Universality Limits for Random Matrices and de Branges Spaces of Entire Functions☆

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Abstract

We prove that de Branges spaces of entire functions describe universality limits in the bulk for random matrices, in the unitary case. In particular, under mild conditions on a measure with compact support, we show that each possible universality limit is the reproducing kernel of a de Branges space of entire functions that equals a classical Paley-Wiener space. We also show that any such reproducing kernel, suitably dilated, may arise as a universality limit for sequences of measures on $[-1, 1]$.

1. Introduction and Results

Let μ be a finite positive Borel measure on $\mathbb{R}$ with all moments $\int x^j d\mu(x)$, $j \geq 0$, finite, and with infinitely many points in its support. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

Throughout we use $\mu'(x) = \frac{d\mu}{dx}$ to denote the almost everywhere existing Radon-Nikodym derivative of μ.

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Orthogonal polynomials play an important role in random matrix theory, especially in the unitary case [2], [5], [12], [27]. One of the key limits there involves the reproducing kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)\, p_k(y)\,. \tag{1.1}$$

Because of the Christoffel-Darboux formula, it may also be expressed as

$$K_n(x,y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(y) - p_{n-1}(x)\, p_n(y)}{x-y}, \quad x \neq y. \tag{1.2}$$

Define the normalized kernel

$$\widetilde{K}_n(x,y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x,y)\,. \tag{1.3}$$

The simplest case of the *universality law* is the limit

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n(\xi,\xi)} = \frac{\sin \pi (a-b)}{\pi (a-b)}, \tag{1.4}$$

involving the sinc kernel. It describes the distribution of spacing of eigenvalues of random matrices. Typically this limit holds uniformly for ξ in the interior of the support of μ and a, b in compact subsets of the real line. Of course, when $a = b$, we interpret $\frac{\sin \pi(a-b)}{\pi(a-b)}$ as 1.

There are a wide variety of methods for establishing universality, and we cannot survey them all here. Perhaps the deepest are Riemann-Hilbert methods, which yield much more than universality, though they require some smoothness properties for the measure [2], [5], [26]. There are a number of methods that use techniques of mathematical physics [7], [31]. Eli Levin observed that first order asymptotics for orthogonal polynomials are sufficient to establish universality [16].

One recently introduced technique [21] (see also [14], [18], [19]) involves a comparison inequality, and allows one to start with universality for a given measure, and extend it to far more general measures. The disadvantage there is that one needs to start with some measure, with a similar support to the given measure, for which universality is known. However, it has been greatly extended, using devices such as polynomial pullbacks by Totik [38], and his student Findley [6]. Simon [34] obtained equally impressive results by combining this method with Jost functions. In particular, Findley and Totik showed that for regular measures, universality holds a.e. in any interval where $\log \mu'$ is integrable. Here regularity in the sense of Stahl and Totik [36] can be defined as the condition

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\text{supp}\,[\mu])},$$

where $cap\,(\text{supp}\,[\mu])$ is the logarithmic capacity of the support of μ.

A perhaps more promising idea was introduced in [22]. It uses classical complex analysis, such as the theory of normal families, entire functions of exponential type, and reproducing kernels for Paley Wiener spaces. Its advantage is that it does not require a base measure for which universality is known, nor regularity. It shows that universality is equivalent to "universality along the diagonal", or alternatively, ratio asymptotics for Christoffel functions $\lambda_n(x) = 1/K_n(x,x)$. Here is a typical result:

Theorem 1.1 *Let μ be a finite positive Borel measure on the real line with compact support. Let $J \subset supp[\mu]$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that μ' is positive and continuous at each point of J. The following are equivalent:*

(I) *Uniformly for $\xi \in J$ and a in compact subsets of the real line,*

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = 1. \tag{1.5}$$

(II) *Uniformly for $\xi \in J$ and a, b in compact subsets of the complex plane, we have*

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{1.6}$$

While it is possible that (1.5) always holds under the initial hypotheses of Theorem 1.1, it has been established only when we assume that μ is regular. In [22], it was also shown that instead of continuity of w, we may assume a Lebesgue point type condition. The method may also be applied to varying and exponential weights, and at the "hard" or "soft" edge of the spectrum, where we obtain a Bessel or Airy kernel [15], [17], [20]. Avila, Last and Simon [1] have shown that this method can be adapted to prove universality for measures whose support is a Cantor set of positive measure, while Simon has extended Theorem 1.1 in a number of other directions [35].

In this paper, we explore the possible limits of subsequences of the sequence $\{f_n\}$, where

$$f_n(a,b) = \frac{K_n\left(\xi_n + \frac{a}{\widetilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\widetilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)}, \tag{1.7}$$

and $\{\xi_n\}$ is a sequence of real numbers. Since the $\{K_n\}$ are reproducing kernels for polynomials, it is scarcely surprising that limits of subsequences of $\{f_n\}$ are reproducing kernels for suitable spaces of entire functions. It turns out that the natural such spaces are de Branges spaces. We can use some of their remarkable theory to characterize universality limits.

de Branges spaces [4, p. 50], [25, p. 983. ff], [30, p. 793 ff.] are built around the Hermite-Biehler class. An entire function E is said to belong to the *Hermite-Biehler class* if it has no zeros in the upper half-plane $\mathbb{C}^+ = \{z : \text{Im } z > 0\}$ and

$$|E(z)| \geq |E(\bar{z})| \text{ for } z \in \mathbb{C}^+. \tag{1.8}$$

We write $E \in \overline{HB}$. Recall that the *Hardy space* $H^2(\mathbb{C}^+)$ is the set of all functions g analytic in the upper-half plane, for which

$$\sup_{y>0} \int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx < \infty.$$

Given an entire function g, we let

$$g^*(z) = \overline{g(\bar{z})}. \tag{1.9}$$

Definition 1.2 The de Branges space $\mathcal{H}(E)$ corresponding to the entire function $E \in \overline{HB}$, is the set of all entire functions g such that both g/E and g^*/E belong to $H^2(\mathbb{C}^+)$, with

$$\|g\|_E = \left(\int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2\right)^{1/2} < \infty. \tag{1.10}$$

$\mathcal{H}(E)$ is a Hilbert space with inner product

$$(g,h) = \int_{-\infty}^{\infty} \frac{g\bar{h}}{|E|^2}.$$

One may construct a reproducing kernel for $\mathcal{H}(E)$ from E [25, p. 984], [30, p. 793]. Indeed, if we let

$$\mathcal{K}(\zeta, z) = \frac{i}{2\pi} \frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{z - \bar{\zeta}}, \tag{1.11}$$

then for all ζ, $\mathcal{K}(\zeta, \cdot) \in \mathcal{H}(E)$ and for all complex ζ and all $g \in \mathcal{H}(E)$,

$$g(\zeta) = \int_{-\infty}^{\infty} \frac{g(t)\overline{\mathcal{K}(\zeta, t)}}{|E(t)|^2} dt. \tag{1.12}$$

We shall later identify $\mathcal{K}\left(\bar{\zeta}, z\right)$ with a function $f(\zeta, z)$ that arises as a universality limit. We emphasize that the standard reproducing kernel $\mathcal{K}$ for a de Branges space involves a conjugate variable, while the standard reproducing kernel K_n for an orthogonal polynomial system does not.

The classical de Branges spaces are the Paley-Wiener spaces PW_σ, consisting of entire functions of exponential type $\leq \sigma$ that are square integrable along the real axis. There one may take $E(z) = \exp(-i\sigma z)$, and the norm is just

$$\|g\|_{L_2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |g|^2\right)^{1/2}.$$

We write

$$\mathcal{H}(E) = PW_\sigma$$

if the two spaces are equal as sets, and have equivalent norms (we do not imply isometric isomorphism). Recall that having equivalent norms means that for some $C > 1$ independent of $g \in PW_\sigma$,

$$C^{-1} \|g\|_{L_2(\mathbb{R})} \leq \|g\|_E \leq C \|g\|_{L_2(\mathbb{R})}. \tag{1.13}$$

The closed graph theorem can be used to show that this norm equivalence follows from mere equality as sets.

Our main conclusion is that, under mild conditions,

Universality limits in the bulk are reproducing kernels of de Branges spaces that equal classical Paley-Wiener spaces.

More precisely:

Theorem 1.3 *Let μ be a measure with compact support. Let Jbe a compact set such that μ is absolutely continuous in an open set O containing J, and for some $C > 1$,*

$$C^{-1} \leq \mu' \leq C \text{ in } O.$$

Choose $\{\xi_n\} \subset J$ *and define* $\{f_n\}$ *by* (1.7).

(a) $\{f_n(\cdot,\cdot)\}$ *is a normal family in compact subsets of* $\mathbb{C}^2$.

(b) *Let* $f(\cdot,\cdot)$ *be the limit of some subsequence* $\{f_n(\cdot,\cdot)\}_{n\in\mathcal{S}}$. *Then* f *is an entire function of two variables, that is real valued in* $\mathbb{R}^2$ *and has* $f(0,0) = 1$. *Moreover, for some* $\sigma > 0$, $f(\cdot,\cdot)$ *is entire of exponential type* σ *in each variable.*

(c) *Define*

$$L(u,v) = (u-v) f(u,v), \quad u, v \in \mathbb{C}. \tag{1.14}$$

Let $a \in \mathbb{C}$ *have* $Im\ a > 0$ *and let*

$$E_a(z) = \sqrt{2\pi} \frac{L(\bar{a}, z)}{|L(a,\bar{a})|^{1/2}}. \tag{1.15}$$

Then f *is a reproducing kernel for* $\mathcal{H}(E_a)$. *In particular, for all* z, ζ,

$$f\left(z,\bar{\zeta}\right) = \frac{i}{2\pi} \frac{E_a(z)\overline{E_a(\zeta)} - E_a^*(z)\overline{E_a^*(\zeta)}}{z-\bar{\zeta}}. \tag{1.16}$$

(d) *Moreover,*

$$\mathcal{H}(E_a) = PW_\sigma \tag{1.17}$$

and the norms $\|\cdot\|_{E_a}$ *of* $\mathcal{H}(E_a)$ *and* $\|\cdot\|_{L_2(\mathbb{R})}$ *of* PW_σ *are equivalent.*

We emphasize that there are many de Branges spaces that equal PW_σ, but their reproducing kernel is not the sinc kernel $\frac{\sin \pi t}{\pi t}$. We shall present some examples after Theorem 1.7. A complete description of such spaces is given in [25].

In the case where there is a little smoothness of w at ξ such as continuity, or a Lebesgue point type condition, and the measure is regular in the sense of Stahl and Totik, indeed f above equals the sinc kernel, as shown in [6], [21], [22], [34], [38]. Nor does the above theorem exclude the possibility that f above is always a sinc kernel. We shall show below, however, that for *sequences* of measures, universality limits can definitely be the reproducing kernel of any de Branges space that equals a classical Paley-Wiener space.

More information about f and L are given in the following result:

Theorem 1.4 *Assume the hypotheses of Theorem* 1.3.

(a) *The function L satisfies the functional equation*

$$L(u,v)\,L(a,b) = L(a,u)\,L(b,v) - L(b,u)\,L(a,v) \tag{1.18}$$

for all complex a,b,u,v. Moreover, the functions $L(\cdot,\cdot)$ and $f(\cdot,\cdot)$are uniquely determined by the functional equation (1.18)*, and the values of the function $f(a,\cdot)$ for one non-real a.*

(b)

$$F(z) = zf(0,z) \tag{1.19}$$

has countably many real simple zeros $\{\rho_j\}$, and no other zeros.

