SERIES REPRESENTATIONS FOR BEST APPROXIMATING ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. Let $\alpha > 0$ not be an even integer. We derive Lagrange type series representations for the entire function of exponential type 1 that minimizes

$$|||x|^{\alpha} - f(x)||_{L_p(\mathbb{R})}$$

amongst all such entire functions f, when $p = 1$ and $p = \infty$. This minimum arises as the scaled limit of the L_p error of polynomial approximation of $|x|^{\alpha}$ on $[-1,1]$, and is one representation of the L_p Bernstein constant.

1. [1]Introduction

Let $\alpha > 0$ be not an even integer. S.N. Bernstein [2], [3] established the limit

$$\Lambda^*_{\infty,\alpha} = \lim_{n\to\infty} n^{\alpha} E_n\left[|x|^{\alpha}; L_{\infty}[-1,1]\right],$$

where

$$E_n\left[f; L_p[a,b]\right] = \inf\left\{\| f - P \|_{L_p[a,b]} : \deg(P) \leq n\right\}$$

denotes the error in best L_p approximation of a function f on $[a,b]$ by polynomials of degree $\leq n$. Bernstein's 1938 method yielded a formulation of the limit as the error in approximation on the whole real axis by entire functions of exponential type, namely

$$\begin{aligned} & \Lambda^*_{\infty,\alpha} \\ = \;& \inf\left\{\| |x|^{\alpha} - f(x) \|_{L_{\infty}(\mathbb{R})} : f \text{ is entire of exponential type} \leq 1\right\}. \end{aligned}$$

Recall here that f is of exponential type $A \geq 0$ means that for each $\varepsilon > 0$, and for $|z|$ large enough,

$$|f(z)| \leq \exp\left(|z|\left(A + \varepsilon\right)\right).$$

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Moreover, A is the smallest number with this property. The exact value of $\Lambda^*_{\infty,\alpha}$ is not known for any α, and the search for it has inspired much research. See [5], [6], [8], [9] for references.

There are extensions to spaces other than L_∞. It is known [6], [9] that for $1 \leq p \leq \infty$, there exists

$$\begin{aligned}\Lambda^*_{p,\alpha} &= \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} E_n[|x|^\alpha ; L_p[-1,1]] \\ &= \inf\left\{\| \, |x|^\alpha - f(x) \, \|_{L_p(\mathbb{R})} : f \text{ is entire of exponential type} \leq 1\right\}.\end{aligned}$$

Only for $p = 1$ and $p = 2$ is $\Lambda^*_{p,\alpha}$ known explicitly. Nikolskii [10] proved that

$$\Lambda^*_{1,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 8\Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-\alpha-2}$$

(for some α, while Bernstein observed this remains true for all α) and Raitsin [11] proved that

$$\Lambda^*_{2,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 2\Gamma(\alpha+1) \sqrt{\pi/(2\alpha+1)}.$$

There is a series representation for the minimizing entire function in L_1. The author is not certain who first derived it, but it appears in a recent paper of Ganzburg [6]. An integral representation is given below. In the case $p = 2$, M. Ganzburg has informed the author that one can can use Fourier transforms, the Paley-Wiener Theorem, and the theory of distributions, to derive a representation. It is the purpose of this paper to derive a series representation for the minimizing entire function in L_∞, using the same method that can be used in L_1.

There is a substantial body of estimates for approximation by entire functions of exponential type, when the approximated function is bounded or has bounded derivatives of some order [1], [4], [12], [13], [14]. With a view to applications in number theory, there are also explicit representations of the best approximating entire function when $p = 1$ and g is one of a special class of functions. For example for $g(x) = sign(x)$, the best L_1 entire function was determined by Vaaler [15]. For other special g, it can be determined using the theory of minimal extrapolations [12, Chapter 7].

Quite recently Littman [7] has used these ideas to determine a representation for the best L_1 entire function when $g(x) = x^n_+$, that is $g(x) = x^n$ in $[0,\infty)$ and is 0 on the negative real axis. Then one can deduce from this the extremal entire function for $g(x) = |x|^n = 2x^n_+ - x^n$. For odd integers n, this gives another approach to the L_1 results of this paper. It is not clear that the approach there can be extended to all $\alpha > 0$.

