Condition Numbers of Hankel Matrices for Exponential Weights

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Abstract

We obtain the rate of growth of the largest eigenvalues and Euclidean condition numbers of the Hankel matrices $\left(\int_I t^{j+k} W^2(t)\,dt\right)_{j,k=0}^n$ for a general class of even exponential weights $W^2 = \exp(-2Q)$ on an interval I. As particular examples, we discuss $Q(x) = |x|^\alpha$ on $I = \mathbb{R}$, and $Q(x) = \left(d^2 - x^2\right)^{-\alpha}$ on $I = [-d, d]$.

Remark 1 *Running Title: Condition Numbers of Hankel Matrices*

1 The Result

Let $I = (-d, d)$ where $0 < d \leq \infty$. Let $Q : I \to [0, \infty)$ be continuous and $W^2 = \exp(-2Q)$ be such that all the moments

$$\int_I t^j W^2(t)\,dt, j = 0, 1, 2, ...,$$

exist. Form the positive definite Hankel matrix

$$H_n = \left(\int_I t^{j+k} W^2(t)\, dt \right)_{j,k=0}^{n}$$

and denote its smallest eigenvalue by λ_n, and its largest eigenvalue by Λ_n. The focus of this paper is the rate of growth of the Euclidean condition number $\kappa_n(H_n)$ of H_n, defined by

$$\kappa(H_n) = \frac{\Lambda_n}{\lambda_n}. \qquad (1)$$

The condition number of H_n provides a measure of the sensitivity of solutions of equations $H_n \underline{x} = \underline{b}$ to perturbations of $\underline{b}$. This may be derived from the Rayleigh-Ritz formulation of $\kappa(H_n)$:

$$\kappa(H_n) = \sup_{\underline{x}} \frac{\| H_n \underline{x} \|}{\| \underline{x} \|} \Big/ \inf_{\underline{x}} \frac{\| H_n \underline{x} \|}{\| \underline{x} \|}, \qquad (2)$$

where both sup and inf are taken over all non-zero vectors $\underline{x} \in \mathbb{R}^{n+1}$. A special Hankel matrix is the Hilbert matrix

$$\left(\int_0^1 t^{j+k} dt \right)_{j,k=0}^{n},$$

whose condition number has been investigated by several authors [1], [10], [11], [13]. More generally, Beckermann examined how rapidly the condition number of

$$\left(\int_0^1 t^{j+k} d\mu(t) \right)_{j,k=0}^{n}$$

can grow when the measure μ is supported in a given interval. If we define $\Gamma_n^2([-1,1])$ to be the smallest possible condition number of such matrices when μ is supported in $[-1,1]$, Beckermann [1, p. 568] proved that

$$\frac{\left(1+\sqrt{2}\right)^{2n}}{8(n+1)} \le \Gamma_n^2([-1,1]) \le (n+1)\left(1+\sqrt{2}\right)^{2n}. \qquad (3)$$

Similar geometric growth is established there for more general intervals, and for Krylov and Vandermonde matrices.

Our focus here is to provide matching upper and lower bounds for $\kappa(H_n)$ when $W^2 = e^{-2Q}$ is an exponential weight. In an earlier paper, the author and Y. Chen [4] obtained upper and lower bounds for λ_n. Our main task here is then to obtain matching upper and lower bounds for Λ_n. Many authors have investigated the asymptotic behaviour of λ_n as $n \to \infty$: [12], [8], [2], [3]. As far as the author is aware, there is less work on Λ_n, though it is easier to analyze than λ_n.

Before we define our class of weights, which is the even case of the weights in [5], we need the notion of a quasi-increasing function. A function $g : (0,d) \to (0,\infty)$ is said to be *quasi-increasing* if there exists $C > 0$ such that

$$g(x) \leq Cg(y), 0 < x \leq y < d.$$

Note that any increasing function is quasi-increasing.

Definition 1.1 General Exponential Weights
Let $I = (-d,d)$, where $0 < d \leq \infty$ and let $W = e^{-Q}$ where $Q : I \to [0,\infty)$ is even and satisfies the following properties:
(a) Q' is continuous in I and $Q(0) = 0$;
(b) Q'' exists and is positive in $I \backslash \{0\}$;
(c)

$$\lim_{t \to d-} Q(t) = \infty; \tag{4}$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, t \neq 0$$

is quasi-increasing in $(0,d)$, with

$$T(t) \geq \Lambda > 1, t \in (0,d); \tag{5}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \text{ a.e. } x \in (0,d). \tag{6}$$

Then we write $W \in \mathcal{F}(C^2)$. If in addition, there exist $c \in (0,d)$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{Q'(x)} \geq C_2 \frac{Q'(x)}{Q(x)}, \text{ a.e. } x \in (c,d), \tag{7}$$

then we write $W \in \mathcal{F}(C^2+)$.

The simplest case of the above definition is when $I = \mathbb{R}$ and T is bounded, the so called Freud case. A typical example is

$$Q(x) = |x|^{\alpha}, x \in \mathbb{R},$$

where $\alpha > 1$. A more general example satisfying the requirements of Definition 1.1 is

$$Q(x) = \exp_{\ell}(|x|^{\alpha}) - \exp_{\ell}(0), \tag{8}$$

where $\alpha > 1$ and $\ell \geq 0$. Here we set $\exp_0(x) := x$ and for $\ell \geq 1$,

$$\exp_{\ell}(x) = \underbrace{\exp(\exp(\exp \ldots \exp(x))))}_{\ell \text{ times}}$$

is the ℓth iterated exponential.

An example on the finite interval $I = (-1, 1)$ is

$$Q(x) = \exp_{\ell}((1-x^2)^{-\alpha}) - \exp_{\ell}(1), x \in (-1, 1),$$

where $\alpha > 0$ and $\ell \geq 0$. Further examples are discussed in [5].

