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Convergence versus correspondence for sequences of rational functions

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ABSTRACT

Let f be meromorphic in the plane and analytic at 0. Then its diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ of Padé approximants need not converge pointwise. We ask whether by reducing the order of contact (or correspondence) of $[n/n]$ with f at 0, namely $2n+1$,we can ensure locally uniform convergence. In particular, we show that there exist rational functions R_n of type $(n,n), n \geq 1$, and a sequence of positive integers $\{\ell_n\}_{n=1}^{\infty}$ with limit ∞, depending on f, such that R_n has contact of order $n+\ell_n+1$ with f at 0, and which converge locally uniformly to f. Moreover, for any given sequence $\{\ell_n\}_{n=1}^{\infty}$, there exists an entire f for which order of contact higher than $n+\ell_n$ is incompatible with convergence.

1. INTRODUCTION AND RESULTS

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

be a formal power series. A rational function of type (n,n) is a rational function whose numerator and denominator degrees are at most n (and of course the denominator polynomial should not be identically 0). For $n \geq 0$, the (n,n) Padé approximant to f is a rational function $[n/n] = P/Q$ of type (n,n) with

$$(fQ - P)(z) = O(z^{2n+1}).$$

We say that $[n/n]$ has order of contact $2n+1$ with f at 0. More generally, a rational function $R = P/Q$ of type (n, n) is said to have order of contact m with f at 0, if

$$(fQ - P)(z) = O(z^m).$$

Note that in some cases R may have several different orders of contact m, as we may multiply both P and Q by a common power of z, provided, of course, we don't exceed the permitted degrees of P and Q.

The convergence theory of $\{[n/n]\}_{n=1}^{\infty}$ is complicated. It is known [1], [5] that if f is meromorphic in $\mathbb{C}$, then $\{[n/n]\}_{n=1}^{\infty}$ converges in measure (and in capacity) in bounded subsets of $\mathbb{C}$. On the other hand, there need not be pointwise convergence: H. Wallin [7] constructed an entire function with

$$\limsup_{n\to\infty} |[n/n](z)| = \infty, \ \forall z \in \mathbb{C}\backslash\{0\}.$$

In this paper, we investigate the following:

Question
To what extent must we weaken the order of contact of $[n/n]$ with f at 0 in order to guarantee pointwise convergence?

We believe that this question is new, relevant and interesting. It has connections with the convergence theory of continued fractions and Padé and Padé-type approximants. The (perhaps disappointing) conclusion is that we must weaken $2n + 1$ to $n + \ell_n + 1$, where $\{\ell_n\}_{n=1}^{\infty}$ may grow arbitrarily slowly to ∞:

Theorem 1.
(a) Let f be meromorphic in $\mathbb{C}$ and analytic at 0. There exists a sequence of positive integers $\{\ell_n\}_{n=1}^{\infty}$ with

(1.1) $$\lim_{n\to\infty} \ell_n = \infty,$$

and for $n \geq 1$, rational functions R_n of type (n, n), having order of contact $n + \ell_n + 1$ with f, and satisfying

(1.2) $$\lim_{n\to\infty} R_n(z) = f(z)$$

uniformly in compact subsets of $\mathbb{C}$ omitting poles of f.

(b) Let $\{\ell_n\}_{n=1}^{\infty}$ be a sequence of positive integers satisfying (1.1). There exists an entire function f with the following property: given for $n \geq 1$, rational functions R_n of type (n, n) having order of contact $n + \ell_n + 1$ with f at 0, then $\{R_n\}_{n=1}^{\infty}$ has every point in the plane as a limit point of its poles, and moreover,

(1.3) $$\limsup_{n\to\infty} |R_n(z)| = \infty, \ \forall z \in \mathcal{A},$$

where $\mathcal{A}$ is a set dense in $\mathbb{C}$.

We note that when applied to functions with finite radius of meromorphy, our methods of proof give the following assertions: let f be meromorphic in the unit ball.

(I) There exists for $n \geq 1$, rational functions R_n of type (n, n), having order of contact $n+2$ with f at 0, and satisfying

$$(1.4) \qquad \lim_{n\to\infty} R_n = f$$

uniformly in compact subsets of $\{z : |z| < 1\}$ omitting poles of f.

(II) Let $\ell \geq 2$. Then $\exists 0 < \rho_\ell < 1$ (independent of f) and for $n \geq 1$, rational functions R_n of type (n, n) having order of contact $n + \ell + 1$ with f at 0 and satisfying (1.4) uniformly in compact subsets of $\{z : |z| < \rho_\ell\}$.

The proofs of (I) and (II) involve de Montessus de Ballore's theorem [1], a theorem of Buslaev, Goncar and Suetin [3], and of Beardon [2], much as in the proof of Theorem 1. We pose one question in connection with (I) and (II):

Problem

Does there exist a function f analytic in $\{z : |z| < 1\}$ with the following property? Given a rational function R_n of type (n, n), $n \geq 1$, such that R_n has order of contact $n+3$ with f at 0, it is not possible that (1.4) holds uniformly in compact subsets of $\{z : |z| < 1\}$.

