{j-1,n}), \end{aligned} \tag{3.8}$$

by Bernstein's inequality. Here for $t \in I_3$, our bounds on the Christoffel function, and Cauchy-Schwarz give

$$\begin{aligned} |\ell_{jn}(t)| &= \lambda_n(\mu, x_{kn}) |K_n(x, x_{jn})| \\ &\leq \lambda_n(\mu, x_{kn}) (K_n(x,x))^{1/2} (K_n(x_j, x_{jn}))^{1/2} \leq \frac{C}{n} n = C, \end{aligned}$$

by (3.3). Then the right-hand inequality in (3.4) follows from (3.8). The left-hand inequality follows easily from the Markov-Stieltjes inequalities [**5**, p. 33]

$$x_{jn} - x_{j-1,n} \leq \lambda_n(x_{j-1,n}) + \lambda_n(x_{jn}).$$

(d) The method is due to Eli Levin [**8**], in a far more general situation than that considered here. We do this first for $j = 1$. By (c), and (3.3),

$$x_{1n} = \xi + \frac{a_n}{\tilde{K}_n(\xi, \xi)},$$

where $a_n \geq 0$ and $a_n = O(1)$. We shall show that

$$\lim_{n \to \infty} a_n = 1. \tag{3.9}$$

Let us choose a subsequence $\{a_n\}_{n \in S}$ with

$$\lim_{n \to \infty, n \in \mathcal{S}} a_n = a.$$

Because of the uniform convergence in (a),

$$\begin{aligned} 0 &= \lim_{n \to \infty, n \in \mathcal{S}} \frac{K_n(x_{1n}, \xi)}{K_n(\xi, \xi)} \\ &= \lim_{n \to \infty, n \in \mathcal{S}} \frac{K_n\left(\xi + \frac{a_n}{\tilde{K}_n(\xi,\xi)}, \xi\right)}{K_n(\xi, \xi)} = S(a) = \frac{\sin \pi a}{\pi a}. \end{aligned}$$

It follows that a is a positive integer. If $a \geq 2$, then as $S(t)$ changes sign at 1, the intermediate value theorem shows that there will be a point

$$y_n = \xi + \frac{b_n}{\tilde{K}_n(\xi, \xi)},$$

with $y_n \in (\xi, x_{1n})$, with $b_n \to 1$, and $K_n(y_n, \xi) = 0$. This contradicts that x_{1n} is the first zero to the right of ξ. Thus necessarily $a = 1$. As this is independent of the subsequence, we have (3.9), and hence the result for $j = 1$. The general case of positive can be completed by induction on j. Negative j is similar. □

We now analyze the main part of the sum in (1.9): in the sequel, the sets I_1, I_2, I_3 are as above.

LEMMA 3.2. *Assume that for* $1 \leq k \leq m$,

$$y_k = y_k(n) = \xi + \frac{a_{n,k}}{\tilde{K}_n(\xi, \xi)}, \tag{3.10}$$

where for $1 \leq k \leq m$,

$$\lim_{n\to\infty} a_{n,k} = a_k,$$

and $a_1, a_2, \ldots, a_m$ *are fixed. Then for each fixed positive integer* L,

$$\lim_{n\to\infty} \sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} \frac{\left(\prod_{k=1}^{m} \lambda_n\left(x_{j_k n}\right)\right)^{\beta-1}}{K_n\left(\xi,\xi\right)^m}$$

$$\times \left|\det \begin{bmatrix} K_n\left(x_{j_1 n}, y_1\right) & \ldots & K_n\left(x_{j_1 n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_m n}, y_1\right) & \ldots & K_n\left(x_{j_m n}, y_m\right) \end{bmatrix}\right|^{\beta}$$

$$= \sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} \left|\det\left(S\left(j_i - a_k\right)\right)\right|^{\beta}. \tag{3.11}$$