(c) *Each $g \in PW_\sigma$ admits the expansion*

$$g(z) = \sum_{j=-\infty}^{\infty} g(\rho_j)\,\frac{f(\rho_j, z)}{f(\rho_j, \rho_j)}, \tag{1.20}$$

which is an orthonormal expansion in $\mathcal{H}(E_a)$, and moreover,

$$\int_{-\infty}^{\infty} \left|\frac{g}{E_a}\right|^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}. \tag{1.21}$$

Remarks (a) Note that the right-hand side of (1.21) is independent of a, which is surprising as E_a appears in the left-hand side. This phenomenon is well understood. Indeed, for a non-negative measure ω, we have [30, p. 794]

$$\int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2 = \int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2 d\omega$$

for all $g \in \mathcal{H}(E)$ iff there is a function A analytic in the interior of $\mathbb{C}^+$, with $|A| \leq 1$ there, and

$$\frac{\operatorname{Im} z}{\pi} \int_{-\infty}^{\infty} \frac{d\omega(t)}{|t-z|^2} = \operatorname{Re}\frac{E + E^* A}{E - E^* A}(z), \quad \operatorname{Im} z > 0.$$

(b) When $\xi_n = \xi$, $n \geq 1$, and ξ is a Lebesgue point of w, then the exponential type of f in each variable is

$$\sigma = \pi \sup_{x \in \mathbb{R}} f(x,x).$$

The proof of this given in [22, Lemma 6.4] goes through without change under the above hypotheses.
(c) As a consequence of (1.20), we can say a lot about the distribution of the $\{\rho_j\}$, which in the special case of the sinc kernel are just the integers. Define the counting function of $\{\rho_j\}$,

$$\nu [a, b] = \# \{j : \rho_j \in [a, b]\} \tag{1.22}$$

and

$$\nu (t) = \begin{cases} \nu ([0, t]) , & t \geq 0 \\ \nu ([t, 0]) , & t \leq 0 \end{cases} . \tag{1.23}$$

Classical complex analysis [13, p. 126 ff.] shows that

$$\lim_{|t|\to\infty} \frac{\nu (t)}{|t|} = \frac{\sigma}{\pi}.$$

Much more is true - roughly speaking, for each $\varepsilon > 0$,

$$\nu (t) - \frac{\sigma}{\pi} t = O(\log |t|)^{1+\varepsilon} :$$

Theorem 1.5 *Let $p > 0$ and $\tau > 1$. Then*

$$\int_{-\infty}^{\infty} \frac{\left|\nu (t) - \frac{\sigma}{\pi} t\right|^p}{(1 + |t|) (\log (2 + |t|))^{p+\tau}} dt < \infty. \tag{1.24}$$

All of the above results can be proven for a sequence of measures $\{\mu_n\}$, rather than a fixed measure μ. The hypotheses (1.26) to (1.28) below in a sense generalize the notion of the bulk of the support to sequences of measures.

Theorem 1.6 *For $n \geq 1$, let μ_n be a measure with support on the real line, for which the power moments $\int x^j d\mu_n (x)$, $0 \leq j \leq 2n - 2$, are finite. Let K_n denote the nth reproducing kernel for the measure μ_n, and $\tilde{K}_n$ its normalized cousin. Let $\{\xi_n\}$ be a sequence of real numbers, and let*

$$f_n (a, b) = \frac{K_n \left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n (\xi_n, \xi_n)}. \tag{1.25}$$

Assume that there exists $a \in \mathbb{C}\backslash\mathbb{R}$, and $C_1, C_2, C_3 > 0$ with the following property: given $A > 0$, there exists n_0 such that for $n \geq n_0$ and $|z| \leq A$,

$$|f_n (a, z)| \leq C_1 e^{C_2|\text{Im } z|}, \tag{1.26}$$

and for $x \in [-A, A]$*,*

$$f_n(x,x) \geq C_3. \tag{1.27}$$

Assume moreover, that for some $C_0 > 0$ *and a.e. real* t*,*

$$\liminf_{n\to\infty} \frac{\mu_n'\left(\xi_n + \frac{t}{\widetilde{K}_n(\xi_n,\xi_n)}\right)}{\mu_n'(\xi_n)} \geq C_0. \tag{1.28}$$

Then the conclusions of Theorems 1.3, 1.4 *and* 1.5 *hold true for* $\{f_n\}$*.*

Barry Simon has shown [35] that one can weaken the growth assumption (1.26) in a number of ways. We shall also prove a partial converse, showing that any reproducing kernel for a de Branges space that equals a classical Paley-Wiener one, can arise as a multiple of a universality limit:

Theorem 1.7 *Let* $\mathcal{H}(E)$ *be a de Branges space that equals* PW_σ *for some* $\sigma > 0$*. Let* $f\left(\bar{\zeta}, z\right)$ *be the reproducing kernel for* $\mathcal{H}(E)$ *normalized so that* $f(0,0) = 1$*. Assume also that* $|E(0)| = 1$*. Then there exists for* $n \geq 1$*, an absolutely continuous measure* μ_n*, with support* $[-1,1]$*, with* μ_n' *infinitely differentiable in* $(-1,1)$*, with* $\mu_n'(0) = 1$*, and for which*

$$f_n(a,b) = \frac{K_n\left(0 + \frac{a}{\widetilde{K}_n(0,0)}, 0 + \frac{b}{\widetilde{K}_n(0,0)}\right)}{K_n(0,0)}$$

satisfies (1.26) *and* (1.27)*, while*

$$\lim_{n\to\infty} f_n(a,b) = f(a,b), \tag{1.29}$$

uniformly for a, b *in compact subsets of* $\mathbb{C}$*.Moreover, given* $R > 0$*,* (1.28) *holds for* $t \in [-R, R]$*. If in addition, there exists* $C_1 > 1$ *such that*

$$C_1^{-1} \leq |E(x)| \leq C_1, \; x \in \mathbb{R}, \tag{1.30}$$

then (1.28) *holds for all* $t \in \mathbb{R}$*.*

Remarks (a) The hypothesis $f(0,0) = 1$ matches the conclusion in Theorem 1.3(b). It can always be achieved by multiplying E by a suitable constant. However, the hypothesis $|E(0)| = 1$ is more problematic. Without it, we have to replace (1.29) by

$$\lim_{n\to\infty} f_n(a,b) = f\left(|E(0)|^2 a, |E(0)|^2 b\right).$$

By a dilation of the variable, we can ensure $|E(0)| = 1$. More precisely, make the substitution $t = s|E(0)|^2$ in the reproducing kernel relation (1.12), and let

$$E_1(z) = \frac{E\left(|E(0)|^2 z\right)}{|E(0)|}.$$

One can check that the reproducing kernel for $\mathcal{H}(E_1)$ is $f_1(\cdot,\cdot) = f(|E(0)|^2\cdot, |E(0)|^2\cdot)$. Then $E_1(0) = 1$ and $f_1(0,0) = 1$, and the above result may be applied to $\mathcal{H}(E_1)$.

(b) Lyubarskii and Seip [25, p. 1005] presented a range of examples of $E(z)$ other than $e^{-i\pi z}$ for which $\mathcal{H}(E) = PW_\pi$. For $0 \leq \delta < \frac{1}{4}$, let

$$E_\delta(z) = (z+i)\prod_{k=1}^{\infty}\left(\left(1 - \frac{z}{k-\delta-ik^{-4\delta}}\right)\left(1 + \frac{z}{k-\delta+ik^{-4\delta}}\right)\right).$$

This is an entire function of exponential type π with $\mathcal{H}(E) = PW_\pi$ that satisfies (1.30) only if $\delta = 0$. In fact, if Λ_δ denotes the zero set of E_δ, then uniformly for all real x, and for some $C_1 > 1$,

$$C_1^{-1} \leq |E_\delta(x)| / \left[(1+|x|)^{2\delta}\, dist\,(x, \Lambda_\delta)\right] \leq C_1.$$

Here $dist\,(x, \Lambda_\delta)$ denotes the distance from x to Λ_δ. For $\delta > 0$, the reproducing kernel f_δ of $\mathcal{H}(E_\delta)$ is not the sinc kernel. Indeed, using (1.16) for f_δ, we see that

$$f(z,-i) = \frac{i}{2\pi}\frac{E_\delta(z)\,\overline{E_\delta(i)}}{z+i}$$

and this has a very different zero set, with respect to z, from $\frac{\sin\pi(z+i)}{\pi(z+i)}$.

(c) If we let

$$E_0(z) = c\sin\pi(z+i),$$

for some normalizing constant c, a straightforward calculation shows that the reproducing kernel f_0 for $\mathcal{H}(E_0)$, given by (1.16), is

$$f(z,\zeta) = c\frac{\sinh(2\pi)}{2}\frac{\sinh\pi(z-\zeta)}{\pi(z-\zeta)}.$$

If we let

$$E_1(z) = \frac{z+2i}{z+i}E_0(z),$$

then on the real line,

$$C_1 \leq |E_1| \leq C_2$$

so (1.30) is satisfied. Moreover, it is easily seen that $\mathcal{H}(E_1) = \mathcal{H}(E_0) = PW_\pi$. However, the reproducing kernel f_1 for $\mathcal{H}(E_1)$ is not the sinc kernel. Indeed, (1.16) shows that for some constant C,

$$f_1(z, -2i) = C \frac{\sin \pi (z+i)}{\pi (z+i)},$$

and this is not a constant multiple of $\frac{\sin \pi(z+2i)}{\pi(z+2i)}$.

This paper is organized as follows. In Section 2, we present notation and general background, such as on orthogonal polynomials. In Section 3, we present background on entire functions and de Branges spaces. In Section 4, we discuss some polynomial de Branges spaces. In Section 5, we use these to examine de Branges spaces of entire functions associated with general measures μ. In Section 6, we prove Theorems 1.3, 1.4, and 1.5. In Section 7, we prove Theorems 1.6 and 1.7.

2. Notation and Background

In this section, we record our notation, though some of it has already been introduced earlier. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, y, s, t. The same symbol does not necessarily denote the same constant in different occurences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. We use $\sim$ in the following sense: given real sequences $\{c_n\}$, $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 with

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

Throughout, μ denotes a finite positive Borel measure with not necessarily compact support on the real line. Its Radon-Nikodym derivative, which exists a.e., is μ'. The corresponding orthonormal polynomials are denoted by $\{p_n\}_{n=0}^{\infty}$, so that

$$\int p_n p_m d\mu = \delta_{mn}.$$

We denote the zeros of p_n by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}. \tag{2.1}$$

The reproducing kernel $K_n(x,t)$ is defined by (1.1), while the normalized reproducing kernel is defined by (1.3). We let

$$\begin{aligned} L_n(x,t) &= (x-t)\,K_n(x,t) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(x)\,p_{n-1}(t) - p_{n-1}(x)\,p_n(t)\right). \end{aligned} \tag{2.2}$$

The nth Christoffel function is [8, p. 25], [29],

$$\lambda_n(x) = 1/K_n(x,x) = \inf_{\deg(P)\le n-1} \frac{\int P^2 d\mu}{P^2(x)}. \tag{2.3}$$

When we need to display dependence of p_n, K_n or λ_n on μ (or some other measure), we use $p_n(\mu,\cdot)$, $K_n(\mu,\cdot,\cdot)$, $\lambda_n(\mu,\cdot)$, and so on. The Gauss quadrature formula asserts that whenever P is a polynomial of degree $\le 2n-1$,

$$\sum_{j=1}^{n} \lambda_n(x_{jn})\,P(x_{jn}) = \int P\ d\mu. \tag{2.4}$$

In addition to this, we shall need another Gauss type of quadrature formula [8, p. 19 ff.]. Given a real number ξ, there are n or $n-1$ points $t_{jn} = t_{jn}(\xi)$, one of which is ξ, such that

$$\sum_{j} \lambda_n(t_{jn})\,P(t_{jn}) = \int P\ d\mu, \tag{2.5}$$

whenever P is a polynomial of degree $\le 2n-2$. The $\{t_{jn}\}$ are zeros of

$$L_n(\xi,t) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(\xi)\,p_{n-1}(t) - p_{n-1}(\xi)\,p_n(t)\right),$$

regarded as a function of t.

Because we consider a sequence $\{\xi_n\}$ of points in J, rather than a fixed ξ, we use the quadrature rule that includes ξ_n, so that

$$t_{jn} = t_{jn}(\xi_n) \text{ for all } j.$$

Moreover, because we wish to focus on ξ_n, we shall set $t_{0n} = \xi_n$, and order the $\{t_{jn}\}$ around ξ_n, treated as the origin:

$$\cdots < t_{-2,n} < t_{-1,n} < t_{0n} = \xi_n < t_{1n} < \cdots . \tag{2.6}$$

Of course the sequence $\{t_{jn}\}$ consists of either $n-1$ or n points, so terminates, and it is possible that all t_{jn} lie to the left or right of ξ_n. It is known [8, p. 19] that when $(p_n p_{n-1})(\xi_n) \neq 0$, then one zero of $L_n(\xi_n, t)$ lies in $(x_{jn}, x_{j-1,n})$ for each j, and the remaining zero lies outside (x_{nn}, x_{1n}).