We shall prove the theorem in the next section.

2. PROOF OF THEOREM 1

Proof of (a) of Theorem 1

We distinguish two cases:

(I) *f has infinitely many poles in $\mathbb{C}$*

In this case, we apply de Montessus de Ballore's theorem: for $\ell \geq 1$, let ρ_ℓ be the largest circle centre 0, inside which f has at most ℓ poles, counted according to order. By de Montessus de Ballore's theorem [1, p. 282 ff.],

$$\lim_{m\to\infty} [m/\ell](z) = f(z),$$

uniformly in compact subsets of $\{z : |z| < \rho_\ell\}$ omitting poles of f. Then by choosing m_j to grow to ∞ sufficiently rapidly with j, we obtain

$$(2.1) \qquad \lim_{j\to\infty} [m_j/j](z) = f(z),$$

uniformly in compact subsets of $\mathbb{C}$ omitting poles of f. We may obviously assume that $m_j \geq j$ for each j and that $m_1 = 1$.

Let us elaborate on the choice of $\{m_j\}_{j=1}^{\infty}$. For $j \geq 2$, let $\mathcal{K}_j$ denote the closed ball centre 0 and radius $\rho_j/2$, but with open balls of radius $1/j$ centred on the poles of f inside that ball deleted. By de Montessus de Ballore,

$$\lim_{m\to\infty} [m/j](z) = f(z)$$

uniformly in $\mathcal{K}_j$. Then if m_j is large enough,

$$\max_{z \in \mathcal{K}_j} |f - [m_j/j]|(z) \leq 2^{-j}.$$

We may clearly also ensure inductively that $m_j > m_{j-1}$ and $m_j \geq j$.

Next, given $n \geq 1$, we let

$$(2.2) \qquad j(n) := \max\{j : m_j \leq n\},$$

and if $[m_{j(n)}/j(n)] = p/q$, where p, q have degree at most $m_{j(n)}, j(n)$ respectively, we define

$$(2.3) \qquad P(z) := z^{n-m_{j(n)}} p(z); Q(z) := z^{n-m_{j(n)}} q(z); R_n := P/Q.$$

Then as $j(n) \leq m_{j(n)}$, R_n is a rational function of type (n, n) with

$$(2.4) \qquad \begin{aligned} (fQ - P)(z) &= z^{n-m_{j(n)}}(fq - p)(z) \\ &= z^{n-m_{j(n)}} O(z^{m_{j(n)}+j(n)+1}) \\ &= O(z^{n+j(n)+1}). \end{aligned}$$

Here the requisite convergence of $\{R_n\}_{n=1}^{\infty}$ follows from (2.1). If we set

$$\ell_n := j(n), n \geq 1$$

we have completed the proof of Theorem 1 in this case.

(II) *f has finitely many poles in* $\mathbb{C}$

In this case, we observe first that it suffices to prove the following apparently weaker assertion: let ℓ be a positive integer exceeding the total order of poles of f in $\mathbb{C}$. Then there exist for $n \geq 1$, rational functions R_n of type (n, n), having order of contact $n + \ell + 1$ with f at 0, and that converge to f uniformly in compact subsets of $\mathbb{C}$ omitting poles of f. (Indeed we may then choose ℓ_n to grow sufficiently slowly to ∞, much as in Case (I)). To prove this weaker assertion, we use a theorem of Buslaev, Goncar and Suetin [3]: for each such ℓ, we can find an infinite subsequence $\{m_j\}_{j=1}^{\infty}$ such that

$$(2.5) \qquad \lim_{j \to \infty} [m_j/\ell](z) = f(z),$$

uniformly in compact subsets of $\mathbb{C}$ omitting poles of f. We may assume that $m_1 = \ell$ and set $R_n = [n/\ell]$ for $n < \ell$. For $n \geq \ell$, we define $j(n)$ by (2.2) and if $[m_{j(n)}/\ell] = p/q$, we define R_n by (2.3). Observe that instead of (2.4), we obtain

$$(fQ - P)(z) = O(z^{n+\ell+1}).$$

The convergence of $\{R_n\}_{n=1}^{\infty}$ follows from (2.5). □

In proving (b), we shall use the following simple lemma:

Lemma 2.1.
Suppose that f is a formal power series and $n, \ell \geq 1$. Write $[\ell/1] = p/q$ where $\deg(p) \leq \ell, \deg(q) \leq 1$ *and suppose that*

(2.6) $(fq - p)(z) = O(z^{n+\ell+1})$.

Let R be a rational function of type (n, n) having order of contact $n + \ell + 1$ with f. We then have

(2.7) $R = [\ell/1]$.