Our approach is based on an integral representation established in [9, Theorem 1.3, and remarks thereafter]:

Theorem 1.1
Let $1 \leq p \leq \infty$ and $\alpha > -\frac{1}{p}$, not an even integer. The unique entire function $H^*_{p,\alpha}$ of exponential type 1 satisfying

$$\begin{aligned} & \| \ |x|^\alpha - H^*_{p,\alpha}(x) \|_{L_p(\mathbb{R})} \\ = & \inf \left\{ \| \ |x|^\alpha - f(x) \|_{L_p(\mathbb{R})} : f \text{ is entire of exponential type} \leq 1 \right\} \end{aligned} \tag{1.1}$$

admits for $\operatorname{Re} z \geq 0$ the representation

$$H^*_{p,\alpha}(z) = z^\alpha + \frac{\sin\frac{\alpha}{2}\pi}{\pi} F^*_{p,\alpha}(z) \int_{-\infty}^{\infty} \frac{|s|^{\alpha+1}}{s^2+z^2} \frac{ds}{F^*_{p,\alpha}(is)}, \tag{1.2}$$

where

$$F^*_{p,\alpha}(z) = \prod_{j=1}^{\infty} \left(1 - \left(\frac{z}{x^*_j} \right)^2 \right), \tag{1.3}$$

and

$$0 < x^*_1 < x^*_2 < x^*_3 < \ldots \tag{1.4}$$

with

$$x^*_j \in \left[\left(j - \frac{3}{2} \right) \pi, \left(j - \frac{1}{2} \right) \pi \right], j \geq 2. \tag{1.5}$$

In the special case $p = 1$, $F^*_{1,\alpha}(z) = \cos z$.

From this, we shall derive for $p = \infty$:

Theorem 1.2
Let $\alpha > 0$, not an even integer. Then

$$H^*_{\infty,\alpha}(z) = F^*_{\infty,\alpha}(z) \{ P^*(z) + 2z^\ell \sum_{j=1}^{\infty} \frac{\left(x^*_j\right)^{\alpha-\ell+1}}{\left(z^2 - x^{*2}_j\right) F^{*\prime}_{\infty,\alpha}\left(x^*_j\right)} \}, \tag{1.6}$$

where ℓ is the even integer in $(\alpha - 1, \alpha + 1]$, and P^* is a polynomial of degree at most $\ell - 2$,

$$P^*(z) = \frac{2\sin\frac{\alpha}{2}\pi}{\pi} \sum_{j=0}^{\ell/2-1} \left(-z^2\right)^j \int_0^\infty \frac{s^{\alpha-1-2j}}{F^*_{\infty,\alpha}(is)} ds. \tag{1.7}$$

For $p = 1$, where $F^*_{\infty,\alpha}(z) = \cos(z)$, the representation is more explicit:

Theorem 1.3
Let $\alpha > -1$, not an even integer. Then

$$H_{1,\alpha}^{*}(z) = (\cos z)\{P^{*}(z) + 2z^{\ell}\sum_{j=1}^{\infty}\frac{(-1)^{j}\left(\left(j-\frac{1}{2}\right)\pi\right)^{\alpha-\ell+1}}{z^{2}-\left(\left(j-\frac{1}{2}\right)\pi\right)^{2}}\}, \tag{1.8}$$

where ℓ is the even integer in $(\alpha-1, \alpha+1]$, and P^{*} is a polynomial of degree at most $\ell-2$,

$$P^{*}(z) = \frac{2\sin\frac{\alpha}{2}\pi}{\pi}\sum_{j=0}^{\ell/2-1}\left(-z^{2}\right)^{j}\int_{0}^{\infty}\frac{s^{\alpha-1-2j}}{\cosh s}ds. \tag{1.9}$$

Remarks
(a) For α an odd integer, the case $p = 1$ may be deduced from results of Littman [7]. However, for all α, the result of Theorem 1.2 is known, and appears in a paper of Ganzburg [6]. The author is not sure where it first appeared.
(b) There are several ways to prove Theorems1.2 and 1.3. The simplest is based on the integral representation (1.2) and the residue theorem. Another approach is to take scaled limit of polynomials of best L_p approximation on $[-1, 1]$.
(c) When $\alpha = 1$, the representation simplifies to

$$H_{1,\alpha}^{*}(z) = \cos(z)\{1 + 2z\sum_{j=1}^{\infty}\frac{(-1)^{j}}{z^{2}-\left(\left(j-\frac{1}{2}\right)\pi\right)^{2}}\}.$$