Proof. Note that for each fixed j, Lemma 3.1(b), (d), and the continuity of μ' give

$$\frac{K_n\left(x_{jn}, x_{jn}\right)}{K_n\left(\xi,\xi\right)} = 1 + o\left(1\right). \tag{3.12}$$

Moreover,

$$\frac{K_n\left(x_{jn}, y_k\right)}{K_n\left(\xi,\xi\right)} = \frac{K_n\left(\xi + \frac{j+o(1)}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_{n,k}}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi,\xi\right)} = S\left(j - a_k\right) + o\left(1\right), \tag{3.13}$$

because of the uniform convergence in Lemma 3.1(a). Hence, for each $m-$tuple of integers $j_1, j_2, \ldots, j_m$,

$$\frac{1}{K_n\left(\xi,\xi\right)^m} \det \begin{bmatrix} K_n\left(x_{j_1 n}, y_1\right) & \ldots & K_n\left(x_{j_1 n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_m n}, y_1\right) & \ldots & K_n\left(x_{j_m n}, y_m\right) \end{bmatrix}$$

$$= \det\left[S\left(j_i - a_k\right)\right]_{1\leq i,k\leq m} + o\left(1\right). \tag{3.14}$$

Then using (3.12),

$$\sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} \frac{\left(\prod_{k=1}^{m} \lambda_n\left(x_{j_k n}\right)\right)^{\beta-1}}{K_n\left(\xi,\xi\right)^m} \left|\det \begin{bmatrix} K_n\left(x_{j_1 n}, y_1\right) & \ldots & K_n\left(x_{j_1 n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_m n}, y_1\right) & \ldots & K_n\left(x_{j_m n}, y_m\right) \end{bmatrix}\right|^{\beta}$$

$$= \left(1 + o\left(1\right)\right) \sum_{|j_1|,|j_2|,\ldots,|j_m|\leq L} K_n\left(\xi,\xi\right)^{-m\beta} \left|\det \begin{bmatrix} K_n\left(x_{j_1 n}, y_1\right) & \ldots & K_n\left(x_{j_1 n}, y_m\right) \\ \vdots & \ddots & \vdots \\ K_n\left(x_{j_m n}, y_1\right) & \ldots & K_n\left(x_{j_m n}, y_m\right) \end{bmatrix}\right|^{\beta},$$

and the lemma follows from (3.14). □

Now we estimate the tail. We assume (3.10) throughout. First we deal with the (known) case $\beta = 2$:

LEMMA 3.3. *As $L \to \infty$,*

$$T_{L,2} = \sum_{\substack{(j_1,j_2,\ldots,j_m): \\ \max_i |j_i| > L}} \frac{\prod_{k=1}^{m} \lambda_n \left(x_{j_k n}\right)}{K_n \left(\xi,\xi\right)^m} \left|\det \left[K_n \left(x_{j_i n}, y_k\right)\right]_{1 \le i,k \le m}\right|^2 \to 0. \tag{3.15}$$

PROOF. Recall that from Theorem 1.1 and Corollary 1.2,

$$\begin{aligned} &\frac{1}{m!} \sum_{j_1 \cdots j_m = -\infty}^{\infty} \frac{\prod_{k=1}^{m} \lambda_n \left(x_{j_k n}\right)}{K_n \left(\xi,\xi\right)^m} \left|\det \left[K_n \left(x_{j_i n}, y_k\right)\right]_{1 \le i,k \le m}\right|^2 \\ &= \det \left[\frac{K_n \left(y_i, y_j\right)}{K_n \left(\xi,\xi\right)}\right]_{1 \le i,j \le m}, \end{aligned}$$

and that from Corollary 1.4 below,

$$\begin{aligned} &\frac{1}{m!} \sum_{j_1 \cdots j_m = -\infty}^{\infty} \left|\det \left[S \left(a_i - a_{j_k}\right)\right]_{1 \le i,k \le m}\right|^2 \\ &= \det \left[S \left(a_i - a_j\right)\right]_{1 \le i,j \le m}. \end{aligned}$$