In analysis of exponential weights, an important role is played by the Mhaskar-Rakhmanov-Saff number $a_u \in (0, d), u > 0$, which is the unique root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u s Q'(a_u s)}{\sqrt{1-s^2}} ds. \tag{9}$$

One of the features that motivates their importance is the Mhaskar-Saff identity [6]

$$\| PW \|_{L_{\infty}(I)} = \| PW \|_{L_{\infty}[-a_n, a_n]},$$

valid for all polynomials P of degree $\leq n$. An older quantity is Freud's number q_u, the root of the equation

$$u = q_u Q'(q_u), u > 0.$$

A little calculus shows that q_n is the place where $x^n e^{-Q(x)}$ attains its maximum in $[0, \infty)$. It is easily seen that

$$q_u \leq a_u.$$

Indeed, if $xQ'(x)$ is strictly increasing, then (9) gives

$$q_u Q'(q_u) = u \leq a_u Q'(a_u).$$

Unfortunately there is no exact asymptotic relation between q_u and a_u for general exponential weights. Both q_u and a_u approach d as $u \to \infty$. For the special case $Q(x) = |x|^\alpha$ on $I = \mathbb{R}$, we have [6]

$$q_u = \left(\frac{u}{\alpha}\right)^{1/\alpha} < C_\alpha u^{1/\alpha} = a_u, u > 0, \tag{10}$$

where

$$C_\alpha = \left(\frac{2^{\alpha-2}\Gamma(\alpha/2)^2}{\Gamma(\alpha)}\right)^{1/\alpha}. \tag{11}$$

Throughout, $C, C_1, C_2, ...$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda), C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. Given sequences of real numbers (c_n) and (d_n) we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that

$$C_1 \leq c_n/d_n \leq C_2$$

for the relevant range of n. Similar notation is used for functions and sequences of functions.We shall prove:

Theorem 1.2
Let W be even and $W \in \mathcal{F}(C^2+)$.
(a) If $d \leq 1$, then for $n \geq 1$,

$$\Lambda_n \sim 1. \tag{12}$$

If $d > 1$, then for $n \geq 1$,

$$\Lambda_n \sim n^{-1} q_n^{2n+1} Q(q_n)^{1/2} e^{-2Q(q_n)}. \tag{13}$$

(b) If $d \leq 1$, then for $n \geq 1$,

$$\kappa(H_n) \sim \sqrt{\frac{a_n}{n}} \exp\left(2\int_0^n \log\left[\frac{1}{a_s} + \sqrt{1 + \frac{1}{a_s^2}}\right] ds\right). \tag{14}$$

If $d>1$, then for $n\geq 1$,and

$$\kappa\left(H_n\right)\sim\exp\left(2\int_0^n\log\left[\frac{1}{a_s}+\sqrt{1+\frac{1}{a_s^2}}\right]ds\right)$$
$$\times n^{-3/2}q_n^{2n+3/2}Q\left(q_n\right)^{1/2}\exp\left(-2Q\left(q_n\right)\right). \qquad (15)$$

It may seem strange that we need both q_n and a_n in describing the asymptotic, but q_n^{2n} may have a very different rate of growth from that of a_n^{2n} for some Q - see for example, Lemmas 3.1 and 3.3 below.

Example 1
Let $\alpha>1$ and

$$Q\left(x\right)=\left|x\right|^{\alpha},x\in\mathbb{R}.$$

Here we have the identities (10) and (11) for q_u and a_u. Theorem 1.2 gives

$$\Lambda_n\sim n^{-\frac{1}{2}+\frac{1}{\alpha}}\left(\frac{n}{\alpha e}\right)^{\frac{2n}{\alpha}}.$$

(A sharper asymptotic for Λ_n will be given after Theorem 1.3). If α is not an odd integer, it was shown in [4] that

$$\lambda_n\sim n^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}\exp\left(-2n\sum_{k=0}^{\left[\frac{\alpha-1}{2}\right]}\left(-1\right)^k\frac{(2k)!}{2^{2k}\left(k!\right)^2\left(2k+1\right)}\frac{a_n^{-2k-1}}{1-\frac{2k+1}{\alpha}}\right).$$

Here $[x]$ denotes the greatest integer $\leq x$. Hence

$$\kappa\left(H_n\right)=\Lambda_n/\lambda_n\sim\exp\left(\begin{array}{c}\frac{2n}{\alpha}\log\frac{n}{\alpha e}+n^{1-\frac{1}{\alpha}}\frac{2C_\alpha^{-1}}{1-\frac{1}{\alpha}}\\+2n\sum_{k=1}^{\left[\frac{\alpha-1}{2}\right]}\left(-1\right)^k\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\frac{a_n^{-2k-1}}{1-\frac{2k+1}{\alpha}}+\left(-1+\frac{3}{2\alpha}\right)\log n\end{array}\right).$$

The leading order term clearly comes from the largest eigenvalue Λ_n:

$$\kappa\left(H_n\right)=\exp\left(\frac{2n}{\alpha}\log\frac{n}{\alpha e}+n^{1-\frac{1}{\alpha}}\frac{2C_\alpha^{-1}}{1-\frac{1}{\alpha}}+\text{lower order terms}\right). \qquad (16)$$

In particular, for the Hermite weight $\alpha=2$, this gives

$$\kappa\left(H_n\right)\sim n^{-\frac{1}{4}}\exp\left(n\log\frac{n}{2e}+4\sqrt{n}\right).$$

A sharper estimate than this for the Hermite weight was already known to Szego [8, p. 668]. When α is an odd integer, there is an extra term in the asymptotic for λ_n,

$$\lambda_n \sim n^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \exp \left(\begin{array}{c} -2n \sum_{k=0}^{\left[\frac{\alpha-3}{2}\right]} (-1)^k \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \frac{a_n^{-2k-1}}{1-\frac{2k+1}{\alpha}} \\ -2\left(\log n\right)(-1)^{\frac{\alpha-1}{2}} \frac{(\alpha-1)!}{2^{\alpha-1}\left(\frac{\alpha-1}{2}!\right)^2 \alpha} C_\alpha^{-\alpha} \end{array} \right)$$

but this is a lower order term, and (16) persists.