For the given sequence $\{\xi_n\}$ in J, we shall define for $n \geq 1$,

$$f_n(a, b) = \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n, \xi_n)} \tag{2.7}$$

and

$$\widetilde{L_n}(a, b) = (a - b) f_n(a, b). \tag{2.8}$$

The zeros of

$$f_n(0, t) = \frac{K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n, \xi_n)}$$

will be denote by $\{\rho_{jn}\}_{j\neq 0}$. Since $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ are the zeros of $L_n(\xi_n, t)$, we have

$$\rho_{jn} = \tilde{K}_n(\xi_n, \xi_n)(t_{jn} - \xi_n). \tag{2.9}$$

We also set

$$\rho_{0n} = 0,$$

corresponding to $t_{0n} = \xi_n$.

For an appropriate subsequence $\mathcal{S}$ of integers, we shall let

$$f(a, b) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a, b). \tag{2.10}$$

The zeros of $f(0, \cdot)$ will be denoted by $\{\rho_j\}_{j\neq 0}$, and we set $\rho_0 = 0$. Our ordering of zeros is

$$\cdots \leq \rho_{-2} \leq \rho_{-1} < \rho_0 = 0 < \rho_1 \leq \rho_2 \leq \cdots . \tag{2.11}$$

In Theorem 5.3, and only in that Theorem, we shall further restrict the $\{\rho_j\}$ to exclude those zeros ρ for which $f(\rho, \rho) = 0$. This eventuality can not happen under the hypotheses of Theorems 1.3, 1.6 or 5.4. We shall denote the (exponential) type of $f(a, \cdot)$ by σ. (We shall show it is independent of a.) We let

$$L(a, b) = (a - b) f(a, b). \tag{2.12}$$

3. Background on Entire functions

We first review some theory that we shall use about entire functions of exponential type. Most of this can be found in the elegant series of lectures of B. Ja. Levin [13]. Recall that if g is entire of order 1, then its *exponential type* σ is

$$\sigma = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log |g(z)|}{r}. \tag{3.1}$$

We say that an entire function g belongs to the *Cartwright class* and write $g \in \mathcal{C}$ if it is of exponential type and

$$\int_{-\infty}^{\infty} \frac{\log^{+} |g(t)|}{1+t^2} dt < \infty. \tag{3.2}$$

Here $\log^{+} s = \max\{0, \log s\}$.

We let $n(g,r)$ denote the number of zeros of g in the ball center 0, radius r, counting multiplicity. An important result is that for $g \in \mathcal{C}$, that is real valued on the real axis,

$$\lim_{r\to\infty} \frac{n(g,r)}{2r} = \frac{\sigma}{\pi}. \tag{3.3}$$

For this, see [13, Theorem 1, p. 127] or [11, p. 66].

When g is entire of exponential type σ and bounded along the real axis, we have [13, p. 38, Theorem 3]

$$|g(z)| \le e^{\sigma|\mathrm{Im}\ z|} \|g\|_{L_\infty(\mathbb{R})}, \quad z \in \mathbb{C}. \tag{3.4}$$

When g is entire of exponential type $\le \sigma$ and $g \in L_2(\mathbb{R})$, we write $g \in PW_\sigma$. (In [13], the notation is $g \in L^2_\sigma$). Here, we have instead of the last inequality, [13, p. 149]

$$|g(z)| \le \left(\frac{2}{\pi}\right)^{1/2} e^{\sigma(|\mathrm{Im}\ z|+1)} \|g\|_{L_2(\mathbb{R})}, \quad z \in \mathbb{C}. \tag{3.5}$$

Another useful result is that if $g \in \mathcal{C}$, has exponential type σ, and has all real zeros, then [13, p. 126, p. 118]

$$\lim_{r\to\infty} \frac{\log \left|g\left(re^{i\theta}\right)\right|}{r} = \sigma |\sin\theta|, \quad 0 < |\theta| < \pi. \tag{3.6}$$

If we do not know that all zeros are real, it is known that [13, p. 118, p. 55, no. 3]

$$\limsup_{r\to\infty}\frac{\log\left|g\left(re^{i\theta}\right)\right|}{r}\leq\sigma\left|\sin\theta\right|,\quad 0<\left|\theta\right|<\pi. \tag{3.7}$$

The Hermite-Biehler class $\overline{HB}$ was defined in Section 1, as was the de Branges space $\mathcal{H}(E)$, for a given entire function $E\in\overline{HB}$. It is possible to give an abstract definition of a de Branges space [4, pp. 56–57]. de Branges' original definition involved the notions of mean type and bounded type. One useful alternative involves the reproducing kernel $\mathcal{K}(\zeta,z)$, defined in terms of E by (1.11). Then [4, p. 53] $\mathcal{H}(E)$ is the set of all entire functions g with

$$\|g\|_E=\left(\int_{-\infty}^{\infty}\left|\frac{g}{E}\right|^2\right)^{1/2}<\infty \tag{3.8}$$

and

$$|g(z)|\leq\mathcal{K}(z,z)^{1/2}\|g\|_E\text{ for all }z\in\mathbb{C}. \tag{3.9}$$

We emphasize that later on, we shall identify $\mathcal{K}(\zeta,z)$ with $f\left(\bar{\zeta},z\right)$.

For real x, and E as above, we define a phase function φ by

$$E(x)=|E(x)|\,e^{-i\varphi(x)}. \tag{3.10}$$

Here φ is an increasing continuous function. We have [4, p. 54], [25, p. 984] if $E(x)\neq 0$,

$$\varphi'(x)=\frac{\pi\mathcal{K}(x,x)}{|E(x)|^2}. \tag{3.11}$$

There is a sampling series determined by φ and a given real number α [4, p. 55], [30, p. 794]. Let $\{s_k\}$ denote the increasing sequence such that

$$\varphi(s_k)=\alpha+k\pi,\quad k=0,\pm1,\pm2,\ldots. \tag{3.12}$$

Assume

$$e^{i\alpha}E-e^{-i\alpha}E^*\notin\mathcal{H}(E). \tag{3.13}$$

Then

$$\left\{\frac{\mathcal{K}(s_k,z)}{\sqrt{\mathcal{K}(s_k,s_k)}}\right\}_k \tag{3.14}$$

is an orthonormal sequence in $\mathcal{H}(E)$, and for all $g \in \mathcal{H}(E)$,

$$\int_{-\infty}^{\infty} \left| \frac{g}{E} \right|^2 = \sum_k \frac{\pi |g(s_k)|^2}{\varphi'(s_k) |E(s_k)|^2} = \sum_k \frac{|g(s_k)|^2}{\mathcal{K}(s_k, s_k)}, \tag{3.15}$$

while for all z,

$$g(z) = \sum_k g(s_k) \frac{\mathcal{K}(s_k, z)}{\sqrt{\mathcal{K}(s_k, s_k)}}. \tag{3.16}$$

Moreover, there is at most one real $\alpha \in [0, \pi)$ for which (3.13) fails.

We shall later show that $\{\rho_j\}$ of Theorem 1.4 is a *complete interpolating sequence for* PW_σ. That is, given any sequence $\{c_j\}$ with

$$\sum_j |c_j|^2 < \infty,$$

there exists a unique $g \in PW_\sigma$ such that

$$g(\rho_j) = c_j \text{ for all } j.$$

Such sequences have been characterized in [10], [24] using the distribution of $\{\rho_j\}$. In particular, if ν is the counting function defined at (1.22)–(1.23), then

$$h(t) = \nu(t) - \frac{\sigma}{\pi} t$$

lies in the class BMO of the real line. That is,

$$\sup_I \frac{1}{|I|} \int_I |h - h_I| < \infty,$$

where for any interval I, with length $|I|$, we let

$$h_I = \frac{1}{|I|} \int_I h.$$

It is known that then [9, p. 233, Corollary 2.3], for each $p > 0$,

$$\sup_I \frac{1}{|I|} \int_I |h - h_I|^p < \infty. \tag{3.17}$$

(Garnett considers only $p \geq 1$, but the case $p < 1$ follows from Hölder's inequality.)

de Branges spaces that equal Paley-Wiener (and more general) spaces have been characterized in [25]. In particular, they showed [25, Theorem 4(ii), p. 982] that if $\mathcal{H}(E) = PW_\sigma$, then uniformly for all real x,

$$\varphi'(x) |E(x)|^2 = \pi \mathcal{K}(x, x) \sim 1. \tag{3.18}$$

4. de Branges Spaces of Polynomials

In this section, $n \geq 1$ is fixed, and μ is a measure on the real line with $\int x^j d\mu(x)$ finite, for $0 \leq j \leq 2n$. We assume the notation of section 2; in particular,

$$\begin{aligned} L_n(u,v) &= (u-v) K_n(u,v) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(u) p_{n-1}(v) - p_{n-1}(u) p_n(v)\right). \end{aligned} \tag{4.1}$$

In [23], we used ideas inspired by de Branges spaces to generate formulae for orthogonal polynomials with a weight that is a reciprocal of a positive polynomial. Here, we begin with some simple identities. The first is inspired by the more general theory of de Branges spaces, and the second is well known [28]:

Lemma 4.1 (a) *For all complex* α, β, z, v*,*

$$L_n(z,v) L_n(\alpha,\beta) = L_n(\alpha,z) L_n(\beta,v) - L_n(\beta,z) L_n(\alpha,v). \tag{4.2}$$

(b)

$$L_n(z,v) = \frac{\gamma_{n-1}}{\gamma_n} p_n(z) p_n(v) \left[G_n(v) - G_n(z)\right], \tag{4.3}$$

where

$$G_n(z) = \frac{p_{n-1}(z)}{p_n(z)} = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \frac{\lambda_n(x_{jn}) p_{n-1}^2(x_{jn})}{z - x_{jn}}. \tag{4.4}$$

Proof. (a) Just substitute (4.1) into the right-hand side of (4.2), then multiply out, cancel common factors, and refactorize. (A slightly simpler manipulation is to substitute the formula (4.3) into the right-hand side of (4.1)).

(b) Let $G_n(z) = \frac{p_{n-1}(z)}{p_n(z)}$. Then (4.3) follows from (4.1). We really only need to prove the second identity in (4.4). We apply the formula for Lagrange interpolation at the zeros of p_n to p_{n-1}. This gives

$$p_{n-1}(z) = \sum_{j=1}^{n} \frac{p_{n-1}(x_{jn}) p_n(z)}{p_n'(x_{jn})(z - x_{jn})}. \tag{4.5}$$

We now use the confluent form of the Christoffel-Darboux formula,

$$\lambda_n^{-1}(x) = K_n(x,x) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n'(x)\,p_{n-1}(x) - p_n(x)\,p_{n-1}'(x)\right).$$

Setting $x = x_{jn}$ gives

$$\lambda_n^{-1}(x_{jn}) = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{jn})\,p_{n-1}(x_{jn}).$$

Substituting this into (4.5) gives the second identity in (4.4). $\square$

Lemma 4.2 (a) *If* $K_n(z,w) = 0$*, then* $\mathrm{Im}\ z$ *and* $\mathrm{Im}\ w$ *have the same sign. In particular,* $\mathrm{Im}\ z > 0 \Rightarrow \mathrm{Im}\ w > 0$*.*

(b) *Let* $\mathrm{Im}\ a > 0$*. Then for* $\mathrm{Im}\ z \geq 0$*,*

$$|K_n(\bar{a},z)| \geq |K_n(a,z)|; \tag{4.6}$$

$$|L_n(\bar{a},z)| \geq |L_n(a,z)|. \tag{4.7}$$

In particular, $L_n(\bar{a},\cdot) \in \overline{HB}$*.*

Proof. (a) If z is real, then it is known [8, p. 19], that all zeros of $K_n(z,\cdot)$ are real. Thus in this case $\mathrm{Im}\ z = \mathrm{Im}\ w = 0$. Now suppose $\mathrm{Im}\ z > 0$. From (4.3), and the fact that all zeros of $p_n p_{n-1}$ are real, we deduce that

$$G_n(z) = G_n(w).$$

Taking imaginary parts in (4.4), we deduce that

$$(\mathrm{Im}\ z)\sum_{j=1}^{n}\frac{\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn})}{|z-x_{jn}|^2} = (\mathrm{Im}\ w)\sum_{j=1}^{n}\frac{\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn})}{|w-x_{jn}|^2}.$$

Since both sums are positive, the result follows.