(Formally, we have not yet proven this, but of course it is independent of the hypotheses here.) Now we split up the sum in the first of these identities, take limits as $n \to \infty$, and use Lemma 3.2 for $\beta = 2$, as well as the limit (3.1), which ensures that

$$\lim_{n \to \infty} \det \left[\frac{K_n \left(y_i, y_j\right)}{K_n \left(\xi,\xi\right)}\right]_{1 \le i,j \le m} = \det \left[S \left(a_i - a_j\right)\right]_{1 \le i,j \le m}.$$

□

LEMMA 3.4. *Assume the hypotheses of Theorem* 1.3*, except for* (1.12) *and* (1.13)*. Then for $n \ge 1$, and $t \in J$,*

$$p_n^2 \left(t\right) \le C \left(p_{n-2}^2 \left(t\right) + p_{n-1}^2 \left(t\right)\right). \tag{3.16}$$

PROOF. We shall show below that

$$\inf_n \frac{\gamma_{n-1}}{\gamma_n} \ge C. \tag{3.17}$$

Once we have this, we can apply the three term recurrence relation in the form

$$\frac{\gamma_{n-1}}{\gamma_n} p_n \left(x\right) = \left(x - b_n\right) p_{n-1} \left(x\right) - \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2} \left(x\right),$$

and the fact that $\{|b_n|\}$ and $\left\{\frac{\gamma_{n-1}}{\gamma_n}\right\}$ are bounded above, (for $J = \operatorname{supp}\left[\mu\right]$ is compact) to deduce (3.16). We turn to the proof of (3.17). From the confluent form of the Christoffel-Darboux formula, we have

$$K_n \left(x_{j_n}, x_{j_n}\right) = \frac{\gamma_{n-1}}{\gamma_n} p_{n-1} \left(x_{jn}\right) p_n' \left(x_{jn}\right).$$

Let I_4 be a non-empty compact subinterval of I_3. By the spacing estimate (3.4), there are at least $C_4 n$ zeros $x_{jn} \in I_4$, so

$$C_4 n \leq \sum_{x_{jn}\in I_4} \lambda_n(x_{jn}) K_n(x_{j_n}, x_{j_n}) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{x_{jn}\in I_4} \lambda_n(x_{jn}) \left|p_{n-1}(x_{jn}) p_n'(x_{jn})\right|$$
$$\leq \frac{\gamma_{n-1}}{\gamma_n} \left(\sum_j \lambda_n(x_{jn}) p_{n-1}^2(x_{jn}) \right)^{1/2} \left(\sum_{x_{jn}\in I_4} \lambda_n(x_{jn}) p_n'(x_{jn})^2 \right)^{1/2}. \tag{3.18}$$

The first quadrature sum is 1. By a theorem of P. Nevai [**12**, p. 167, Thm. 23], followed by Bernstein's inequality, the second sum may be estimated as

$$\left(\sum_{x_{jn}\in I_4} \lambda_n(x_{jn}) p_n'(x_{jn})^2 \right)^{1/2} \leq C \left(\int_{I_4'} p_n'(t)^2\,dt \right)^{1/2}$$
$$\leq Cn \left(\int_{I_4''} p_n^2(t)\,dt \right)^{1/2} \leq Cn,$$

recall that μ' is bounded above and below in I_3. We also use I_4' and I_4'' to denote nested intervals containing I_4 but inside I_3. Substituting in (3.18) gives (3.17). □

Next we handle the case $\beta > 2$:

Lemma 3.5. *Assume all the hypotheses of Theorem* 1.3*, except* (1.12) *and* (1.13)*. Instead of those, assume*

$$\sup_{t\in J, x\in I_2} \lambda_n(t) \left|K_n(x,t)\right| \leq C, \quad n \geq 1, \tag{3.19}$$

where I_2 is a compact subinterval of I_1 containing I in its interior. Let $\beta > 2$. Then as $L \to \infty$,

$$T_{L,\beta} = \sum_{\substack{(j_1,j_2,\ldots,j_m):\\ \max_i |j_i|>L}} \frac{\prod_{k=1}^m \lambda_n(x_{j_k n})^{\beta-1}}{K_n(\xi,\xi)^m} \left|\det\left[K_n(x_{j_i n}, y_k)\right]_{1\leq i,k\leq m}\right|^\beta \to 0. \tag{3.20}$$