Example 2
Let $\alpha > 0$ and

$$Q(x) = \left(1-x^2\right)^{-\alpha}, x \in (-1,1).$$

Let

$$D_\alpha = \left[2^{-\alpha+\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha)}\right]^{\frac{1}{\alpha+\frac{1}{2}}}.$$

Since here $d = 1$, we have

$$\Lambda_n \sim 1.$$

In Section 3, we shall show that

$$\kappa(H_n) \sim \lambda_n^{-1} \sim \left(1+\sqrt{2}\right)^{2n} n^{-1/2} \exp\left(\sqrt{2} D_\alpha \left(\int_1^n s^{-\frac{1}{\alpha+\frac{1}{2}}} ds\right)(1+o(1))\right).$$

This should be compared to Beckermann's result (3). In particular, if
(a) $\alpha < \frac{1}{2}$

$$\kappa(H_n) \sim \left(1+\sqrt{2}\right)^{2n} n^{-1/2};$$

(b) $\alpha = \frac{1}{2}$

$$\kappa(H_n) \sim \left(1+\sqrt{2}\right)^{2n} n^{\left(-\frac{1}{2}+\frac{\sqrt{2}}{\pi}\right)(1+o(1))};$$

(c) $\alpha > \frac{1}{2}$

$$\kappa(H_n) \sim \left(1+\sqrt{2}\right)^{2n} n^{-1/2} \exp\left(\sqrt{2} D_\alpha \frac{2\alpha+1}{2\alpha-1} n^{\frac{2\alpha-1}{2\alpha+1}} (1+o(1))\right).$$

Example 3
Now let $\alpha > 0, d > 1$, and

$$Q(x) = \left(d^2-x^2\right)^{-\alpha}, x \in (-d,d).$$

Also let

$$D_\alpha = \left[(2d)^{-\alpha+\frac{1}{2}} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha)}\right]^{\frac{1}{\alpha+\frac{1}{2}}}; \tag{17}$$

$$E_\alpha = \left(\frac{d^{2\alpha}}{2\alpha}\right)^{\frac{-1}{1+\alpha}}. \tag{18}$$

In Section 3, we show that

$$\Lambda_n \sim d^{2n} n^{-\frac{1}{2}\left(\frac{\alpha+2}{\alpha+1}\right)} \exp\left(-A_1 n^{\frac{\alpha}{1+\alpha}} - A_2 n^{\frac{\alpha-1}{\alpha+1}} + O\left(n^{\frac{\alpha-2}{\alpha+1}}\right)\right), \tag{19}$$

where

$$A_1 = 2(1+\alpha) d^{-2\alpha} E_\alpha^{-\alpha}; \tag{20}$$

$$A_2 = \alpha d^{-2\alpha} E_\alpha^{1-\alpha}. \tag{21}$$

(Of course, if $\alpha > 2$, the order term in the exponent in (19) will swamp the power of n outside). There we also show that

$$\lambda_n^{-1} \sim n^{-\frac{1}{2}} \left(\frac{1}{d} + \sqrt{1+\frac{1}{d^2}}\right)^{2n} \exp\left((1+o(1)) B_1 \int_1^n s^{-\frac{1}{\alpha+\frac{1}{2}}} ds\right),$$

where

$$B_1 = \frac{2}{d^2\sqrt{1+d^{-2}}} D_\alpha. \tag{22}$$

Then

$$\kappa(H_n) = \Lambda_n/\lambda_n \sim \left(1+\sqrt{d^2+1}\right)^{2n} n^{-\frac{1}{2}\left(\frac{2\alpha+3}{\alpha+1}\right)} \times \exp\left(-A_1 n^{\frac{\alpha}{1+\alpha}} - A_2 n^{\frac{\alpha-1}{\alpha+1}} + (1+o(1)) B_1 \int_1^n s^{-\frac{1}{\alpha+\frac{1}{2}}} ds + O\left(n^{\frac{\alpha-2}{\alpha+1}}\right)\right).$$

Again the dominant term is the geometric factor arising from geometric factors in Λ_n and λ_n^{-1}.

If $I = \mathbb{R}$ and we assume more smoothness of Q, we can obtain finer asymptotics for the largest eigenvalue:

Theorem 1.3
Let W be even and $W \in \mathcal{F}(C^2+)$. Assume in addition that $d = \infty$, and that for some $c \in (0, \infty)$, Q''' exists in (c, ∞) and satisfies there

$$\left|\frac{Q'''(x)}{Q'(x)}\right| \leq C \left(\frac{Q'(x)}{Q(x)}\right)^2. \tag{23}$$

Let

$$T_1(x) = 1 + \frac{xQ''(x)}{Q'(x)}, x \in (0, \infty). \tag{24}$$

Then as $n \to \infty$,

$$\Lambda_n = q_n^{2n+1} e^{-2Q(q_n)} \sqrt{\frac{\pi}{nT_1(q_n)}} (1 + o(1)). \tag{25}$$

Note that we do not have an asymptotic of matching precision for λ_n.

Example 4
Let $\alpha > 1$ and

$$Q(x) = |x|^\alpha, x \in \mathbb{R}.$$

Here

$$T_1(x) = \alpha \text{ in } (0, \infty)$$

so we obtain

$$\Lambda_n = \sqrt{\frac{\pi}{\alpha}} \alpha^{-\frac{1}{\alpha}} n^{-\frac{1}{2}+\frac{1}{\alpha}} \left(\frac{n}{\alpha e}\right)^{\frac{2n}{\alpha}} (1 + o(1)).$$

I am not sure if this is known. In this special case $T = T_1$ identically. However, they are different in general, although our hypotheses ensure that $T(x) \sim T_1(x)$ for large x. Other Q to which Theorem 1.3 may be applied include that in (8).

This paper is organised as follows: in Section 2, we prove Theorems 1.2 and 1.3. In Section 3, we present the calculations for Examples 2 and 3.

2 Proof of Theorems 1.2 and 1.3

We begin with some simple estimates. Related estimates appear in [9] and [13, Section 3.5]. Throughout we assume that $W \in \mathcal{F}(C^2)$ and we use the notation

$$\mu_j = \int_I t^j W^2(t)\,dt, j \geq 0.$$

Lemma 2.1
(a)

$$\max_{0 \leq j \leq n} \mu_{2j} \leq \Lambda_n \leq \sum_{j=0}^{n} \mu_{2j}. \tag{26}$$