Proof.
Write $R = P/Q$, where P, Q have deg $\leq n$. By hypothesis,

$$(fQ - P)(z) = O(z^{n+\ell+1}).$$

Multiplying (2.6) by Q and substracting q times this last relation gives

$$(-pQ + qP)(z) = O(z^{n+\ell+1}).$$

But the left-hand side is a polynomial of degree $\leq n + \ell$, and (2.7) follows. □

Proof of Theorem 1(b)
We use a construction that goes back to Perron [6] and that has been widely used in Padé approximation. Let $\{\ell_n\}_{n=1}^{\infty}$ have limit ∞. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of non-zero complex numbers, dense in $\mathbb{C}$, such that each point in the sequence appears infinitely often in the sequence, and let

$$\mathcal{A} := \{z_1, z_2, z_3, \ldots\}.$$

We shall construct an entire function

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

and a subsequence $\{\ell_{n_k}\}_{k=1}^{\infty}$ of $\{\ell_n\}_{n=1}^{\infty}$ such that

(I) $[\ell_{n_k}/1]$ has pole $z_k, k \geq 1$;

(II)

$$(f - [\ell_{n_k}/1])(z) = O(z^{n_k+\ell_{n_k}+1}), k \geq 1.$$

This last relation of course implies (2.6) with $\ell = \ell_{n_k}$. From the lemma, given rational functions R_n of type (n, n) with order of contact $n + \ell_n + 1$ with f at 0, $n \geq 1$, we then have

$$R_{n_k} = [\ell_{n_k}/1],$$

so that $\{R_{n_k}\}_{k=1}^{\infty}$ has every point in $\mathbb{C}$ as a limit point of its poles, and also then, if $\mathcal{A} := \{z_1, z_2, z_3, \ldots\}$,

$$\limsup_{k\to\infty} |R_{n_k}(z)| = \infty, z \in \mathcal{A}.$$

We now turn to establishing (I) and (II).

We set $\ell_0 := 1$ and $n_0 := 1$ and choose $\{\ell_{n_k}\}_{k=1}^{\infty}$ to grow so rapidly that

(2.8) $\ell_{n_k} \geq n_{k-1} + \ell_{n_{k-1}} + 1, k \geq 1$.

We set $a_j := 1, 0 \leq j < \ell_{n_1}$. Now fix $k \geq 1$ and define $a_j, \ell_{n_k} \leq j < \ell_{n_{k+1}}$ as follows: set

$$\eta_k := 2^{-\ell_{n_{k+1}}^2} (\min\{1, |z_k|\})^{\ell_{n_{k+1}} - \ell_{n_k}};$$

and

$$a_j := \eta_k z_k^{\ell_{n_k} - j}, \ell_{n_k} \leq j < \ell_{n_{k+1}}.$$

Note that given $r > 1$,

$$\sum_{j=\ell_{n_k}}^{\ell_{n_{k+1}}-1} |a_j| r^j \leq \eta_k \ell_{n_{k+1}} r^{\ell_{n_{k+1}}} (\min\{1, |z_k|\})^{\ell_{n_k} - \ell_{n_{k+1}}}$$
$$\leq 2^{-\ell_{n_{k+1}}^2} \ell_{n_{k+1}} r^{\ell_{n_{k+1}}}.$$

Then since $\ell_{n_k} > k + 1$, we deduce that

$$\sum_{j=\ell_{n_1}}^{\infty} |a_j| r^j \leq \sum_{k=1}^{\infty} 2^{-\ell_{n_{k+1}}^2} \ell_{n_{k+1}} r^{\ell_{n_{k+1}}} < \infty.$$

Hence f is entire.

Next, given $k \geq 1$, we use a well known formula for $[\ell_{n_k}/1]$:

$$[\ell_{n_k}/1](z) = \sum_{j=0}^{\ell_{n_k}-1} a_j z^j + \frac{a_{\ell_{n_k}} z^{\ell_{n_k}}}{1 - z a_{\ell_{n_k}+1}/a_{\ell_{n_k}}}$$
$$= \sum_{j=0}^{\ell_{n_k}-1} a_j z^j + \frac{a_{\ell_{n_k}} z^{\ell_{nk}}}{1 - z/z_k},$$

so $[\ell_{n_k}/1]$ has a pole at z_k, and

$$[\ell_{n_k}/1](z) = \sum_{j=0}^{\ell_{n_k}-1} a_j z^j + \sum_{j=\ell_{n_k}}^{\ell_{n_{k+1}}-1} \eta_k z_k^{\ell_{n_k} - j} z^j + O(z^{\ell_{n_{k+1}}})$$
$$= f(z) + O(z^{\ell_{n_{k+1}}})$$
$$= f(z) + O(z^{n_k + \ell_{n_k} + 1}),$$

by (2.8). □

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