In particular, (3.19) *holds when* (1.12) *or* (1.13) *holds.*

Proof. We see that

$$T_{L,\beta} \leq T_{L,2} \left\{ \max_{\substack{(j_1,j_2,\ldots,j_m):\\ \max_i |j_i|>L}} \left[\prod_{k=1}^m \lambda_n(x_{j_k n})\right] \left|\det\left[K_n(x_{j_i n}, y_k)\right]_{1\leq i,k\leq m}\right| \right\}^{\beta-2}, \tag{3.21}$$

where by Lemma 3.3, $T_{L,2} \to 0$ as $L \to \infty$. Next, if σ denotes a permutation of $\{1, 2, \dots, m\}$, we see that

$$\begin{aligned}
&\left[\prod_{k=1}^{m} \lambda_n\left(x_{j_k n}\right)\right] \left|\det\left[K_n\left(x_{j_i n}, y_k\right)\right]_{1\le i,k\le m}\right| \\
&\le \sum_{\sigma} \prod_{k=1}^{m} \lambda_n\left(x_{j_k n}\right)\left|K_n\left(x_{j_k n}, y_{\sigma(k)}\right)\right| \\
&\le m!\left(\sup_{t\in J, y\in I_2} \lambda_n(t)\left|K_n(t,y)\right|\right)^m \le C,
\end{aligned}$$

by our hypothesis (3.19). Combined with (3.21), this gives the result. We turn to proving (3.19) under (1.12) or (1.13). Recall that $I \subset I_2 \subset I_3 \subset I_1$. If firstly $t \in I_3$ and $x \in I_2$,

$$\lambda_n(t)\left|K_n(x,t)\right| \le \lambda_n(t) K_n(x,x)^{1/2} K_n(t,t)^{1/2} \le C,$$

by (3.3). In the sequel, we let

$$A_n(t) = p_n^2(t) + p_{n-1}^2(t).$$

From the Christoffel-Darboux formula,

$$\left|K_n(x,t)\right| \le \frac{\gamma_{n-1}}{\gamma_n} \frac{A_n(t)^{1/2} A_n(x)^{1/2}}{|x-t|}. \tag{3.22}$$

Here $\left\{\frac{\gamma_{n-1}}{\gamma_n}\right\}$ is bounded as μ has compact support. If next, $t \notin I_3$ and $x \in I_2$, we have $|x-t| \ge C$, so

$$\lambda_n(t)\left|K_n(x,t)\right| \le C\lambda_n(t) A_n^{1/2}(t) A_n^{1/2}(x).$$

Here by Lemma 3.4, $\lambda_n(t) A_n(t) \le C\lambda_n(t) A_{n-1}(t) \le C$, so

$$\lambda_n(t)\left|K_n(x,t)\right| \le C\left(\lambda_n(t) A_n(x)\right)^{1/2}.$$

If (1.12) holds, then $A_n(x) \le C$, while $\lambda_n(t) \le \int d\mu$, so (3.19) follows. If instead (1.13) holds, then

$$\begin{aligned}
\lambda_n(t)\left|K_n(x,t)\right| &\le C\left(n^{-1} A_n(x)\right)^{1/2} \\
&\le C\left(n^{-1} K_{n+1}(x,x)\right)^{1/2} \le C.
\end{aligned}$$

This in all cases, we have (3.19). □

The case $\beta < 2$ is more difficult:

Lemma 3.6. *Assume all the hypotheses of Theorem* 1.3, *including* (1.12) *and* (1.15). *Let* $\beta < 2$. *Then as* $L \to \infty$, (3.20) *holds.*