(b) For $n \geq 1$,

$$\Lambda_n \sim 1 + \mu_{2n}. \tag{27}$$

(c) If $d = \infty$,

$$\lim_{n \to \infty} \Lambda_n / \mu_{2n} = 1. \tag{28}$$

Proof
(a) We begin with the Rayleigh-Ritz formula

$$\Lambda_n = \sup_{\underline{x} \neq 0} \frac{\underline{x}^T H_n \underline{x}}{\underline{x}^T \underline{x}}, \tag{29}$$

where the sup is taken over all $\underline{x} \neq 0$ in $\mathbb{R}^{n+1}$. Taking $\underline{x}$ to have a 1 in the $(j+1)$th position, and 0's elsewhere, gives

$$\Lambda_n \geq \mu_{2j},$$

for $0 \leq j \leq n$. Then the left inequality in (26) follows. Next, if

$$\underline{x} = [x_0\ x_1\ x_2\ ...x_n]^T,$$

we see that

$$\begin{aligned} \underline{x}^T H_n \underline{x} &= \int_{-d}^{d} \left(\sum_{j=0}^{n} x_j t^j \right)^2 W^2(t)\,dt \\ &\leq \int_{-d}^{d} \left(\sum_{j=0}^{n} x_j^2 \right) \left(\sum_{j=0}^{n} t^{2j} \right) W^2(t)\,dt \\ &= \left(\sum_{j=0}^{n} \mu_{2j} \right) \underline{x}^T \underline{x} \end{aligned}$$

and then the right-hand inequality in (26) follows.
(b) We consider two cases:
(A) $d > 1$
Let $1 < \alpha < \beta < d$. Then

$$\begin{aligned}
\sum_{j=0}^{n} \mu_{2j} &= 2\left(\int_0^\alpha + \int_\alpha^d\right)\left(\sum_{j=0}^{n} t^{2j}\right) W^2(t)\,dt \\
&\leq 2(n+1)\,\alpha^{2n}\int_0^\alpha W^2(t)\,dt + 2\int_\alpha^d t^{2n}\left(\sum_{j=0}^{n} t^{-2j}\right) W^2(t)\,dt \\
&\leq 2(n+1)\left(\frac{\alpha}{\beta}\right)^{2n}\frac{\int_0^\alpha W^2(t)\,dt}{\int_\beta^d W^2(t)\,dt}\int_\beta^d t^{2n} W^2(t)\,dt \\
&\quad + \frac{2}{1-\alpha^{-2}}\int_\alpha^d t^{2n} W^2(t)\,dt.
\end{aligned}$$

Since $\alpha/\beta < 1$, we see that

$$\limsup_{n\to\infty}\left(\sum_{j=0}^{n}\mu_{2j}\right)/\mu_{2n} \leq \frac{1}{1-\alpha^{-2}}.$$

Here this is valid for any $1 < \alpha < d$, so we obtain from (a),

$$\limsup_{n\to\infty}\Lambda_n/\mu_{2n} \leq \frac{1}{1-d^{-2}}. \tag{30}$$

(If $d = \infty$, we interpret $d^{-2} = 0$.) Moreover, from (a),

$$\liminf_{n\to\infty}\Lambda_n/\mu_{2n} \geq 1. \tag{31}$$

Since μ_{2n} grows to ∞ as $n \to \infty$ in this case, we then obtain the result.
(B) $d \leq 1$
Here we use the inequality

$$\sum_{j=0}^{n} t^{2j} \leq \frac{1}{1-t^2}, t \in [0,1)$$

to obtain

$$\sum_{j=0}^{2n}\mu_{2j} \leq 2\int_0^d \frac{1}{1-t^2}e^{-2Q(t)}dt.$$

If $d < 1$, the integral is trivially finite. If $d = 1$, the integral on the right converges, since for some $\rho > 0, C > 0$,

$$Q(t) \geq C(1-t)^{-\rho}, t \in (C, 1).$$

See Lemma 3.2(f) in [5, p. 65]. Thus in this case

$$\Lambda_n \leq C_1, n \geq 1.$$

In the other direction, we have

$$\Lambda_n \geq \mu_0 > 0,$$

so

$$\Lambda_n \sim 1 \sim 1 + \mu_{2n}.$$

(c) This follows directly from (30) and (31). ■

Next, we present some technical estimates.

Lemma 2.2
(a)

$$T(q_n) = \frac{n}{Q(q_n)} = o(n), n \to \infty. \tag{32}$$

(b) Fix $\beta \in (0,1)$. For $j = 0, 1$ and $n \geq 1$,

$$Q^{(j)}(q_n) \sim Q^{(j)}(q_{\beta n}). \tag{33}$$

Moreover,

$$T(q_n) \sim T(q_{\beta n}) \tag{34}$$

and

$$q_n \sim q_{\beta n}. \tag{35}$$

(c) There exists n_0 such that uniformly for $n \geq n_0$, and $\gamma \in (1, 2]$,

$$q_{\gamma n} - q_n \sim (\log \gamma) \frac{q_n}{T(q_n)}. \tag{36}$$

(d) Fix $n \geq 1$ and let

$$f(t) = 2n \log t - 2Q(t), t \in (0, d).$$

Then f has a maximum at $t = q_n$, and f' is positive and decreasing in $(0, q_n)$ and negative and decreasing in (q_n, d).

Proof

(a) Now

$$T(q_n) = \frac{q_n Q'(q_n)}{Q(q_n)} = \frac{n}{Q(q_n)}.$$

Here $xQ'(x)$ is continuous in $[0, d)$, and hence finite valued there. It also approaches ∞ as $x \to d-$. So necessarily $q_n \to d-$ as $n \to \infty$. Then (c) of Definition 1.1 shows that $Q(q_n) \to \infty$ as $n \to \infty$. So we obtain (32).

(b) Firstly as T is quasi-increasing, and $q_{\beta n} \leq q_n$,

$$Q(q_{\beta n}) \leq Q(q_n) = \frac{n}{T(q_n)} \leq C \frac{\beta n}{T(q_{\beta n})} = CQ(q_{\beta n}).$$

So (33) is true for $j = 0$. Then (32) gives (34) for $T(q_n)$. Next, as Q' is increasing, as is q_n,

$$Q'(q_{\beta n}) \leq Q'(q_n) = \frac{n}{q_n} \leq \beta \frac{n/\beta}{q_{\beta n}} = \beta Q'(q_{\beta n}).$$

So (33) is true for $j = 1$. Finally,

$$q_{\beta n} = \frac{T(q_{\beta n}) Q(q_{\beta n})}{Q'(q_{\beta n})} \sim \frac{T(q_n) Q(q_n)}{Q'(q_n)} = q_n,$$

so we have (35).