(b) The rational function

$$h(z) := K_n(a,z)/K_n(\bar{a},z)$$

is analytic for z in the closed upper-half plane $\{z : \mathrm{Im}\ z \geq 0\}$, and for real x,

$$|h(x)| = 1.$$

Moreover, as a polynomial in z, the coefficients of the Taylor expansion about 0 of $K_n(\bar{a}, z)$ are the conjugates of those of $K_n(a, z)$. Then, as $z \to \infty$, $|h(z)| \to 1$. The maximum-modulus principle now shows that

$$|h(z)| \leq 1 \text{ for } \operatorname{Im} z \geq 0.$$

Since for $\operatorname{Im} z \geq 0$, also $|\bar{a} - z| \geq |a - z|$, we obtain (4.7) as well. $\square$

From the above, we obtain some de Branges spaces that consist of polynomials. Recall that K_n is the orthogonal polynomial reproducing kernel arising from the measure μ, while $\mathcal{K}$ denotes the reproducing kernel for a de Branges space. Recall too, the * notation introduced at (1.9).

Theorem 4.3 *Fix a with* $\operatorname{Im} a > 0$*, and let*

$$E_{n,a}(z) = \sqrt{2\pi} \frac{L_n(\bar{a}, z)}{|L_n(a, \bar{a})|^{1/2}}. \tag{4.8}$$

(a) *Then*

$$K_n\left(z, \bar{\zeta}\right) = \frac{i}{2\pi} \frac{E_{n,a}(z)\overline{E_{n,a}(\zeta)} - E_{n,a}^*(z)\overline{E_{n,a}^*(\zeta)}}{z - \bar{\zeta}}. \tag{4.9}$$

(b) *The de Branges space $\mathcal{H}(E_{n,a})$ corresponding to $E_{n,a}$ is the space of polynomials of degree $\leq n - 1$.*

(c) *For all polynomials P of degree $\leq n - 1$, and all $z \in \mathbb{C}$, we have*

$$P(z) = \int_{-\infty}^{\infty} P(t) \frac{K_n(t, z)}{|E_{n,a}(t)|^2} dt. \tag{4.10}$$

(d) *For all polynomials R of degree $\leq 2n - 2$,*

$$\int_{-\infty}^{\infty} \frac{R}{|E_{n,a}|^2} = \int R \, d\mu. \tag{4.11}$$

Proof. (a) The identity (4.2), with $\alpha = a; \beta = \bar{a}; v = \bar{\zeta}$ gives

$$L_n\left(z, \bar{\zeta}\right) L_n(a, \bar{a}) = L_n(a, z) L_n\left(\bar{a}, \bar{\zeta}\right) - L_n(\bar{a}, z) L_n\left(a, \bar{\zeta}\right). \tag{4.12}$$

Since

$$L_n(a, \bar{a}) = 2i \operatorname{Im} a K_n(a, \bar{a}) = i |L_n(a, \bar{a})|,$$

we obtain

$$K_n\left(z,\bar{\zeta}\right)=\frac{i}{\left|L_n\left(a,\bar{a}\right)\right|}\frac{L_n\left(\bar{a},z\right)L_n\left(a,\bar{\zeta}\right)-L_n\left(a,z\right)L_n\left(\bar{a},\bar{\zeta}\right)}{z-\bar{\zeta}},$$

and (4.9) follows on taking account of (4.8).

(b) Note first that $E_{n,a}\in\overline{HB}$ by Lemma 4.2(b), so that $\mathcal{H}\left(E_{n,a}\right)$ is well defined. By definition, it consists of all entire functions g for which both $g/E_{n,a}$ and $g^*/E_{n,a}$ lie in the Hardy class of the upper-half plane, and the norm $\|g\|_{E_{n,a}}$ is finite. The reproducing kernel $\mathcal{K}$ for this space is given by (1.11), with $E=E_{n,a}$:

$$\mathcal{K}\left(\zeta,z\right)=\frac{i}{2\pi}\frac{E_{n,a}\left(z\right)\overline{E_{n,a}\left(\zeta\right)}-E_{n,a}^*\left(z\right)\overline{E_{n,a}^*\left(\zeta\right)}}{z-\bar{\zeta}}. \tag{4.13}$$

For $g\in\mathcal{H}\left(E_{n,a}\right)$, the reproducing kernel relation (1.12) and Cauchy-Schwarz give, as at (3.9),

$$\left|g\left(z\right)\right|\leq\mathcal{K}\left(z,\bar{z}\right)^{1/2}\|g\|_{E_{n,a}},\quad z\in\mathbb{C}.$$

Inasmuch as $E_{n,a}$ is a polynomial of degree $\leq n-1$, we see that as $|z|\to\infty$,

$$\mathcal{K}\left(z,z\right)=O\left(|z|^{2n-1}\right).$$

Indeed, if we write

$$E_{n,a}\left(t\right)=\sum_{j=0}^{n-1}c_jt^j,$$

a calculation shows that

$$\mathcal{K}\left(\zeta,z\right)=\frac{i}{2\pi}\sum_{0\leq j<k\leq n}\left(c_j\overline{c_k}-\overline{c_j}c_k\right)\frac{z^j\bar{\zeta}^k-z^k\bar{\zeta}^j}{z-\bar{\zeta}}$$

and then the estimate above follows. Consequently, for $g\in\mathcal{H}\left(E_{n,a}\right)$, as $|z|\to\infty$,

$$\left|g\left(z\right)\right|=O\left(|z|^{n-1/2}\right),$$

so g is a polynomial of degree $\leq n-1$. Conversely, if g is a polynomial of degree $\leq n-1$, then $g\left(z\right)/E_{n,a}\left(z\right)=O\left(|z|^{-1}\right)$ as $|z|\to\infty$, and it follows easily that $g/E_{n,a},g^*/E_{n,a}\in H^2\left(\mathbb{C}^+\right)$, so $g\in\mathcal{H}\left(E_{n,a}\right)$.

(c) From (4.9) and (4.13), we see that

$$\mathcal{K}(\zeta, z) = K_n\left(z, \bar{\zeta}\right).$$

The reproducing kernel relation (1.12) gives, for polynomials P of degree $\leq n-1$,

$$\begin{aligned} P(\zeta) &= \int_{-\infty}^{\infty} \frac{P(t)\,\overline{\mathcal{K}(\zeta, t)}}{|E_{n,a}(t)|^2} dt \\ &= \int_{-\infty}^{\infty} \frac{P(t)\,\overline{K_n\left(t, \bar{\zeta}\right)}}{|E_{n,a}(t)|^2} dt \\ &= \int_{-\infty}^{\infty} \frac{P(t)\,K_n(t, \zeta)}{|E_{n,a}(t)|^2} dt. \end{aligned}$$

(d) We can write $R = PS$ where both P and S are polynomials of degree $\leq n-1$. We multiply the identity in (c) by S and then integrate with respect to μ. We obtain

$$\begin{aligned} \int R\,d\mu &= \int (PS)(z)\,d\mu(z) \\ &= \int S(z)\left[\int_{-\infty}^{\infty} P(t)\,\frac{K_n(t, z)}{|E_{n,a}(t)|^2} dt\right] d\mu(z) \\ &= \int_{-\infty}^{\infty} P(t)\,\frac{1}{|E_{n,a}(t)|^2}\left[\int S(z)\,K_n(t, z)\,d\mu(z)\right] dt \\ &= \int_{-\infty}^{\infty} P(t)\,\frac{1}{|E_{n,a}(t)|^2} S(t)\,dt \\ &= \int_{-\infty}^{\infty} \frac{R}{|E_{n,a}|^2}. \end{aligned}$$

Here, we have used the reproducing kernel formula for the measure μ. Moreover, the interchange of integrals is justified by absolute convergence of all integrals involved. □

Remark The identity in (d) is a real line analogue of a unit circle formula much used in Szegő theory [8, p. 198, Theorem 2.2], but I am not sure it is new. It seems similar to identities in the theory of orthogonal rational functions [3, p. 145], and seems in spirit similar to identities used by Simon [33, p. 456, Theorem 2.1].

5. de Branges Spaces of Entire Functions

Recall the notation

$$f_n(a,b) = \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)}.$$

We shall prove four general theorems in this section, and we begin by stating them. Throughout this section, we do not assume the hypotheses of Theorem 1.3.

Theorem 5.1 *Let μ be a measure with support on the real line, with all power moments $\int x^j d\mu(x)$, $j \geq 0$ finite, and with infinitely many points in its support. Let $\{\xi_n\}$ be a sequence of real numbers. Assume that there is a non-real complex number a, and an infinite sequence of integers $\mathcal{S}$, for which there exists*

$$f(a,z) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,z), \tag{5.1}$$

uniformly in compact subsets of $\mathbb{C}$, and that

$$f(a,\bar{a}) \neq 0. \tag{5.2}$$

Then

(a) *There exists, for all $z, v \in \mathbb{C}$,*

$$f(z,v) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(z,v),$$

and the limit is uniform for z, v in compact subsets of $\mathbb{C}$.

(b) *Let*

$$L(z,v) = (z-v) f(z,v). \tag{5.3}$$

For all complex α, β, z, v,

$$L(z,v) L(\alpha,\beta) = L(\alpha,z) L(\beta,v) - L(\beta,z) L(\alpha,v). \tag{5.4}$$

(c) *Let* $\mathrm{Im}\ a > 0$. *Then for* $\mathrm{Im}\ z > 0$,

$$|f(\bar{a},z)| \geq |f(a,z)|; \tag{5.5}$$

$$|L(\bar{a},z)| > |L(a,z)|. \tag{5.6}$$

In particular, for $\mathrm{Im}\ z > 0$,

$$|L(z,\bar{z})| > 0 \text{ and } f(z,\bar{z}) > 0. \tag{5.7}$$

(d) *If* $f(z,v)=0$, *then* $\operatorname{Im} z$ *and* $\operatorname{Im} v$ *have the same sign. In particular,* $\operatorname{Im} z>0 \Rightarrow \operatorname{Im} v>0$. *Consequently, for* $\operatorname{Im} a>0$, $L(\bar{a},\cdot)\in\overline{HB}$.

The assumption (5.2) is satisfied if $a=iy$, some $y\neq 0$. Indeed, as p_n has all real zeros,

$$
\begin{aligned}
& K_n\left(\xi_n+\frac{iy}{\tilde{K}_n(\xi_n,\xi_n)},\xi_n-\frac{iy}{\tilde{K}_n(\xi_n,\xi_n)}\right) \\
&= \sum_{k=0}^{n-1}\left|p_k\left(\xi_n+\frac{iy}{\tilde{K}_n(\xi_n,\xi_n)}\right)\right|^2 \\
&\geq \sum_{k=0}^{n-1}\left|p_k(\xi_n)\right|^2 = K_n(\xi_n,\xi_n),
\end{aligned}
$$

so

$$f_n(iy,-iy)\geq 1,$$

and also, for all real y,

$$f(iy,-iy)\geq 1. \tag{5.8}$$

Of course, it then follows from (5.7) that $f(z,\bar{z})>0$ for all non-real z.

Theorem 5.2 *Assume the hypotheses of Theorem* 5.1. *Fix* a *with* $\operatorname{Im} a>0$, *and let*

$$E_a(z)=\sqrt{2\pi}\frac{L(\bar{a},z)}{|L(a,\bar{a})|^{1/2}}. \tag{5.9}$$

(a) *Then all zeros of* E_a *lie in the lower half plane, and* $E_a\in\overline{HB}$. *Moreover,*

$$f\left(z,\bar{\zeta}\right)=\frac{i}{2\pi}\frac{E_a(z)\,\overline{E_a(\zeta)}-E_a^*(z)\,\overline{E_a^*(\zeta)}}{z-\bar{\zeta}}. \tag{5.10}$$

(b) *For all* $g\in\mathcal{H}(E_a)$, *and all* $z\in\mathbb{C}$, *we have*

$$g(z)=\int_{-\infty}^{\infty}g(t)\frac{f(z,t)}{|E_a(t)|^2}dt. \tag{5.11}$$

Moreover, $f(z,\cdot)\in\mathcal{H}(E_a)$ *for all* $z\in\mathbb{C}$.

(c) *For any* a,b, *with* $\operatorname{Im} a>0$, $\operatorname{Im} b>0$, $\mathcal{H}(E_a)=\mathcal{H}(E_b)$ *and the norms* $\|\cdot\|_{E_a}$ *and* $\|\cdot\|_{E_b}$ *are equivalent.*

Theorem 5.3 *Assume the hypotheses of Theorem* 5.1*. Fix* a *with* $\operatorname{Im} a > 0$.