Proof. Each term in $T_{L,\beta}$ has the form

$$\frac{\prod_{k=1}^{m} \lambda_n\left(x_{j_k n}\right)^{\beta-1}}{K_n(\xi,\xi)^m} \left|\det\left[K_n\left(x_{j_i n}, y_k\right)\right]_{1\le i,k\le m}\right|^{\beta}$$

$$\leq \frac{C}{n^m} \sum_\sigma \prod_{k=1}^m \left(\lambda_n \left(x_{j_k n}\right)^{\beta-1} \left| K_n \left(x_{j_k n}, y_{\sigma(k)}\right) \right|^\beta \right), \tag{3.23}$$

Here the sum is over all permutations σ. If first $x_{j_k n} \in I_3$, then by the estimate (3.3) for λ_n, and by (3.22),

$$\begin{aligned} &\frac{1}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} \left| K_n \left(x_{j_k n}, y_{\sigma(k)}\right) \right|^\beta \\ &\leq \frac{C}{n^\beta} \frac{A_n^{\beta/2} \left(x_{j_k n}\right) A_n^{\beta/2} \left(y_{\sigma(k)}\right)}{\left| x_{j_k n} - y_{\sigma(k)} \right|^\beta} \\ &\leq \frac{C}{\left(n \left| x_{j_k n} - y_{\sigma(k)} \right|\right)^\beta}, \end{aligned}$$

by our bound (1.12) on p_n. Here, recalling (3.10),

$$\begin{aligned} \left| x_{j_k n} - y_{\sigma(k)} \right| &= \left| x_{j_k n} - \xi - \frac{a_{n,\sigma(k)}}{\tilde{K}_n (\xi, \xi)} \right| \\ &\geq C_1 \frac{|j_k|}{n} - C_2 \frac{\max_i |a_i|}{n}, \end{aligned}$$

by (3.4) and (3.3). It follows that there exists $B > 0$ depending only on $\max_i |a_i|$ such that for $|j_k| \geq B$,

$$\left| x_{j_k n} - y_{\sigma(k)} \right| \geq C_3 \frac{|j_k|}{n}.$$

In particular, B is independent of L. Then for $|j_k| \geq B$, and $x_{j_k n} \in I_3$,

$$\frac{1}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} \left| K_n \left(x_{j_k n}, y_{\sigma(k)}\right) \right|^\beta \leq \frac{C}{\left(1 + |j_k|\right)^\beta}. \tag{3.24}$$

Now if $|j_k| \leq B$, we can just use our bounds (3.3) on λ_n and Cauchy-Schwarz to deduce that

$$\frac{1}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} \left| K_n \left(x_{j_k n}, y_{\sigma(k)}\right) \right|^\beta \leq C \frac{1}{n^\beta} n^\beta \leq \frac{C}{\left(1 + |j_k|\right)^\beta}.$$

Thus again (3.24) holds, so we have (3.24) for all j_k with $x_{j_k n} \in I_3$. Next if $x_{j_k n} \notin I_3$, then $\left| x_{j_k n} - y_{\sigma(k)} \right| \geq C$, so

$$\begin{aligned} &\frac{1}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} \left| K_n \left(x_{j_k n}, y_{\sigma(k)}\right) \right|^\beta \\ &\leq \frac{C}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} A_n^{\beta/2} \left(x_{j_k n}\right) A_n^{\beta/2} \left(y_{\sigma(k)}\right) \\ &\leq \frac{C}{n} \lambda_n \left(x_{j_k n}\right)^{\beta-1} A_n^{\beta/2} \left(x_{j_k n}\right), \end{aligned}$$

by (1.12). Note that there is no dependence on σ in the bound in this last inequality nor in (3.24). Then

$$T_{L,\beta} \leq C \sum_{\substack{(j_1,j_2,\dots,j_m): \\ \max_i|j_i|>L}} \left(\prod_{x_{j_k n}\in I_3} (1+|j_k|)^{-\beta} \right) \prod_{x_{j_k n}\notin I_3} \left(\frac{1}{n} \lambda_n\left(x_{j_k n}\right)^{\beta-1} A_n^{\beta/2}\left(x_{j_k n}\right) \right).$$