(c) Let T_1 be defined by (24). For some $c \in (0, d)$, (6) and (7) show that

$$T_1(x) \sim T(x), x \in (c, d).$$

Differentiating the relation

$$q_u Q'(q_u) = u$$

leads to, for large enough u,

$$\frac{q_u'}{q_u} = \frac{1}{uT_1(q_u)} \sim \frac{1}{uT(q_u)}.$$

Then

$$\log \frac{q_{\gamma u}}{q_u} = \int_u^{\gamma u} \frac{dt}{tT_1(q_t)} \sim \frac{\log \gamma}{T(q_u)},$$

recall (34). This holds uniformly in $\gamma \in (1,2]$ and u large enough. Then

$$q_{\gamma u} - q_u = q_u \left[\frac{q_{\gamma u}}{q_u} - 1 \right] \sim (\log \gamma) \frac{q_u}{T(q_u)}.$$

(d) We see that

$$f'(t) = \frac{2}{t} \left[n - tQ'(t) \right]$$

and

$$f''(t) = -\frac{2n}{t^2} - 2Q''(t).$$

These two relations imply the result. ■

Proof of Theorem 1.2(a)

If $d \leq 1$, this follows immediately from Lemma 2.1(b). So we assume that $d > 1$. We shall use a crude form of Laplace's method in estimating μ_{2n}. Let

$$f(t) = 2n \log t - 2Q(t), t \in [0, d),$$

so that

$$\begin{aligned} \mu_{2n} &= \left(\int_0^{q_{n/2}} + \int_{q_{n/2}}^{q_{2n}} + \int_{q_{2n}}^{d} \right) e^{f(t)} dt \\ &= : I_1 + I_2 + I_3. \end{aligned}$$

Estimation of I_2

The maximum of f occurs at $t = q_n$, and the main contribution comes from I_2. For $t \in \left[q_{n/2}, q_{2n} \right]$, we expand

$$f(t) = f(q_n) + 0 + \frac{1}{2} f''(\xi) (t - q_n)^2,$$

where ξ is between t and q_n. Here using our hypotheses (6), (7) on Q'', we have (at least for n large enough),

$$Q''(\xi) \sim \frac{Q'(\xi)^2}{Q(\xi)},$$

and since both Q and Q' are increasing, and for $j = 0, 1$,

$$Q^{(j)}\left(q_{n/2} \right) \sim Q^{(j)}\left(q_{2n} \right),$$

we obtain (uniformly in $n, t \in \left[q_{n/2}, q_{2n}\right], \xi$ between t and q_n),

$$Q''(\xi) \sim \frac{Q'(q_n)^2}{Q(q_n)} = \frac{n}{q_n^2} T(q_n),$$

so (uniformly in n, t, ξ),

$$f''(\xi) = -\frac{2n}{\xi^2} - 2Q''(\xi) \sim -\frac{n}{q_n^2} T(q_n).$$

(Recall that $T \geq \Lambda > 1$ and $q_n \sim q_{2n}$). We use these estimates in

$$I_2 = e^{f(q_n)} \int_{q_{n/2}}^{q_{2n}} e^{\frac{1}{2} f''(\xi)(t-q_n)^2} dt.$$

Making the substitution $u = C\sqrt{\frac{n}{q_n^2} T(q_n)}\,(t - q_n)$ (with different C for lower and upper bounds) gives

$$I_2 \sim \left(e^{f(q_n)} / \sqrt{\frac{n}{q_n^2} T(q_n)}\right) \int_{C\sqrt{\frac{n}{q_n^2} T(q_n)}\left(q_{n/2}-q_n\right)}^{C\sqrt{\frac{n}{q_n^2} T(q_n)}(q_{2n}-q_n)} e^{-\frac{1}{2}u^2 du}.$$

Here by Lemma 2.2(c), (d) as $n \to \infty$,

$$\sqrt{\frac{n}{q_n^2} T(q_n)}\left(q_{n/2} - q_n\right) \sim -\sqrt{\frac{n}{T(q_n)}} = -\sqrt{Q(q_n)} \to -\infty.$$

Similarly the upper limit of integration approaches ∞. Thus

$$\begin{aligned} I_2 &\sim \left(e^{f(q_n)} / \sqrt{\frac{n}{q_n^2} T(q_n)}\right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \\ &\sim q_n^{2n+1} e^{-2Q(q_n)} \left(nT(q_n)\right)^{-1/2}. \end{aligned} \tag{37}$$

Here

$$T(q_n) = \frac{n}{Q(q_n)},$$

so

$$I_2 \sim n^{-1} q_n^{2n+1} Q(q_n)^{1/2} e^{-2Q(q_n)}. \tag{38}$$

Estimation of I_1
Since f' is decreasing and positive in $[0, q_{n/2})$, we obtain for t there,

$$f(t) - f\left(q_{n/2}\right) \leq f'(q_{n/2})(t - q_{n/2})$$

so

$$\begin{aligned} I_1 &\leq e^{f(q_{n/2})} \int_0^{q_{n/2}} e^{f'(q_{n/2})(t-q_{n/2})} dt \\ &\leq e^{f(q_{n/2})}/f'(q_{n/2}). \end{aligned}$$

Here for any $\beta > 0$,

$$\begin{aligned} & f'(q_{\beta n}) \\ &= \frac{2n}{q_{\beta n}} - 2Q'(q_{\beta n}) \\ &= \frac{2n}{q_{\beta n}} - \frac{2\beta n}{q_{\beta n}} = 2(1-\beta)\frac{n}{q_{\beta n}}. \end{aligned} \tag{39}$$

Combining the above inequalities gives

$$I_1 \leq C\frac{q_n}{n} e^{f(q_{n/2})}$$

so from (37),

$$I_1/I_2 \leq C\sqrt{\frac{T(q_n)}{n}} e^{f(q_{n/2})-f(q_n)}.$$

Here

$$\begin{aligned} f(q_{n/2}) - f(q_n) &\leq f(q_{n/2}) - f(q_{3n/4}) \\ &\leq f'(q_{3n/4})(q_{n/2} - q_{3n/4}) \\ &\leq -C\frac{n}{q_n}\frac{q_n}{T(q_n)}, \end{aligned}$$

by (39), (36). Thus

$$\begin{aligned} I_1/I_2 &\leq C\sqrt{\frac{T(q_n)}{n}} \exp\left(-C_1\frac{n}{T(q_n)}\right) \\ &= CQ(q_n)^{-1/2} \exp(-C_1 Q(q_n)) \\ &\to 0, n \to \infty. \end{aligned} \tag{40}$$