(d) Let $\sigma > 0$. We note that because of homogeneity of $|x|^{\alpha}$, the entire function H of type $\leq \sigma$ that best approximates $|x|^{\alpha}$ in $L_p(\mathbb{R})$ is just

$$H(z) = \sigma^{-\alpha}H_{p,\alpha}^{*}(\sigma z), \tag{1.10}$$

This follows as $\sigma^{-\alpha}H(\sigma z)$ is of exponential type $\leq \sigma$ whenever H is of exponential type ≤ 1..
(e) One may also derive a representation for $H_{2,\alpha}^{*}$ based on the observation that the best L_2 polynomial approximation of $|x|^{\alpha}$ is just a partial sum of its orthonormal expansion in Legendre polynomials. For example, for $-\frac{1}{2} < \alpha < 1$, the representation is

$$H_{2,\alpha}^{*}(z) = \frac{2}{\pi}\int_{0}^{\infty}s^{\alpha}\mathbb{J}(x, s)\,ds,$$

where $\mathbb{J}(x,s)$ is a kernel familiar in universality theory,

$$\begin{aligned}\mathbb{J}(x,s) &= \frac{1}{2}\left[\frac{\sin(x+s)}{x+s}+\frac{\sin(x-s)}{x-s}\right] \\ &= \frac{x\sin x\cos s - s\sin s\cos x}{x^2-s^2}.\end{aligned}$$

There is an analogue for all $\alpha > 0$, and this provides a different representation to the Fourier transform one that may be derived from Raitsin's work.

We prove the results in Section 2. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of variables x, z, s, t and indices $j, k, m, n, \ldots$. The same symbol may denote different constants in different occurrences.

2. Proofs

We shall concentrate on the proof for $p = \infty$ and then indicate the differences for $p = 1$. We begin by summarizing results from [9].

Lemma 2.1
Let $\alpha > 0$, not an even integer. Then
(a) There exist alternation points $\left\{y_j^*\right\}_{j=0}^{\infty}$ with

$$0 = y_0^* < x_1^* < y_1^* < x_2^* < y_2^* < \ldots$$

and

$$y_j^{*\alpha} - H_{\infty,\alpha}^*\left(\pm y_j^*\right) = (-1)^{j+\overline{\frac{\alpha}{2}}}\left\||x^\alpha| - H_{\infty,\alpha}^*(x)\right\|_{L_\infty(\mathbb{R})}. \tag{2.1}$$

Here $\overline{\frac{\alpha}{2}}$ is the least integer exceeding $\frac{\alpha}{2}$.
(b) For $j \geq 1$,

$$(-1)^j F_{\infty,\alpha}^*\left(y_j^*\right) \geq C y_j^{*2}. \tag{2.2}$$

(c) There exist C_1 and C_2 such that for $|\mathrm{Im}\, z| \geq 1$,

$$\left|F_{\infty,\alpha}^*\right|(z) \geq C_1 |\cos z|\, |z|^{-C_2}. \tag{2.3}$$

Proof
(a) This is part of Theorem 1.2 in [9].
(b) In [9], we transformed approximation over the whole real line to the non-negative axis $[0,\infty)$, involving a factor $x^{-1/2p}$ in the L_p norm. In

the case of the L_∞ norm, this factor is 1, and we considered an entire function $H_{\infty,\alpha/2}$ such that

$$\left\|x^{\alpha/2}-H_{\infty,\alpha/2}(x)\right\|_{L_\infty[0,\infty)}=\inf\left\|x^{\alpha/2}-f(x)\right\|_{L_\infty[0,\infty)},$$

where the inf is over all entire functions f such that $f(x^2)$ is of exponential type ≤ 2. In Section 12 of [9], we showed that

$$H^*_{\infty,\alpha}(z)=2^\alpha H_{\infty,\alpha/2}\left(\left(\frac{z}{2}\right)^2\right).$$

$H_{\infty,\alpha/2}$ admits an integral representation like that in (1.2), involving a function $F_{\infty,\alpha/2}$ and also has alternation points $\{y_j\}$. The relationship between these quantities and the corresponding ones for $H^*_{\infty,\alpha}$ was established in [9, Section 12], and is

$$\begin{aligned} F^*_{\infty,\alpha}(z) &= F_{\infty,\alpha/2}\left(\left(\frac{z}{2}\right)^2\right); \\ y^*_j &= 2\sqrt{y_j}. \end{aligned} \tag{2.4}$$