(a) *Let*
$$F(z) = L(z, 0) = zf(0, z), \tag{5.12}$$
and let $\{\rho_j\}$ *be the zeros* ρ *of* F *for which* $f(\rho, \rho) \neq 0$*. These are all real and simple.*

(b) *The set* $\left\{ \frac{f(\rho_j, \cdot)}{\sqrt{f(\rho_j, \rho_j)}} \right\}_j$ *is an orthonormal sequence in* $\mathcal{H}(E_a)$ *and for all* $g \in \mathcal{H}(E_a)$,
$$\sum_j \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \leq \int \left| \frac{g}{E_a} \right|^2, \tag{5.13}$$
while
$$G[g] = \sum_j g(\rho_j) \frac{f(\rho_j, z)}{f(\rho_j, \rho_j)} \in \mathcal{H}(E_a). \tag{5.14}$$

(c) *Assume that* $F \notin \mathcal{H}(E_a)$*. Then for all* $g, h \in \mathcal{H}(E_a)$*, we have*
$$\int_{-\infty}^{\infty} \frac{g\bar{h}}{|E_a|^2} = \sum_j \frac{\left(g\bar{h}\right)(\rho_j)}{f(\rho_j, \rho_j)}, \tag{5.15}$$
and
$$G[g] = g. \tag{5.16}$$

Remarks (a) Note that if ρ is a zero of F, then ρ is necessarily real, but we have not excluded the possibility that $f(\rho, \rho) = 0$. If this is the case, then $g(\rho) = 0$ for all $g \in \mathcal{H}(E_a)$. This follows easily from the reproducing kernel relation (5.11) and Cauchy-Schwarz.

(b) The possibility that $f(\rho, \rho) = 0$ occurs only in the above theorem. It cannot happen under the hypotheses of Theorems 1.3, 1.6, and 5.4 below.

Theorem 5.4 *Assume, in addition to the hypothesis of Theorem* 5.1*, that* $f(a, \cdot)$ *is an entire function of exponential type* σ *and*
$$f(t, t) \sim 1 \text{ for } t \in \mathbb{R}. \tag{5.17}$$

(a) *Then for all complex* b, $f(b, \cdot)$ *is an entire function of exponential type* σ.

(b) *For all* $g \in PW_\sigma$,
$$g = G[g] \in \mathcal{H}(E_a). \tag{5.18}$$
In particular,
$$PW_\sigma \subset \mathcal{H}(E_a).$$

(c) *Assume that there exists* $C_0 > 0$ *such that for a.e.* $t \in \mathbb{R}$,
$$\liminf_{n\to\infty} \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} \geq C_0, \tag{5.19}$$
or, assume that for each $r > 0$,
$$\lim_{n\to\infty} \int_{-r}^{r} \left| \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} - 1 \right| dt = 0. \tag{5.20}$$
Then
$$PW_\sigma = \mathcal{H}(E_a).$$

We note that we do not assume that μ_n is absolutely continuous in the above result. Recall from (2.2) and (2.8) our notations
$$L_n(u, v) = (u - v) K_n(u, v)$$
and
$$\begin{aligned} \widetilde{L_n}(a, b) &= (a - b) f_n(a, b) \\ &= \mu_n'(\xi_n) L_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n, \xi_n)}\right). \end{aligned} \tag{5.21}$$

Lemma 5.5 *For all complex* α, β, z, v,
$$\widetilde{L_n}(z, v) \widetilde{L_n}(\alpha, \beta) = \widetilde{L_n}(\alpha, z) \widetilde{L_n}(\beta, v) - \widetilde{L_n}(\beta, z) \widetilde{L_n}(\alpha, v). \tag{5.22}$$

PROOF. This is immediate from (5.21) and (4.2). $\square$

Proof of Theorem 5.1. (a) From Lemma 5.5, we have

$$\widetilde{L_n}(z,v)\,\widetilde{L_n}(a,\bar{a}) = \widetilde{L_n}(a,z)\,\widetilde{L_n}(\bar{a},v) - \widetilde{L_n}(\bar{a},z)\,\widetilde{L_n}(a,v)\,. \tag{5.23}$$

Our hypothesis (5.1), the conjugate relation $f_n(\bar{a},z) = \overline{f_n(a,\bar{z})}$ and the symmetry $f_n(a,b) = f_n(b,a)$ give, uniformly for z in compact subsets of $\mathbb{C}$,

$$\begin{aligned}
\lim_{n\to\infty, n\in\mathcal{S}} \widetilde{L_n}(a,z) &= (a-z)\,f(a,z) = L(a,z)\,; \\
\lim_{n\to\infty, n\in\mathcal{S}} \widetilde{L_n}(\bar{a},z) &= (\bar{a}-z)\,f(\bar{a},z) = L(\bar{a},z)\,; \\
\lim_{n\to\infty, n\in\mathcal{S}} \widetilde{L_n}(a,\bar{a}) &= L(a,\bar{a})\,.
\end{aligned}$$

By our hypothesis (5.2), and (5.3),

$$L(a,\bar{a}) = 2i\,(\mathrm{Im}\ a)\,f(a,\bar{a}) \neq 0,$$

so (5.23) gives, uniformly for z, v in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \widetilde{L_n}(z,v) = \frac{1}{L(a,\bar{a})}\left[L(a,z)\,L(\bar{a},v) - L(\bar{a},z)\,L(a,v)\right].$$

That is, there exists,

$$f(z,v) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(z,v) = \frac{L(a,z)\,L(\bar{a},v) - L(\bar{a},z)\,L(a,v)}{L(a,\bar{a})\,(z-v)}, \tag{5.24}$$

and the limit is uniform for z, v in compact sets with $z \neq v$. For the case $z = v$, we can use convergence continuation theorems and the maximum-modulus principle.

(b) This follows directly from (5.22), by taking limits.

(c), (d) Taking limits in Lemma 4.2(b) gives for $\mathrm{Im}\ z \geq 0$,

$$|f(\bar{a},z)| \geq |f(a,z)| \ \text{ and } \ |L(\bar{a},z)| \geq |L(a,z)|\,. \tag{5.25}$$

We must show strict inequality in the second inequality. We first show the assertion on the zeros. Suppose $\mathrm{Im}\ v > 0$ and $f(z,v) = 0$. Hurwitz's

Theorem and Lemma 4.2(a), show that there exist $\{z_n\}$ with $f_n(z_n, v) = 0$ and

$$\lim_{n\to\infty, n\in\mathcal{S}} z_n = z.$$

By Lemma 4.2(a), Im $z_n > 0$. Then Im $z \geq 0$. To prove that it is positive, we use our functional relation (5.24). Assume Im $z = 0$. Then the numerator in (5.24) can be written as

$$\begin{aligned} 0 &= L(a, z) L(\bar{a}, v) - L(\bar{a}, z) L(a, v) \\ &= L(a, z) L(\bar{a}, v) - \overline{L(a, z)} L(a, v). \end{aligned}$$

Defining

$$h(u) = \frac{L(a, u)}{L(\bar{a}, u)} \text{ for Im } u \geq 0,$$

we have that h is meromorphic in the upper-half plane, satisfying there by (5.25),

$$|h(u)| \leq 1,$$

except perhaps at isolated poles. But these are removable singularities, because of the local boundedness, so we obtain that h is analytic in the upper half-plane. Also, $|h(x)| = 1$ for real x (again, we can remove isolated singularities), while

$$|h(v)| = \left|\frac{L(a, v)}{L(\bar{a}, v)}\right| = \left|\frac{L(a, z)}{\overline{L(a, z)}}\right| = 1.$$

Since Im $v > 0$, the maximum-modulus principle shows that $h = c$ in the upper-half plane, for some unimodular constant c. Then for all u in the upper-half plane, (5.24) gives

$$f(u, v) = \frac{cL(\bar{a}, u) L(\bar{a}, v) - L(\bar{a}, u) cL(\bar{a}, v)}{L(a, \bar{a})(u - v)} = 0.$$

Hence $f(u, v) = 0$ for all complex u, and by conjugate symmetry, $f(u, \bar{v}) = 0$ for all complex u. It follows that for each u in the upper-half plane, $f(u, \cdot)$ has a zero in the upper-half plane. The exact same argument we just used shows that $f(u, z) = 0$ for all complex z. Hence, f is identically 0 as a function of two complex variables, contradicting that $f(0, 0) = 1$. So Im $z > 0$, as desired.

It remains to prove strict inequality in (5.6). Suppose we have equality in (5.6) for some z. As above, we form

$$h(u) = \frac{L(a,u)}{L(\bar{a},u)},$$

which is analytic for u in the upper-half plane, and has $|h| \leq 1$ there. We are also assuming $|h(z)| = 1$, so by the maximum-modulus principle, $h = c$ for some unimodular constant c. As above, we obtain a contradiction. $\square$

PROOF OF THEOREM 5.2(A), (B). (a) First, Theorem 5.1(d) shows that all zeros of E_a must lie in the open lower half-plane. Moreover, (5.6) shows that $|E_a(z)| > |E_a(\bar{z})|$ for Im $z > 0$. So $E_a \in \overline{HB}$. Next,

$$L(a,\bar{a}) = 2i(\operatorname{Im} a) f(a,\bar{a}) = i|L(a,\bar{a})|,$$

so the functional equation (5.4) gives

$$L\left(z,\bar{\zeta}\right) i|L(a,\bar{a})| = L(a,z) L\left(\bar{a},\bar{\zeta}\right) - L(\bar{a},z) L\left(a,\bar{\zeta}\right)$$

$$\Rightarrow \left(z-\bar{\zeta}\right) f\left(z,\bar{\zeta}\right)|L(a,\bar{a})| = i\left(L(\bar{a},z)\overline{L(\bar{a},\zeta)} - \overline{L(\bar{a},\bar{z})} L\left(\bar{a},\bar{\zeta}\right)\right).$$

Taking account of the definition (5.9) of E_a, and recalling that $E_a^*(z) = \overline{E_a^*(\bar{z})}$, gives (5.10).

(b) By Theorem 5.1, the function $E_a \in \overline{HB}$, so $\mathcal{H}(E_a)$ is well-defined. If $\mathcal{K}$ denotes its reproducing kernel, (1.11) and (5.10) show that $f\left(z,\bar{\zeta}\right) = \mathcal{K}(\zeta,z)$ and (1.12) gives (5.11). By de Branges' theory, outlined in Section 3, also $f(z,\cdot) \in \mathcal{H}(E_a)$. $\square$

For the proof of Theorem 5.2(c), we need:

Lemma 5.6 (a) *For* Im $a > 0$, Im $b > 0$, *and* Im $z \geq 0$,

$$\left|\frac{L\left(z,\bar{b}\right)}{L(z,\bar{a})}\right| \leq 2\frac{\left|L\left(a,\bar{b}\right)\right|}{|L(a,\bar{a})|}. \tag{5.26}$$

(b) *For all* $u, v \in \mathbb{C}$,

$$|f(u,v)|^2 \leq f(u,\bar{u}) f(v,\bar{v}). \tag{5.27}$$

(c) *For all* $a, b \in \mathbb{R}$*, with* $L(a,b) \neq 0$*, and all* $z \in \mathbb{C}$*,*

$$f(z,\bar{z}) \leq \left(\frac{|b-z|}{|\mathrm{Im}\ z|} \frac{|L(a,z)|}{|L(a,b)|} \right)^2 f(b,b). \tag{5.28}$$

Proof. (a) The functional equation (5.4) gives

$$L\left(z,\bar{b}\right) L(a,\bar{a}) = L(a,z) L\left(\bar{a},\bar{b}\right) - L(\bar{a},z) L\left(a,\bar{b}\right).$$

If $\mathrm{Im}\ z \geq 0$, we obtain from Theorem 5.1(c), that $|L(a,z)| \leq |L(\bar{a},z)|$ and $\left|L\left(\bar{a},\bar{b}\right)\right| = |L(a,b)| \leq \left|L\left(a,\bar{b}\right)\right|$. Thus

$$\left|L\left(z,\bar{b}\right) L(a,\bar{a})\right| \leq 2\left|L(\bar{a},z) L\left(a,\bar{b}\right)\right|.$$

(b) By the Cauchy-Schwarz inequality,

$$|K_n(z,w)|^2 \leq K_n(z,\bar{z}) K_n(w,\bar{w}).$$

After appropriate substitutions in variable, and division by $K_n(\xi_n,\xi_n)$, this leads to

$$|f_n(u,v)|^2 \leq f_n(u,\bar{u}) f_n(v,\bar{v}).$$

Now let $n \to \infty$ through $\mathcal{S}$.