Estimation of I_3

This is similar to that of I_1, but we provide the details. Since f' is decreasing and negative in $[q_{2n}, d)$, we obtain for t there,

$$f(t) - f(q_{2n}) \leq f'(q_{2n})(t - q_{2n})$$

so

$$\begin{aligned} I_3 &\leq e^{f(q_{2n})} \int_{q_{2n}}^{d} e^{f'(q_{2n})(t-q_{2n})} dt \\ &\leq e^{f(q_{2n})} / \left| f'(q_{2n}) \right| = \frac{q_{2n}}{2n} e^{f(q_{2n})}, \end{aligned}$$

by (39). Then

$$I_3/I_2 \leq C\sqrt{\frac{T(q_n)}{n}} e^{f(q_{2n})-f(q_n)}.$$

Here

$$\begin{aligned} f(q_{2n}) - f(q_n) &\leq f(q_{2n}) - f\left(q_{3n/2}\right) \\ &\leq f'(q_{3n/2})(q_{2n} - q_{3n/2}) \\ &\leq -C\frac{n}{q_n}\frac{q_n}{T(q_n)}, \end{aligned}$$

by (39), (36). Thus

$$\begin{aligned} I_3/I_2 &\leq C\sqrt{\frac{T(q_n)}{n}} \exp\left(-C_1 \frac{n}{T(q_n)}\right) \\ &= CQ(q_n)^{-1/2} \exp\left(-C_1 Q(q_n)\right) \to 0, n \to \infty. \end{aligned} \tag{41}$$

Completion of the proof

The above estimates and the positivity of the integrand give

$$\mu_{2n} = I_2(1 + o(1)) \sim n^{-1} q_n^{2n+1} Q(q_n)^{1/2} e^{-2Q(q_n)}.$$

Then Lemma 2.1(b) gives (13). ■

Proof of Theorem 1.2(b)

It was shown in Theorem 1.2 of [4] that

$$\lambda_n \sim \sqrt{\frac{n}{a_n}} \exp\left(-2\int_0^n \log\left(\frac{1}{a_s} + \sqrt{1 + \frac{1}{a_s^2}}\right) ds\right).$$

If $d \leq 1$, we then obtain

$$\kappa(H_n) = \Lambda_n/\lambda_n \sim \lambda_n^{-1},$$

and (14) follows. If $d > 1$, we use instead (13) for Λ_n to obtain (15). Note here that $a_n \sim q_n, n \geq 1$. ■

Proof of Theorem 1.3

Let us assume the notation for $I_j, j = 1, 2, 3$ above. We already know from the proof of Theorem 1.2 and Lemma 2.1(c) that

$$\Lambda_n = \mu_{2n}\left(1 + o\left(1\right)\right) = I_2\left(1 + o\left(1\right)\right). \tag{42}$$

Now we choose small $\varepsilon \in \left(0, \frac{1}{2}\right)$ and split

$$\begin{aligned} I_2 &= \left(\int_{q_{n/2}}^{q_n\left(1-\frac{\varepsilon}{T(q_n)}\right)} + \int_{q_n\left(1-\frac{\varepsilon}{T(q_n)}\right)}^{q_n\left(1+\frac{\varepsilon}{T(q_n)}\right)} + \int_{q_n\left(1+\frac{\varepsilon}{T(q_n)}\right)}^{q_{2n}}\right) e^{f(t)}dt \\ &= : I_{21} + I_{22} + I_{23}. \end{aligned} \tag{43}$$

Using Lemma 2.2(c), and also (33), (34), we see that for some $\beta < 1$, and n large enough,

$$q_n\left(1 - \frac{\varepsilon}{T\left(q_n\right)}\right) \leq q_{\beta n}$$

and

$$q_n\left(1 + \frac{\varepsilon}{T\left(q_n\right)}\right) \geq q_{\beta^{-1}n}.$$

Then almost exactly as in the proof of Theorem 1.2, for I_1 and I_3,

$$I_{21}/I_2 \to 0, n \to \infty \text{ and } I_{23}/I_2 \to 0 \text{ as } n \to \infty. \tag{44}$$

The main contribution comes from I_{22}. Let

$$\mathcal{I}_n = \left(q_n\left(1 - \frac{\varepsilon}{T\left(q_n\right)}\right), q_n\left(1 + \frac{\varepsilon}{T\left(q_n\right)}\right)\right).$$

For $t \in \mathcal{I}_n$, we expand

$$f\left(t\right) = f\left(q_n\right) + 0 + \frac{f''\left(q_n\right)}{2}\left(t - q_n\right)^2 + \frac{f'''\left(\xi\right)}{6}\left(t - q_n\right)^3,$$

where ξ is between t and q_n. We shall show that for $t \in \mathcal{I}_n$, the term involving f''' is small. A calculation (recall (24)) shows that

$$f''\left(q_n\right) = -\frac{2n}{q_n^2}T_1\left(q_n\right).$$

Also, by our hypothesis (23),

$$\begin{aligned}\left|Q'''(\xi)\right| &\leq CQ'(\xi)\left(\frac{Q'(\xi)}{Q(\xi)}\right)^2 = CQ'(\xi)\left(\frac{T(\xi)}{\xi}\right)^2 \\ &\leq C_1 Q'(q_n)\left(\frac{T(q_n)}{q_n}\right)^2 = C_1\frac{n}{q_n^3}T(q_n)^2,\end{aligned}$$

by Lemma 2.2 (b). Then

$$\left|f'''(\xi)\right| = \left|\frac{4n}{\xi^3} - 2Q'''(\xi)\right| \leq C\frac{n}{q_n^3}T(q_n)^2,$$

recall that T is bounded below. Then for $t \in \mathcal{I}_n$,

$$\left|\frac{f'''(\xi)}{6}(t-q_n)^3\right| / \left|\frac{f''(q_n)}{2}(t-q_n)^2\right| \leq C\frac{T(q_n)^2}{q_n T_1(q_n)}\frac{\varepsilon q_n}{T(q_n)} \leq C_2\varepsilon,$$

since (6) and (7) show that $T(q_n) \sim T_1(q_n)$ for large enough n. Here it is crucial that C_2 is independent of ε, n, t. Thus uniformly in n and $t \in \mathcal{I}_n$,

$$f(t) = f(q_n) + \frac{f''(q_n)}{2}(t-q_n)^2(1+\Delta),$$

where $\Delta = \Delta(n,t)$ and satisfies

$$|\Delta| \leq C_2\varepsilon. \tag{45}$$

Then the substitution $v = \sqrt{\frac{|f''(q_n)|}{2}}(t-q_n)$ gives

$$\begin{aligned}I_{22} &= e^{f(q_n)}\int_{q_n\left(1-\frac{\varepsilon}{T(q_n)}\right)}^{q_n\left(1+\frac{\varepsilon}{T(q_n)}\right)} e^{\frac{f''(q_n)}{2}(t-q_n)^2(1+\Delta)}dt \\ &= e^{f(q_n)}\sqrt{\frac{2}{|f''(q_n)|}}\int_{-\sqrt{\frac{|f''(q_n)|}{2}}\frac{\varepsilon q_n}{T(q_n)}}^{\sqrt{\frac{|f''(q_n)|}{2}}\frac{\varepsilon q_n}{T(q_n)}} e^{-v^2(1+\Delta)}dv.\end{aligned}$$