We can bound this above by a sum of m terms, such that in the kth term, the index j_k exceeds L in absolute value, while all remaining indices may assume any integer value. As each such term is identical, we may assume that j_1 is the index with $|j_1| \geq L$, and deduce that

$$\begin{aligned} T_{L,\beta} \leq C &\left(\sum_{|j_1|\geq L} (1+|j_1|)^{-\beta} + \sum_{x_{j_1 n}\notin I_3} \frac{1}{n} \lambda_n\left(x_{j_1 n}\right)^{\beta-1} A_n^{\beta/2}\left(x_{j_1 n}\right) \right) \\ &\times \left(\sum_{j=-\infty}^{\infty} (1+|j|)^{-\beta} + \sum_{x_{jn}\notin I_3} \frac{1}{n} \lambda_n\left(x_{jn}\right)^{\beta-1} A_n^{\beta/2}\left(x_{jn}\right) \right)^{m-1}. \end{aligned}$$

Here by Hölder's inequality with parameters $p = \frac{2}{\beta}$ and $q = \left(1 - \frac{\beta}{2}\right)^{-1}$,

$$\begin{aligned} &\sum_{x_{j_1 n}\notin I_3} \frac{1}{n} \lambda_n\left(x_{j_1 n}\right)^{\beta-1} A_n^{\beta/2}\left(x_{j_1 n}\right) \\ &\leq \frac{1}{n} \sum_{j_1} \left(\lambda_n\left(x_{j_1 n}\right) A_n\left(x_{j_1 n}\right)\right)^{\beta/2} \lambda_n\left(x_{j_1 n}\right)^{\beta/2-1} \\ &\leq \frac{C}{n} \left(\sum_{j_1} \lambda_n\left(x_{j_1 n}\right) A_n\left(x_{j_1 n}\right) \right)^{\beta/2} \left(\sum_{j_1} \lambda_n\left(x_{j_1 n}\right)^{-1} \right)^{1-\beta/2}. \end{aligned}$$

Here by Lemma 3.4,

$$\sum_{j_1} \lambda_n\left(x_{j_1 n}\right) A_n\left(x_{j_1 n}\right) \leq C \sum_{j_1} \lambda_n\left(x_{j_1 n}\right) A_{n-1}\left(x_{j_1 n}\right) \leq 2C,$$

while

$$\left(\sum_{j_1} \lambda_n\left(x_{j_1 n}\right)^{-1} \right)^{1-\beta/2} = O(n)$$

by our hypothesis (1.15). Thus

$$T_{L,\beta} \leq C \left(L^{1-\beta} + o(1) \right),$$

and the lemma follows. □

PROOF OF THEOREM 1.3. This follows directly from Lemmas 3.2, 3.5 and 3.6: we can choose L so large that the tail in Lemma 3.5 or 3.6 is as small as we please. Note that in (3.10),

$$y_k = \xi + \frac{a_{n,k}}{\tilde{K}_n(\xi,\xi)} = \xi + \frac{\tilde{a}_{n,k}}{n\omega_J(\xi)},$$

where $\tilde{a}_{n,k} \to a_k$ as $n \to \infty$, in view of (3.2). This allows us to prove the universality limit in both the forms (1.14) and (1.16). □

PROOF OF COROLLARY 1.4. We have to prove that

$$\sum_{j_1,j_2\cdots j_m=-\infty}^{\infty} \det\left[S(a_i - j_k)\right]^2_{1\le i,k\le m} = m! \det\left[S(a_i - a_k)\right]_{1\le i,k\le m}.$$