(c) Let $a, b \in \mathbb{R}$. The functional equation (5.4) gives

$$L(z,\bar{z}) L(a,b) = L(a,z) L(b,\bar{z}) - L(b,z) L(a,\bar{z}).$$

Then

$$\begin{aligned} 2|\mathrm{Im}\ z| f(z,\bar{z}) |L(a,b)| &\leq 2|L(a,z)| |L(b,z)| \\ &\leq 2|L(a,z)| |b-z| f(b,b)^{1/2} f(z,\bar{z})^{1/2}, \end{aligned}$$

by (b). Rearranging this gives the result. □

Proof of Theorem 5.2(c). From (a) of the lemma, we see that for all z in the upper-half plane,

$$|E_b(z)/E_a(z)| \leq 2\frac{\left|L\left(a,\bar{b}\right)\right|}{|L(a,\bar{a})|^{1/2}\left|L\left(b,\bar{b}\right)\right|^{1/2}}.$$

Recall that the denominator is positive, in view of (5.7). To show $\mathcal{H}(E_a) = \mathcal{H}(E_b)$, let $g \in \mathcal{H}(E_b)$. Then g/E_b, $g^*/E_b \in H^2(\mathbb{C}^+)$. The last inequality shows that also g/E_a, $g^*/E_a \in H^2(\mathbb{C}^+)$. Thus $\mathcal{H}(E_a) \supseteq \mathcal{H}(E_b)$, and the converse inclusion is then obvious. Finally it follows that for all g,

$$\|g\|_{E_b} \leq 2\frac{\left|L\left(a,\bar{b}\right)\right|}{\left|L\left(a,\bar{a}\right)\right|^{1/2}\left|L\left(b,\bar{b}\right)\right|^{1/2}}\|g\|_{E_a},$$

and the inequality is reversible, and thus the two norms are equivalent. □

PROOF OF THEOREM 5.3(A). First note that F cannot have any non-real zeros, for it is a uniform limit as $n \to \infty$ through $\mathcal{S}$, of $zf_n(0,z)$, which has only real zeros. Define, as at (3.10), the phase function φ by

$$E_a(x) = |E_a(x)|\, e^{-i\varphi(x)}.$$

From (5.10), for real x,

$$\begin{aligned} F(x) &= xf(x,0) \\ &= \frac{i}{2\pi}\left(E_a(x)\overline{E_a(0)} - E_a^*(x)\overline{E_a^*(0)}\right) \\ &= \frac{1}{\pi}|E_a(x)|\,|E_a(0)|\sin(\varphi(x)-\varphi(0)). \end{aligned} \tag{5.29}$$

Also,

$$\begin{aligned} F'(x) &= \frac{1}{\pi}\left(\frac{d}{dx}|E_a(x)|\right)|E_a(0)|\sin(\varphi(x)-\varphi(0)) \\ &\quad +\frac{1}{\pi}|E_a(x)|\,|E_a(0)|\cos(\varphi(x)-\varphi(0))\,\varphi'(x). \end{aligned} \tag{5.30}$$

It follows from (5.29) and the fact that E_a has non-real zeros, that,

$$F(x) = 0 \Longleftrightarrow \sin(\varphi(x)-\varphi(0)) = 0.$$

Let $\alpha = \varphi(0)$ and recall that $\{s_j\}$ were defined at (3.12) by $\varphi(s_j) = \alpha + j\pi$, $j = 0, \pm 1, \pm 2, \ldots$. It follows that after reordering, the $\{\rho_j\}$ are just the $\{s_k\}$. We next show that these zeros with $f(\rho_j,\rho_j) \neq 0$ are simple. If ρ_j is not simple, it follows from (5.29) and (5.30) that both $\varphi(\rho_j) = \alpha + k\pi$ for some k, and $\varphi'(\rho_j) = 0$. Then (3.11) with $\mathcal{K}$ taken as f shows that

$$f(\rho_j,\rho_j) = \frac{1}{\pi}\varphi'(\rho_j)\,|E_a(\rho_j)|^2 = 0,$$

a contradiction. Thus, all zeros $\{\rho_j\}$ of F with $f(\rho_j,\rho_j) \neq 0$ are simple. □

Proof of Theorem 5.3(b). Recall that $\alpha = \varphi(0)$. The second equation in (5.29) shows that for some constant C,

$$e^{i\alpha} E_a(z) - e^{-i\alpha} E_a^*(z) = CF(z). \tag{5.31}$$

Of course $C \neq 0$, as E_a and E_a^* have zeros in opposite half-planes. If we knew that (3.13) holds, then we could simply apply the de Branges theory, but we do not. So we proceed as follows: we know that $f(\cdot,\cdot)$ is the locally uniform limit of $f_n(\cdot,\cdot)$, so the $\{\rho_j\}$ are limits of the zeros $\{\rho_{jn}\}$ of f_n. Here if $j' \neq k'$,

$$f_n(\rho_{j'n}, \rho_{k'n}) = \frac{K_n(t_{j'n}, t_{k'n})}{K_n(\xi_n, \xi_n)} = 0.$$

(Recall (2.2) and (2.9)). Taking appropriate limits with appropriate $j' = j'(n), k' = k'(n)$, and using Hurwitz' Theorem, leads to

$$f(\rho_j, \rho_k) = 0, \ j \neq k. \tag{5.32}$$

The reproducing kernel relation (5.11) gives

$$0 = \int_{-\infty}^{\infty} \frac{f(t, \rho_j) f(t, \rho_k)}{|E_a(t)|^2} dt.$$

It follows then that $\left\{ \frac{f(\rho_k,\cdot)}{\sqrt{f(\rho_k,\rho_k)}} \right\}_k$ is an orthonormal sequence in $\mathcal{H}(E_a)$ and for all $g \in \mathcal{H}(E_a)$, we have (in view of (5.11)), the orthonormal expansion

$$G[g](z) = \sum_j \frac{g(\rho_j)}{\sqrt{f(\rho_j,\rho_j)}} \frac{f(\rho_j, z)}{\sqrt{f(\rho_j,\rho_j)}}.$$

By Bessel's inequality,

$$\sum_j \frac{|g(\rho_j)|^2}{f(\rho_j,\rho_j)} \leq \|g\|_{E_a}^2 = \int_{-\infty}^{\infty} \left| \frac{g}{E_a} \right|^2.$$

Moreover, every partial sum of $G[g] \in \mathcal{H}(E_a)$, and the convergence of the series in the last inequality easily yields that $G[g]$ is the limit of these partial sums in the norm of $\mathcal{H}(E_a)$. As the latter space is a Hilbert space, we obtain (5.14). □

Proof of Theorem 5.3(c). Since $F \notin \mathcal{H}(E_a)$, then recalling that $\alpha = \varphi(0)$, (5.31) shows that

$$e^{i\alpha} E_a - e^{-i\alpha} E_a^* \notin \mathcal{H}(E_a) . \tag{5.33}$$

This allows one to apply the theory in Section 3. We identified the $\{\rho_j\}$ with the $\{s_k\}$, and can just apply (3.14) to (3.16). $\square$

Proof of Theorem 5.4(a). We are assuming for a given a, with $\mathrm{Im}\ a > 0$, that $f(a, z)$ is of exponential type σ. Then the same is true of $L(a, z) = (z - a) f(a, z)$ and $L(\bar{a}, z)$. By Lemma 5.6(a), if $\mathrm{Im}\ b > 0, \mathrm{Im}\ z \geq 0$,

$$\left|L\left(z, \bar{b}\right)\right| \leq 2 \frac{\left|L\left(a, \bar{b}\right)\right|}{\left|L(a, \bar{a})\right|} \left|L(z, \bar{a})\right| ,$$

and by Theorem 5.1(c),

$$\left|L\left(\bar{z}, \bar{b}\right)\right| = \left|L(z, b)\right| \leq \left|L\left(z, \bar{b}\right)\right| .$$

It follows easily that the exponential type of $L\left(\bar{b}, \cdot\right)$ is no greater than that of $L(\bar{a}, \cdot)$. The same is then true for $f\left(\bar{b}, \cdot\right)$ and $f(\bar{a}, \cdot)$, and hence also $f(b, \cdot)$ and $f(a, \cdot)$. The reverse assertion follows by symmetry. By conjugate symmetry, the same is true when $\mathrm{Im}\ a < 0$ or $\mathrm{Im}\ b < 0$. Thus when a is non-real, $L(a, \cdot)$ and $f(a, \cdot)$ have exponential type σ.

It remains to show that if a is real, $L(a, z)$ has type σ. From the functional relation (5.4), if $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Im}\ \alpha, \mathrm{Im}\ \beta \neq 0$, and $L(\alpha, \beta) \neq 0$,

$$\begin{aligned} \left|L(a, z)\right| &= \left|L(z, a)\right| \\ &= \frac{1}{\left|L(\alpha, \beta)\right|} \left|L(\alpha, z) L(\beta, a) - L(\beta, z) L(\alpha, a)\right| . \end{aligned}$$

As both $L(\alpha, z)$ and $L(\beta, z)$ are of exponential type σ, it follows that $L(a, z)$ is of type at most σ. To show that it is of type $\geq \sigma$, we let b be real with $L(a, b) \neq 0$, c be non-real, and use Lemma 5.6(b), (c):

$$\begin{aligned} \left|f(c, z)\right| &\leq f(c, \bar{c})^{1/2} f(z, \bar{z})^{1/2} \\ &\leq f(c, \bar{c})^{1/2} \frac{\left|b - z\right|}{\left|\mathrm{Im}\ z\right|} \frac{\left|L(a, z)\right|}{\left|L(a, b)\right|} f(b, b)^{1/2} . \end{aligned}$$

Thus for $\left|\mathrm{Im}\ z\right| \geq 1$, $\left|f(c, z)\right|$ grows no faster than $C\left|z\right|\left|L(a, z)\right|$. Since both $f(c, \cdot)$ and $L(a, \cdot)$ are entire of order $\leq \sigma$, the Phragmen-Lindelöf principle allows one to estimate $f(c,z)$ on the strip $|\text{Im } z| \leq 1$. We deduce that the exponential type of $f(c,\cdot)$ is no smaller than that of $L(a,\cdot)$. Consequently $L(a,\cdot)$, and hence $f(a,\cdot)$, have exponential type $\geq \sigma$. Thus they have type σ. $\square$

For the proof of Theorem 5.4(b), we need:

Lemma 5.7 *Assume in addition to the hypotheses of Theorem* 5.1, *that* $f(a,\cdot)$ *is of type* σ *and* (5.17) *holds. Then*

(a) *There exists* $C > 0$ *such that the zeros* $\{\rho_j\}$ *of* $L(z,0)$ *satisfy for all* j,

$$\rho_{j+1} - \rho_j \geq C. \tag{5.34}$$

(b) *There exists* $C > 0$ *such that for all* $g \in PW_\sigma$,

$$\sum_j |g(\rho_j)|^2 \leq C \|g\|^2_{L_2(\mathbb{R})}. \tag{5.35}$$

(c) *For all* $z \in \mathbb{C}$,

$$\sum_{j=1}^{\infty} \frac{|f(\rho_j,z)|^2}{f(\rho_j,\rho_j)} \leq f(z,\bar{z}). \tag{5.36}$$

Proof. (a) Recall from (5.32) that

$$f(\rho_{j+1},\rho_j) = 0.$$

Next, by hypothesis, $f(\rho_{j+1},\cdot)$ is entire of exponential type, and bounded on the real axis. Indeed, our hypothesis (5.17), and (5.27) give

$$|f(\rho_{j+1},x)| \leq f(\rho_{j+1},\rho_{j+1})^{1/2} f(x,x)^{1/2} \leq C_1.$$

Bernstein's inequality for entire functions of exponential type [13, p. 227] gives for all real t,

$$\left|\frac{\partial}{\partial t} f(\rho_{j+1},t)\right| \leq C_1\sigma.$$

Then using our (5.17) again, for some ξ between ρ_j and ρ_{j+1},

$$\begin{aligned} C_2 &\leq f(\rho_{j+1},\rho_{j+1}) \\ &= f(\rho_{j+1},\rho_{j+1}) - f(\rho_{j+1},\rho_j) \\ &= \left(\frac{\partial}{\partial t} f(\rho_{j+1},t)_{|t=\xi}\right)(\rho_{j+1}-\rho_j) \\ &\leq C_1\sigma(\rho_{j+1}-\rho_j). \end{aligned}$$

(b) This is an immediate consequence of (a) and a well known estimate [13, p. 150, no. 4.].