It was also shown in [9, Theorem 10.2] that

$$(-1)^j F_{\infty,\alpha/2}(y_j)\ge Cy_j.$$

Then the identities (2.4) give the result.
(c) In [9, Theorem 7.2(d)], it was shown that

$$\left|\log\left|F_{\infty,\alpha/2}(z)\right|-\log\left|\cos\left(2\sqrt{z}\right)\right|\right|\le\left|\log|z-c|\right|+\log^+|z|+C,$$

where c is the closest zero of $\cos 2\sqrt{\cdot}$ or $F_{\infty,\alpha/2}(\cdot)$ to z. Now the substitution $z\to\left(\frac{z}{2}\right)^2$ easily yields (2.3). ■

Proof of Theorem 1.2
Assume that P^* is given by (1.7) and $H^*_{\infty,\alpha}$ by (1.2). We start with the identity

$$\sum_{j=0}^{\ell/2-1}\left(-\left(\frac{z}{s}\right)^2\right)^j=\frac{s^2}{z^2+s^2}+(-1)^{\ell/2+1}\frac{\left(\frac{z}{s}\right)^\ell s^2}{z^2+s^2},$$

which follows from the formula for a finite geometric sum. We multiply by $\frac{2\sin\frac{\alpha\pi}{2}}{\pi}s^{\alpha-1}/F^*_{\infty,\alpha}(is)$, integrate over $(0,\infty)$, and use (1.2) and (1.7). This gives for $\mathrm{Re}(z)>0$,

$$P^*(z)=\frac{H^*_{\infty,\alpha}(z)-z^\alpha}{F^*_{\infty,\alpha}(z)}+\frac{2\sin\frac{\alpha\pi}{2}}{\pi}(-1)^{\ell/2+1}z^\ell\int_0^\infty\frac{s^{\alpha-\ell+1}}{(z^2+s^2)F^*_{\infty,\alpha}(is)}ds. \tag{2.5}$$

We now express the integral in the last right-hand side as a contour integral over the imaginary axis, assuming that $\operatorname{Re}(z) > 0$. Since

$$i^{\alpha-\ell+1} + (-i)^{\alpha-\ell+1} = 2\sin\frac{\alpha\pi}{2}(-1)^{\ell/2+1},$$

we see that

$$\begin{aligned}
&\frac{2\sin\frac{\alpha\pi}{2}}{\pi}(-1)^{\ell/2+1}\int_0^\infty \frac{s^{\alpha-\ell+1}}{(z^2+s^2)F^*_{\infty,\alpha}(is)}ds \\
&= \frac{1}{\pi}\int_0^\infty \frac{(is)^{\alpha-\ell+1}+(-is)^{\alpha-\ell+1}}{\left(z^2-(is)^2\right)F^*_{\infty,\alpha}(is)}ds \\
&= \frac{1}{i\pi}\int_{\mathcal{L}} \frac{t^{\alpha-\ell+1}}{(z^2-t^2)F^*_{\infty,\alpha}(t)}dt, \qquad (2.6)
\end{aligned}$$

where $\mathcal{L}$ is the directed contour along the imaginary axis from $-i\infty$ to $i\infty$. We next re-express this as an infinite series. To start, let $R = y_n^*$ for some large n, and consider a rectangular positively oriented contour in the right-half plane determined as follows: let $\mathcal{L}_R$ be a vertical line segment along the imaginary axis from iR to $-iR$. Let $\mathcal{M}_R$ denote a vertical line segment from $R-iR$ to $R+iR$. Finally, let $\mathcal{H}_R$ denote the two horizontal line segments with $\operatorname{Im}(z) = \pm R$ joining $\mathcal{L}_R$ and $\mathcal{M}_R$. Note that $\mathcal{L}_R$ has the opposite orientation to $\mathcal{L}$. If R is large enough so that this contour encloses z, the integrand in (2.6) has simple poles at z and at $x_j^*, 1 \le j \le n$. The residue theorem gives

$$\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{\mathcal{L}_r} + \int_{\mathcal{M}_R} + \int_{\mathcal{H}_R}\right)\frac{t^{\alpha-\ell+1}}{(z^2-t^2)F^*_{\infty,\alpha}(t)}dt \\
&= -\frac{z^{\alpha-\ell+1}}{2zF^*_{\infty,\alpha}(z)} + \sum_{j=1}^n \frac{\left(x_j^*\right)^{\alpha-\ell+1}}{\left(z^2-x_j^{*2}\right)F^{*\prime}_{\infty,\alpha}\left(x_j^*\right)}. \qquad (2.7)
\end{aligned}$$