Here

$$\sqrt{\frac{|f''(q_n)|}{2}}\frac{\varepsilon q_n}{T(q_n)} = \sqrt{\frac{n}{T(q_n)}}\sqrt{\frac{T_1(q_n)}{T(q_n)}}\varepsilon \to \infty, n \to \infty,$$

by (32). We obtain in view of (44), (45),

$$\limsup_{n\to\infty} I_2 / \left[e^{f(q_n)} \sqrt{\frac{2}{|f''(q_n)|}} \right] \le \int_{-\infty}^{\infty} e^{-v^2(1-C_2\varepsilon)} dv;$$
$$\liminf_{n\to\infty} I_2 / \left[e^{f(q_n)} \sqrt{\frac{2}{|f''(q_n)|}} \right] \ge \int_{-\infty}^{\infty} e^{-v^2(1+C_2\varepsilon)} dv.$$

As $\varepsilon > 0$ is arbitrary and both I_2 and C_2 are independent of ε, we obtain

$$I_2 / \left[e^{f(q_n)} \sqrt{\frac{2}{|f''(q_n)|}} \right] = \sqrt{\pi} \left(1 + o(1)\right),$$

whence

$$I_2 = \frac{q_n^{2n+1}}{\sqrt{nT_1(q_n)}} e^{-2Q(q_{2n})} \sqrt{\pi} \left(1 + o(1)\right).$$

■

3 Calculations for Examples 2 and 3

Throughout this section, we let

$$Q(x) = \left(d^2 - x^2\right)^{-\alpha}, x \in (-d, d).$$

Moreover, we let D_α and E_α be given by (17) and (18). Note that when $d = 1$, this agrees with the choice of D_α in Example 2. First we describe the asymptotic behaviour of q_n as $n \to \infty$.

Lemma 3.1
Let

$$\varepsilon_n = 1 - \left(\frac{q_n}{d}\right)^2.$$

Then

$$\varepsilon_n = E_\alpha n^{-\frac{1}{1+\alpha}} \left(1 - \frac{E_\alpha}{1+\alpha} n^{-\frac{1}{1+\alpha}} + O\left(n^{-\frac{2}{1+\alpha}}\right)\right). \tag{46}$$

Proof
We have

$$\begin{aligned} n &= q_n Q'(q_n) = 2\alpha q_n^2 (d^2 - q_n^2)^{-\alpha-1} \\ &= 2\alpha d^{-2\alpha} (1 - \varepsilon_n) \varepsilon_n^{-\alpha-1}. \end{aligned} \tag{47}$$

Then from (18),

$$\begin{aligned}\delta_n &:= n^{-\frac{1}{1+\alpha}} E_\alpha = \varepsilon_n \left(1-\varepsilon_n\right)^{-\frac{1}{1+\alpha}} \\ &= \varepsilon_n \left(1 + \frac{1}{1+\alpha}\varepsilon_n + O\left(\varepsilon_n^2\right)\right). \end{aligned} \tag{48}$$

Write for some $c_n \in \mathbb{R}$,

$$\varepsilon_n = \delta_n \left(1 + c_n \delta_n\right). \tag{49}$$

Substituting in (48) gives

$$\begin{aligned}\delta_n &= \delta_n \left(1 + c_n \delta_n\right) \left(1 + \frac{1}{1+\alpha}\delta_n + O\left(\delta_n^2\right)\right) \\ &\Rightarrow 1 = 1 + \delta_n \left(c_n + \frac{1}{1+\alpha}\right) + O\left(\delta_n^2\right), \end{aligned}$$

so

$$c_n = -\frac{1}{1+\alpha} + O\left(\delta_n\right).$$

Then (46) follows. ■

Of course, we could revert (47) to obtain a complete asymptotic expansion for ε_n in terms of powers of $n^{-\frac{1}{1+\alpha}}$. Now we may establish the asymptotics for Λ_n if $d > 1$:

Lemma 3.2
Let $d > 1$. Then

$$\Lambda_n \sim n^{-\frac{\alpha+2}{2(\alpha+1)}} d^{2n} \exp\left(-A_1 n^{\frac{\alpha}{1+\alpha}} - A_2 n^{\frac{\alpha-1}{\alpha+1}} + O\left(n^{\frac{\alpha-2}{\alpha+1}}\right)\right),$$

where A_1 and A_2 are given by (20), (21).

Proof
Recall that Theorem 1.2 gives

$$\Lambda_n \sim n^{-1} q_n^{2n+1} Q\left(q_n\right)^{1/2} \exp\left(-2Q\left(q_n\right)\right).$$

From Lemma 3.1,

$$\begin{aligned} Q\left(q_n\right) &= d^{-2\alpha}\left(1 - \left(\frac{q_n}{d}\right)^2\right)^{-\alpha} \\ &= d^{-2\alpha} E_\alpha^{-\alpha} n^{\frac{\alpha}{1+\alpha}} \left(1 + \frac{\alpha}{1+\alpha} E_\alpha n^{-\frac{1}{1+\alpha}} + O\left(n^{-\frac{2}{1+\alpha}}\right)\right). \end{aligned}$$

Also, with the notation for ε_n from Lemma 3.1,

$$\begin{aligned} q_n^{2n} &= d^{2n}(1-\varepsilon_n)^n \\ &= d^{2n}\exp\left(-n\varepsilon_n - \frac{n\varepsilon_n^2}{2} + O\left(n^{1-\frac{3}{1+\alpha}}\right)\right) \\ &= d^{2n}\exp(-E_\alpha n^{\frac{\alpha}{1+\alpha}} + \frac{E_\alpha^2}{1+\alpha} n^{\frac{\alpha-1}{\alpha+1}} + O\left(n^{\frac{\alpha-2}{\alpha+1}}\right) - \frac{E_\alpha^2}{2} n^{\frac{\alpha-1}{\alpha+1}}). \end{aligned}$$