(c) This follows by applying Bessel's inequality (5.13) to

$$g(t) = f(t, z),$$

and using the reproducing kernel identity (5.11). $\square$

Proof of Theorem 5.4(b). Let $g \in PW_\sigma$ and

$$G(z) = G[g](z) = \sum_{j=-\infty}^{\infty} g(\rho_j) \frac{f(\rho_j, z)}{f(\rho_j, \rho_j)}.$$

We claim that $G \in \mathcal{H}(E_a)$. Indeed, by (b) of the previous lemma, and as $f(\rho_j, \rho_j) \sim 1$ uniformly in j,

$$\sum_j \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} < \infty,$$

and as in the proof of Theorem 5.3(b), this gives $G \in \mathcal{H}(E_a)$.We are going to show that $G = g$. To this end, let

$$\Psi(z) = \frac{g(z) - G(z)}{F(z)}.$$

As $G(\rho_j) = g(\rho_j)$, (recall (5.32)) and F has simple zeros at each ρ_j (recall Theorem 5.3(a)), so Ψ is entire. As both numerator and denominator are of exponential type, so is Ψ [13, Theorem 5, p. 80]. Next, we claim that also

$$G(z) = \sum_{j=-\infty}^{\infty} g(\rho_j) \frac{F(z)}{F'(\rho_j)(z - \rho_j)}. \tag{5.37}$$

Let $F(\alpha) = L(\alpha, 0) \neq 0$. Since $L(0, \rho_j) = 0$, the functional equation (5.4) gives

$$L(z, \rho_j) L(\alpha, 0) = L(\alpha, z) L(0, \rho_j) - L(0, z) L(\alpha, \rho_j) = F(z) L(\alpha, \rho_j)$$

$$\Rightarrow f(z, \rho_j) = \frac{F(z) L(\alpha, \rho_j)}{F(\alpha)(z - \rho_j)}.$$

Letting $z \to \rho_j$, we obtain

$$f(\rho_j, \rho_j) = F'(\rho_j) \frac{L(\alpha, \rho_j)}{F(\alpha)}.$$

Combining these last two identities, we see that

$$\frac{f(\rho_j, z)}{f(\rho_j, \rho_j)} = \frac{F(z)}{F'(\rho_j)(z - \rho_j)},$$

and we have (5.37). Next, that identity shows that

$$\left| \frac{G(z)}{F(z)} \right| \leq \left(\sum_{j=-\infty}^{\infty} |g(\rho_j)|^2 \right)^{1/2} \left(\sum_{j=-\infty}^{\infty} \frac{1}{|F'(\rho_j)(z - \rho_j)|^2} \right)^{1/2}.$$

Let $\varepsilon \in \left(0, \frac{\pi}{2}\right)$. Here in the cut (double) sector $\mathcal{A}_\varepsilon = \{z : |z| \geq 1$ and $\varepsilon \leq |\arg z| \leq \pi - \varepsilon\}$, there exists C_ε such that for all j,

$$|z - \rho_j| \geq C_\varepsilon |i - \rho_j|.$$

Moreover,

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \frac{1}{|F'(\rho_j)(i - \rho_j)|^2} &= \frac{1}{|F(i)|^2} \sum_{j=-\infty}^{\infty} \frac{|F(i)|^2}{|F'(\rho_j)(i - \rho_j)|^2} \\ &= \frac{1}{|F(i)|^2} \sum_{j=-\infty}^{\infty} \left| \frac{f(\rho_j, i)}{f(\rho_j, \rho_j)} \right|^2 \\ &\leq \frac{1}{|F(i)|^2 \inf_{x \in \mathbb{R}} f(x, x)} f(i, \bar{\imath}) < \infty, \end{aligned}$$

by (5.36). Then for any $n \geq 1$, we see that

$$\limsup_{z \to \infty, z \in \mathcal{A}_\varepsilon} \left| \frac{G(z)}{F(z)} \right| \leq \left(\sum_{j=-\infty}^{\infty} |g(\rho_j)|^2 \right)^{1/2} \left(\frac{1}{C_\varepsilon^2} \sum_{|j| \geq n} \frac{1}{|F'(\rho_j)(i - \rho_j)|^2} \right)^{1/2}.$$

Since this has limit 0 as $n \to \infty$, we have shown that

$$\lim_{z \to \infty, z \in \mathcal{A}_\varepsilon} \left| \frac{G(z)}{F(z)} \right| = 0. \tag{5.38}$$

Next, F is of exponential type σ, has real zeros, and

$$|F(x)| = |xf(0,x)| \leq |x| f(0,0)^{1/2} f(x,x)^{1/2} \leq C|x|$$

by (5.17) and (5.27). Thus it lies in the Cartwright class. From (3.6), for $\theta \in (-\pi, \pi) \setminus \{0\}$,

$$\lim_{r\to\infty} \frac{\log \left|F\left(re^{i\theta}\right)\right|}{r} = \sigma |\sin\theta|.$$

Let us now assume g has type $\tau < \sigma$. Since it is square integrable along the real axis, g also lies in the Cartwright class. By (3.7), for $\theta \in (-\pi, \pi) \setminus \{0\}$,

$$\limsup_{r\to\infty} \frac{\log \left|g\left(re^{i\theta}\right)\right|}{r} \leq \tau |\sin\theta|.$$

Then for $\theta \in (-\pi, \pi) \setminus \{0\}$, as $r \to \infty$,

$$\left|\frac{g}{F}\right| \left(re^{i\theta}\right) \leq \exp\left((\tau - \sigma) r |\sin\theta| + o(r)\right).$$

In particular, for such θ,

$$\lim_{r\to\infty} \left|\frac{g}{F}\right| \left(re^{i\theta}\right) = 0.$$

Then for $\theta \in (-\pi, \pi) \setminus \{0\}$, this and (5.38) show that

$$\lim_{r\to\infty} |\Psi| \left(re^{i\theta}\right) = 0.$$

Inasmuch as Ψ is an entire function of exponential type, the Phragmen-Lindelöf principle (applied on sectors of opening angle less than π) shows that it is bounded in the plane, and hence constant. As it has limit 0 at ∞, we have $\Psi \equiv 0$, so

$$g = G \in \mathcal{H}(E_a).$$

Finally, if g has type σ, then for $\varepsilon \in (0,1)$, the scaled function $g_\varepsilon(z) = g(\varepsilon z)$ has type $\varepsilon\sigma < \sigma$, so

$$g_\varepsilon = G[g_\varepsilon].$$

It is easily seen that we can let $\varepsilon \to 1-$ in both sides of this identity. $\square$

We note that for several of the proofs in this section, one can avoid using de Branges' machinery, and instead take limits in results that hold for the original reproducing kernels K_n. For the proof of Theorem 5.4(c), we seem to be forced to do the latter.

Proof of Theorem 5.4(c). Let $g \in \mathcal{H}(E_a)$. Since $g/E_a, g/E_a^* \in H^2(\mathbb{C}^+)$, while E_a is of exponential type σ, it follows that g has exponential type at most σ. Next, recall that $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ are the quadrature points for μ including ξ_n. Fix $\ell \geq 1$. The Gauss quadrature formula (2.5) and the fact that $K_n(t_{jn}, t_{kn}) = 0$ for $j \neq k$, gives

$$\int \left| \sum_{|j| \leq \ell} g(\rho_j) \frac{K_n(t_{jn}, s)}{K_n(t_{jn}, t_{jn})} \right|^2 d\mu(s) = \sum_{|j| \leq \ell} \frac{|g(\rho_j)|^2}{K_n(t_{jn}, t_{jn})}.$$

Let $r > 0$ and make the substitution

$$s = \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} = \xi_n + \frac{t}{K_n(\xi_n, \xi_n)\, \mu'(\xi_n)}$$

and recall (2.7), (2.9). By dropping the singular part of μ, we obtain for, large enough n,

$$\int_{-r}^{r} \left| \sum_{|j| \leq \ell} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{\mu'(\xi_n)} dt \leq \sum_{|j| \leq \ell} \frac{|g(\rho_j)|^2}{f_n(\rho_{jn}, \rho_{jn})}. \tag{5.39}$$

As $n \to \infty$ through $\mathcal{S}$, the right-hand side converges to

$$\sum_{|j| \leq \ell} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \leq \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}.$$

Recall from Theorem 5.3(b) that this series converges. Next, as $n \to \infty$ through $\mathcal{S}$, uniformly for t in compact sets,

$$\sum_{|j| \leq \ell} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \to \sum_{|j| \leq \ell} g(\rho_j) \frac{f(\rho_j, t)}{f(\rho_j, \rho_j)} =: G_\ell(t)$$

and we have also used the uniform convergence of $f_n(0, z)$ to $f(0, z)$, which forces the zeros $\{\rho_{jn}\}$ of f_n to converge to those of f. By Fatou's Lemma,

$$\begin{aligned}
&\liminf_{n\to\infty, n\in\mathcal{S}} \int_{-r}^{r} \left| \sum_{|j|\le\ell} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} dt \\
&\ge \int_{-r}^{r} |G_\ell(t)|^2 \liminf_{n\to\infty, n\in\mathcal{S}} \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} dt \\
&\ge C_0 \int_{-r}^{r} |G_\ell(t)|^2 dt,
\end{aligned}$$

under our hypothesis (5.19). Alternatively, if we assume our Lebesgue point type condition (5.20), we write the left-hand side of (5.39) as

$$\begin{aligned}
&\int_{-r}^{r} \left| \sum_{|j|\le\ell} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 dt \\
&+ \int_{-r}^{r} \left| \sum_{|j|\le\ell} g(\rho_j) \frac{f_n(\rho_{jn}, t)}{f_n(\rho_{jn}, \rho_{jn})} \right|^2 \left\{ \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} - 1 \right\} dt \\
&= \int_{-r}^{r} |G_\ell(t)|^2 dt + o(1) + O\left(\int_{-r}^{r} \left| \frac{\mu'\left(\xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu'(\xi_n)} - 1 \right| dt \right) \\
&= \int_{-r}^{r} |G_\ell(t)|^2 dt + o(1).
\end{aligned}$$

Thus

$$C_0 \int_{-r}^{r} |G_\ell(t)|^2 dt \le \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)},$$

where $C_0 = 1$ if we have the Lebesgue point type condition. As in the proof of Theorem 5.4(b), $\int_{-r}^{r} |g - G_\ell|^2 \to 0$ as $\ell \to \infty$, so we obtain

$$C_0 \int_{-r}^{r} |g(t)|^2 dt \le \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}.$$

Letting $r \to \infty$ gives

$$C_0 \int_{-\infty}^{\infty} |g|^2 \le \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)}. \tag{5.40}$$

Thus $g \in L_2(\mathbb{R})$, and g is of exponential type at most σ, so also $g \in PW_\sigma$. We have shown that $\mathcal{H}(E_a) \subset PW_\sigma$, and hence $\mathcal{H}(E_a) = PW_\sigma$. It remains to prove equivalence of the norms. First observe that $F \notin \mathcal{H}(E_a)$. Indeed, if $F \in \mathcal{H}(E_a)$, as $F(\rho_j) = 0$ for all j, Theorem 5.4(b) shows that identically

$$F = G[F] = 0,$$

a contradiction. Then (5.15) shows that

$$\|g\|_{E_a}^2 = \int_{-\infty}^{\infty} \left| \frac{g}{E_a} \right|^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \geq C_0 \|g\|_{L_2(\mathbb{R})}^2$$

by (5.40). In the other direction, as $f(\rho_j, \rho_j) \sim 1$ uniformly in j, (5.35) shows that

$$\|g\|_{E_a}^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{f(\rho_j, \rho_j)} \leq C_2 \|g\|_{L_2(\mathbb{R})}^2 . \qquad \square$$

An alternative proof of the norm equivalence uses the closed graph theorem. Let I denote the identity operator from $\mathcal{H}(E_a)$ to PW_σ. Its graph $\{(f, If) : f \in \mathcal{H}(E_a)\}$ is all of $\mathcal{H}(E_a) \times PW_\sigma$, so is closed. Then the operator I is a continuous linear operator, and so is bounded.

6. Proof of Theorems 1.3, 1.4, 1.5

Lemma 6.1 *Assume the hypotheses of Theorem* 1.3.