We next show that the integral over $\mathcal{M}_R \cup \mathcal{H}_R \to 0$ as $R = y_n^* \to \infty$. Firstly, since $F^*_{\infty,\alpha}$ has only real zeros, we see that for $t = R + is \in \mathcal{M}_R$,

$$\left|F^*_{\infty,\alpha}(R+is)\right| \ge \left|F^*_{\infty,\alpha}(R)\right| \ge CR^2,$$

by Lemma 2.1(b). Then

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{\mathcal{M}_R}\frac{t^{\alpha-\ell+1}}{(z^2-t^2)F^*_{\infty,\alpha}(t)}dt\right| &\le C\frac{R^{\alpha-\ell+1}}{R^2-|z|^2}\int_{-R}^R \frac{ds}{\left|F^*_{\infty,\alpha}\right|(R+is)} \\
&\le CR^{\alpha-\ell-2} \to 0,
\end{aligned}$$

as $\alpha-\ell-2<0$, recall our choice of ℓ. Next by Lemma 2.1(c), and as $|\cos(s\pm iR)|\geq|\sinh(R)|$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\mathcal{H}_R}\frac{t^{\alpha-\ell+1}}{(z^2-t^2)\,F^*_{\infty,\alpha}(t)}dt\right| &\leq C\frac{R^{\alpha-\ell+1}}{R^2-|z|^2}\int_0^R\frac{ds}{\left|F^*_{\infty,\alpha}\right|(s\pm iR)}\\ &\leq CR^{C_3}/\sinh(R)\to 0,\end{aligned}$$

$R\to\infty$. Thus, letting $R=y_n^*\to\infty$ in (2.7), and recalling the opposite orientation of $\mathcal{L}_R$ and $\mathcal{L}$ gives

$$-\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{t^{\alpha-\ell+1}}{(z^2-t^2)\cos(t)}dt=-\frac{z^{\alpha-\ell+1}}{2zF^*_{\infty,\alpha}(z)}+\sum_{j=1}^{\infty}\frac{\left(x_j^*\right)^{\alpha-\ell+1}}{\left(z^2-x_j^{*2}\right)F^{*\prime}_{\infty,\alpha}\left(x_j^*\right)}.$$

Combining this, (2.5) and (2.6), gives

$$P^*(z)=\frac{H^*_{\infty,\alpha}(z)-z^{\alpha}}{F^*_{\infty,\alpha}(z)}+\frac{z^{\alpha+1}}{zF^*_{\infty,\alpha}(z)}-2z^{\ell}\sum_{j=1}^{\infty}\frac{\left(x_j^*\right)^{\alpha-\ell+1}}{\left(z^2-x_j^{*2}\right)F^{*\prime}_{\infty,\alpha}\left(x_j^*\right)}.$$

So we obtain (1.6) for $\mathrm{Re}(z)>0$. Its validity follows for all complex z, by analytic continuation. ■

Proof of Theorem 1.3

Here we follow exactly the same steps, except that the calculations are easier because

$$F^*_{1,\alpha}(z)=\cos z,$$

and so the zeros $\left\{x_j^*\right\}$ are

$$x_j^*=\left(j-\frac{1}{2}\right)\pi, j\geq 1,$$

while

$$F^{*\prime}_{1,\alpha}\left(x_j^*\right)=(-1)^j.$$

For the dimensions of the contour above, we choose $R=n\pi$, and in estimating the integral over $\mathcal{M}_R$, use

$$|\cosh(R+is)|\geq\sinh(|s|)$$

as well as

$$|\cosh(R+is)|\geq\frac{1}{2}, |s|\leq C,$$

some C independent of $R=n\pi$. Then

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\mathcal{M}_R}\frac{t^{\alpha-\ell+1}}{(z^2-t^2)\cos(t)}dt\right| &\leq C\frac{R^{\alpha-\ell+1}}{R^2-|z|^2}\int_{-R}^{R}\frac{ds}{|\cos|(R+is)}\\ &\leq CR^{\alpha-\ell-1}\int_{-\infty}^{\infty}\frac{ds}{\max\left\{\frac{1}{2},\sinh(|s|)\right\}}\to 0,\end{aligned}$$

$R \to \infty$ as $\alpha - \ell - 1 < 0$. ■

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