We use the identity [**19**, p. 91]

$$\sum_{k=-\infty}^{\infty} S(a-k)\,S(b-k) = S(a-b).$$

The left-hand side is

$$\begin{aligned}
&\sum_{j_1,j_2\cdots j_m=-\infty}^{\infty} \det\left[S(a_i - j_k)\right]^2_{1\le i,k\le m} \\
&= \sum_{\sigma,\eta} \varepsilon_\sigma\varepsilon_\eta \sum_{j_1,j_2\cdots j_m=-\infty}^{\infty} \prod_{k=1}^{m} S\left(a_{\sigma(k)} - j_k\right) S\left(a_{\eta(k)} - j_k\right) \\
&= \sum_{\sigma,\eta} \varepsilon_\sigma\varepsilon_\eta \prod_{k=1}^{m} \sum_{j_k=-\infty}^{\infty} S\left(a_{\sigma(k)} - j_k\right) S\left(a_{\eta(k)} - j_k\right) \\
&= \sum_{\sigma,\eta} \varepsilon_\sigma\varepsilon_\eta \prod_{k=1}^{m} S\left(a_{\sigma(k)} - a_{\eta(k)}\right) \\
&= \sum_{\sigma,\eta} \varepsilon_\sigma\varepsilon_\eta \prod_{j=1}^{m} S\left(a_j - a_{\eta\circ\sigma^{-1}(j)}\right),
\end{aligned}$$

where σ^{-1} denotes the inverse permutation of σ. Now [**6**, p. 189, p. 190]

$$\varepsilon_\sigma\varepsilon_\eta = \varepsilon_{\eta\circ\sigma^{-1}},$$

and we may replace the sum over all permutations $\omega = \eta\circ\sigma^{-1}$ by a sum over all permutations ω, so we continue this as

$$\begin{aligned}
&= \sum_{\sigma}\sum_{\omega} \varepsilon_\omega \prod_{j=1}^{m} S\left(a_j - a_{\omega(j)}\right) \\
&= m! \det\left[S(a_i - a_j)\right]_{1\le i,j\le m}.
\end{aligned}$$

□

References

[1] P. Bourgade, L. Erd̋os, H-T Yau, *Universality of General $\beta-$Ensembles*, manuscript.

[2] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[3] P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality,* Courant Institute Lecture Notes, Vol. 18, New York University Pres, New York, 2009.

[4] P. Forrester, *Log-Gases and Random Matrices*, Princeton University Press, Princeton, 2010.

[5] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[6] S. Lang, *Linear Algebra*, Second Edition, Addison Wesley, Reading, 1970.

[7] E. Levin, D. S. Lubinsky, *Orthogonal Polynomials*, Springer, New York, 2001.

[8] Eli Levin and D. S. Lubinsky, *Applications of Universality Limits to Zeros and Reproducing Kernels of Orthogonal Polynomials*, Journal of Approximation Theory, 150(2008), 69–95.

[9] D. S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915-939.

[10] D. S. Lubinsky, *Bulk Universality Holds in Measure for Compactly Supported Measures*, to appear in J. d' Analyse de Mathematique.

[11] D. S. Lubinsky, *A Variational Principle for Correlation Functions for Unitary Ensembles, with Applications*, manuscript.

[12] P. Nevai, Orthogonal Polynomials, Memoirs of the Amer. Math. Soc. no. 213, (1979).

[13] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[14] E. B. Saff, V. Totik, *Logarithmic Potentials with External Fields*, Springer, New York, 1997.

[15] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, J. d'Analyse Mathematique, 105 (2008), 345–362.

[16] B. Simon, *Szegő's Theorem and its Descendants*, Princeton University Press, Princeton, 2011.

[17] C. Sinclair, *Ensemble Averages when β is a Square Integer*, manuscript.

[18] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[19] F. Stenger, *Numerical Methods Based on Sinc and Analytic Function*s, Springer, New York, 1993.

[20] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d' Analyse de Mathematique, 81(2000), 283–303.

[21] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361–391.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: lubinsky@math.gatech.edu