Combining these estimates gives

$$\Lambda_n \sim n^{-\frac{\alpha+2}{2(\alpha+1)}} d^{2n} \exp\begin{pmatrix} -\left[2d^{-2\alpha}E_\alpha^{-\alpha} + E_\alpha\right] n^{\frac{\alpha}{1+\alpha}} \\ -\left[2d^{-2\alpha}E_\alpha^{1-\alpha}\frac{\alpha}{1+\alpha} - \frac{E_\alpha^2}{1+\alpha} + \frac{E_\alpha^2}{2}\right] n^{\frac{\alpha-1}{\alpha+1}} + O\left(n^{\frac{\alpha-2}{\alpha+1}}\right) \end{pmatrix}.$$

Some elementary manipulations, and the definition (17) of E_α show that the coefficient of $n^{\frac{\alpha}{1+\alpha}}$ in the exponent is A_1 and that of $n^{\frac{\alpha-1}{\alpha+1}}$ is A_2. ■

Next, we obtain an asymptotic for the Mhaskar-Rakhmanov-Saff number a_u :

Lemma 3.3

Let $d \geq 1$ and D_α be as in (17). Then as $u \to \infty$,

$$a_u = d - D_\alpha u^{-\frac{1}{\alpha+\frac{1}{2}}}(1+o(1)). \tag{50}$$

Proof

The defining relation (9) for a_u gives

$$u = \frac{4\alpha}{\pi} a_u^2 \int_0^1 \frac{s^2\left(d^2-(a_u s)^2\right)^{-\alpha-1}}{\sqrt{1-s^2}} ds,$$

so

$$\frac{\pi u}{4\alpha}\left(\frac{d}{a_u}\right)^2 d^{2\alpha} = \int_0^1 \frac{s^2\left(1-\left(\frac{a_u}{d}s\right)^2\right)^{-\alpha-1}}{\sqrt{1-s^2}} ds =: I. \tag{51}$$

We make the substitution $\left(1-\frac{a_u}{d}\right)v = 1-s$ in I, giving

$$I = \left(1-\frac{a_u}{d}\right)^{-\alpha-\frac{1}{2}} \int_0^{\frac{1}{1-a_u/d}} f_u(v)\,dv,$$

where in $[0, \left(1-\frac{a_u}{d}\right)^{-1})$,

$$f_u(v) = \frac{\left[1-\left(1-\frac{a_u}{d}\right)v\right]^2\left[1+\frac{a_u}{d}v\right]^{-\alpha-1}\left[2-\left(1-\frac{a_u}{d}\right)\left(1+\frac{a_u}{d}v\right)\right]^{-\alpha-1}}{\sqrt{v}\sqrt{2-\left(1-\frac{a_u}{d}\right)v}}$$

and $f_u(v) = 0$ elsewhere. Since $a_u \to d$ as $u \to \infty$, we have for each $v \in (0,\infty)$,

$$\lim_{u\to\infty} f_u(v) = \frac{[1+v]^{-\alpha-1} 2^{-\alpha-1}}{\sqrt{v}\sqrt{2}}.$$

Moreover, for large enough u, and all $v \in (0,\infty)$,

$$0 \le f_u(v) \le \frac{\left[1+\frac{1}{2}v\right]^{-\alpha-1}}{\sqrt{v}}.$$

By Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} I &= \left(1-\frac{a_u}{d}\right)^{-\alpha-\frac{1}{2}} \int_0^\infty \frac{[1+v]^{-\alpha-1} 2^{-\alpha-1}}{\sqrt{v}\sqrt{2}} dv\,(1+o(1)) \\ &= \left(1-\frac{a_u}{d}\right)^{-\alpha-\frac{1}{2}} \frac{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)} 2^{-\alpha-\frac{3}{2}}(1+o(1)). \end{aligned}$$

Substituting in (51), gives (50), after some elementary manipulations. ■

Now we can give the asymptotic for λ_n :

Lemma 3.4

$$\lambda_n^{-1} \sim n^{-\frac{1}{2}} \left(\frac{1}{d}+\sqrt{1+\frac{1}{d^2}}\right)^{2n} \exp\left((1+o(1))\, B_1 \int_1^n s^{-\frac{1}{\alpha+\frac{1}{2}}} ds\right), \qquad (52)$$

where B_1 is given by (22).

Proof

From the previous lemma,

$$\frac{1}{a_u} = \frac{1}{d}\left(1+\frac{D_\alpha}{d} u^{-\frac{1}{\alpha+\frac{1}{2}}}(1+o(1))\right),$$

and hence

$$\begin{aligned} &\frac{1}{a_u}+\sqrt{1+\frac{1}{a_u^2}} \\ = &\left(\frac{1}{d}+\sqrt{1+\frac{1}{d^2}}\right)\left(1+\frac{D_\alpha}{d^2\sqrt{1+\frac{1}{d^2}}} u^{-\frac{1}{\alpha+\frac{1}{2}}}(1+o(1))\right). \end{aligned}$$

Then

$$\begin{aligned}&\int_0^n \log\left(\frac{1}{a_s}+\sqrt{1+\frac{1}{a_s^2}}\right)ds\\ =\ & n\log(\frac{1}{d}+\sqrt{1+\frac{1}{d^2}})+\frac{D_\alpha}{d^2\sqrt{1+\frac{1}{d^2}}}\int_1^n s^{-\frac{1}{\alpha+\frac{1}{2}}}ds\,(1+o(1))+O(1).\end{aligned}$$

Here the $O(1)$ term comes from $\int_0^1 \log\left(\frac{1}{a_s}+\sqrt{1+\frac{1}{a_s^2}}\right)ds$, which is convergent (cf. (2.34) in [5, p. 46]). Since [4, Theorem 1.2] gives

$$\lambda_n^{-1}\sim n^{-1/2}\exp\left(2\int_0^n \log\left(\frac{1}{a_s}+\sqrt{1+\frac{1}{a_s^2}}\right)ds\right),$$

we obtain (52). ■

To obtain the estimate in Example 3 for $\kappa(H_n)$, combine Lemma 3.2 and 3.4. In the case $d=1$, Lemma 3.4 alone gives the estimate in Example 2. The cases $\alpha <,=,> \frac{1}{2}$ follow easily.

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