(a) $\{f_n(u,v)\}_{n=1}^{\infty}$ *is uniformly bounded for* u, v *in compact subsets of the plane.*

(b) *Let* $f(u,v)$ *denote the locally uniform limit of some subsequence* $\{f_n(u,v)\}_{n\in\mathcal{S}}$ *of* $\{f_n(u,v)\}_{n=1}^{\infty}$. *Then for each fixed* $u \in \mathbb{C}$, $f(u,\cdot)$ *is an entire function of exponential type. Moreover, for some* C_1 *and* C_2 *independent of* u, v, *and the subsequence* $\mathcal{S}$,

$$|f(u,v)| \leq C_1 e^{C_2 (|Im\ u| + |Im\ v|)}. \tag{6.1}$$

Proof. This is exactly the same as that of Lemma 5.2 in [22], but we provide some details. We assumed that $\mu' \sim 1$ in some open set O containing compact J. It follows that J is covered by finitely many open intervals in O.

By increasing the size of J, we may assume that J consists of finitely many compact intervals. It then suffices to consider the case where J is just one interval, and we now assume this. Since μ is absolutely continuous in the larger open set O, and μ' is bounded above and below there, we have the well known bound [28, Theorem 20, p. 116]

$$K_n (x, x)^{-1} = \lambda_n (x) \sim \frac{1}{n}, \tag{6.2}$$

uniformly in n and in each compact subset of O. By reducing O, we can assume this holds in O. By Cauchy-Schwarz, we have

$$\frac{1}{n} |K_n (\xi, t)| \leq \left(\frac{1}{n} K_n (\xi, \xi) \right)^{1/2} \left(\frac{1}{n} K_n (t, t) \right)^{1/2} \leq C$$

for $\xi, t \in O$. By Bernstein's growth lemma in the plane, [22, Lemma 5.1], applied separately in each variable, we then have for $\xi, t \in O, |a|, |b| \leq A$ and $n \geq n_0 (A)$,

$$\frac{1}{n} \left| K_n \left(\xi + i\frac{a}{n}, t + i\frac{b}{n} \right) \right| \leq C e^{C_2 (|a| + |b|)}. \tag{6.3}$$

(Strictly speaking, we have to take a slightly smaller set than O, but can relabel.) C and C_2 are independent of A, ξ, t, a, b. Of course if u, v lie in a bounded subset of the plane, and $\xi \in O$, then for n large enough, we may write $\xi + \frac{u}{n} = \xi + \frac{\mathrm{Re}(u)}{n} + i\frac{\mathrm{Im}(u)}{n}$, where $\xi + \frac{\mathrm{Re}(u)}{n}$ is contained in a slightly large open set than O. By relabelling, we may assume it lies in O. Then we may recast (6.3) in the form

$$\frac{1}{n} \left| K_n \left(\xi + \frac{u}{n}, \xi + \frac{v}{n} \right) \right| \leq C e^{C_2 (|\mathrm{Im}\ u| + |\mathrm{Im}\ v|)}. \tag{6.4}$$

Since

$$\tilde{K}_n (\xi_n, \xi_n) \sim n,$$

we see also that for $|u|, |v| \leq A$ and $n \geq n_0 (A)$

$$|f_n (u, v)| \leq C_1 e^{C_2 (|\mathrm{Im}\ u| + |\mathrm{Im}\ v|)},$$

where C_1, C_2 are independent of n, u, v, A. The stated uniform boundedness follows.

(b) Now $\{f_n(u,v)\}_{n=1}^{\infty}$ is a normal family of two variables u,v. If $f(u,v)$ is the locally uniform limit through the subsequence $\mathcal{S}$ of integers, we see that $f(u,v)$ is an entire function in u,v satisfying for all complex u,v,

$$|f(u,v)| \leq C_1 e^{C_2(|\operatorname{Im} u|+|\operatorname{Im} v|)}. \tag{6.5}$$

In particular, $f(u,v)$ is bounded for $u,v \in \mathbb{R}$, and is an entire function of exponential type in each variable. $\square$

Proof of Theorem 1.3. (a) This follows directly from Lemma 6.1.

(b) This follows from Lemma 6.1(b) and Theorem 5.4. Note that the hypothesis $f(t,t) \sim 1$ there is an easy consequence of (6.2) and the fact the values of f are limits of ratios of K_n taken over smaller and smaller neighborhoods of ξ_n.

(c) This follows from Theorem 5.2.

(d) This follows from Theorem 5.4. The hypothesis (5.19) follows easily from our hypothesis $\mu' \sim 1$ in an open set containing J. $\square$

Proof of Theorem 1.4. (a) The functional relation (1.18) is (5.4) in Theorem 5.1. Next, once we know $f(a,z)$ for all z, we also know $f(\bar{a},z) = \overline{f(a,\bar{z})}$ for all z. Moreover, as shown after Theorem 5.1, $f(iy,-iy) \geq 1$ for all real y, and then (5.7) shows that $L(a,\bar{a}) \neq 0$. Then for all z,v,

$$L(z,v) = \frac{1}{L(a,\bar{a})}\left\{L(a,z)\,L(\bar{a},v) - L(\bar{a},z)\,L(a,v)\right\}.$$

So $L(z,v)$ and hence $f(z,v)$ is uniquely determined.

(b) This follows from Theorem 5.3.

(c) The expansions were established in Theorem 5.3(c) and 5.4(b). $\square$

Proof of Theorem 1.5. The expansion (1.20) ensures that $\{\rho_j\}$ is a complete interpolating sequence for PW_σ, as defined in Section 3. Indeed (1.20) shows that each $g \in PW_\sigma$ is uniquely determined by its values on $\{\rho_j\}$, and we cannot drop a single ρ_k, since $f(\rho_k,z)$ vanishes at all ρ_j with $j \neq k$. By a Theorem of Hruscev, Nikolskii, and Pavlov [10, p. 286], [30, p. 791], the function $h(t) = \nu(t) - \frac{\sigma}{\pi}t$ belongs to BMO. By (3.17), this ensures that for each $p > 0$,

$$\sup_I \frac{1}{|I|} \int_I |h - h_I|^p < \infty. \tag{6.6}$$

Next, we apply a well known inequality [9, p. 223, Lemma 1.1]: if I and J are intervals with $|J| > 2|I|$, then

$$|h_I - h_J| \leq C \log(|J| / |I|),$$

where C is independent of I and J. This leads easily to the estimate

$$\left|h_{[-r,r]}\right| \leq C \log r,\ r \geq 2.$$

Together with (6.6), this yields for $j \geq 1$,

$$\int_{2^j}^{2^{j+1}} |h|(t)^p \, dt \leq C 2^j j^p$$

and hence

$$\int_{2^j}^{2^{j+1}} \frac{|h(t)|^p}{(1 + |t|)(\log(2 + |t|))^{p+\tau}} dt \leq C j^{-\tau}.$$

Adding over $j \geq 1$, gives

$$\int_2^{\infty} \frac{|h(t)|^p}{(1 + |t|)(\log(2 + |t|))^{p+\tau}} dt < \infty.$$

The range $(-\infty, -2)$ can be treated similarly. □

7. Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6. The assumption (1.26) implies that $\{f_n(a, \cdot)\}_{n=1}^{\infty}$ is uniformly bounded in compact subsets of the plane. Using the functional relation (5.23), we deduce that the same is true of $\{f_n(\cdot, \cdot)\}_{n=1}^{\infty}$. Note too that if $z = x + iy$, the fact that each p_n has real zeros, ensures that,

$$f_n(z, \bar{z}) \geq f_n(x, x).$$

Our hypothesis (1.27) ensures that if we fix a non-real a, $\{f_n(a, \bar{a})\}_{n=1}^{\infty}$ is bounded below. Next, let $A > 0$. The exponential bound (1.26) together with the functional relation (5.22) ensures that for $n \geq n_0(A)$ and $|z|, |v| \leq A$,

$$|f_n(z, v)| \leq C_1 e^{C_1(|\mathrm{Im}\, z| + |\mathrm{Im}\, v|)}.$$

Here C_1 and C_2 are independent of n, A, z, v. If we take some subsequential limit f, then it follows that the hypotheses of Theorems 5.1 and 5.4 are fulfilled. Indeed, the hypothesis (5.17) in Theorem 5.4 follows from (1.27), while (1.28) is the requisite modification of (5.19). The proofs of Theorems 5.1 to 5.4 then go through without change. $\square$

Proof of Theorem 1.7. **Step 1: The functions f and E:** We assume that $f\left(z, \bar{\zeta}\right) = \mathcal{K}\left(\zeta, z\right)$, the reproducing kernel for $\mathcal{H}\left(E\right) = PW_\sigma$ and that $f\left(0,0\right) = 1$. Recall that equality of the spaces implies norm equivalence, and in turn this implies from (3.18),

$$f\left(x,x\right) = \mathcal{K}\left(x,x\right) \sim 1 \text{ in } \mathbb{R}. \tag{7.1}$$

We know from the de Branges theory (cf. (1.11)) that if $z \neq v$,

$$f\left(z,v\right) = \frac{i}{2\pi} \frac{E\left(z\right) E^*\left(v\right) - E^*\left(z\right) E\left(v\right)}{z - v}, \tag{7.2}$$

while

$$f\left(z,z\right) = \frac{i}{2\pi}\left(E'\left(z\right) E^*\left(z\right) - E^{*\prime}\left(z\right) E\left(z\right)\right). \tag{7.3}$$

By definition of a de Branges space, E has no zeros in $\{z : \operatorname{Im} z > 0\}$. It follows from (7.1) and (7.3) that it also has no real zeros. For if $E\left(x\right) = 0$, then also $E^*\left(x\right) = 0$, and (7.3) gives $f\left(x,x\right) = 0$, contradicting (7.1).

Next, as f is a reproducing kernel for $\mathcal{H}\left(E\right)$, for each fixed u, $f\left(u,\cdot\right) \in \mathcal{H}\left(E\right) = PW_\sigma$. Thus $f\left(u,\cdot\right)$ is of exponential type at most σ. We use this to show that E is of exponential type at most σ. As usual, define

$$L\left(z,v\right) = \left(z - v\right) f\left(z,v\right) = \frac{i}{2\pi}\left(E\left(z\right) E^*\left(v\right) - E^*\left(z\right) E\left(v\right)\right).$$

A little manipulation shows that for complex u, v, z,

$$E\left(z\right) L\left(u,v\right) = L\left(z,v\right) E\left(u\right) - L\left(z,u\right) E\left(v\right). \tag{7.4}$$

Choose u, v with $L\left(u,v\right) \neq 0$. Such a choice is possible, for otherwise $f\left(u,v\right) = 0$ for all $u \neq v$, and continuity yields a contradiction to (7.1). Since $L\left(\cdot,u\right)$ and $L\left(\cdot,v\right)$ are of exponential type at most σ, it follows from (7.4) that $E\left(\cdot\right)$ is of exponential type $\leq \sigma$.

Step 2: The construction of E_n: Next, as E belongs to the Hermite-Biehler class, for Im $z > 0$,

$$\left|\frac{E^*(z)}{E(z)}\right| \leq 1,$$

recall (1.8). This implies that the function E^* belongs to the class P, studied in detail in [13, p. 217 ff.]. See Corollary 3 in [13, p. 218]. As a consequence, [13, Corollary 6, p. 219] there is a sequence of polynomials $\{P_n\}$ without zeros in the (closed) lower-half plane, that converges to E^*, uniformly in compact sets. Define

$$E_n(z) = P_n^*(z) = \overline{P_n(\bar{z})},$$

a polynomial with zeros only in the open lower-half plane. Then for Im$z > 0$, $|E_n(z)| \geq |E_n(\bar{z})|$, so $E_n \in \overline{HB}$. We see that uniformly in compact subsets of the plane,

$$\lim_{n\to\infty} E_n(z) = E(z). \tag{7.5}$$

We may assume that E_n has degree n. Indeed, it is obvious that we can assume E_n has degree at most n, and we can multiply by factors $1 - \frac{z}{n^2 - i}$ to make it up to full degree. Next, for $n \geq 1$,let

$$\Omega_n(t) = \frac{1}{|E_n(t)|^2}, \quad t \in (-\infty, \infty).$$

The measure $\Omega_n(t)\,dt$ has the first $2n - 1$ finite power moments, and so we can define corresponding orthonormal polynomials $\{p_j(\Omega_n, \cdot)\}_{j=0}^{n-1}$. Let $K_n(\Omega_n, \cdot, \cdot)$ denote the nth reproducing kernel formed from these orthogonal polynomials. Then

$$K_n(\Omega_n, z, v) = \frac{i}{2\pi} \frac{E_n(z) E_n^*(v) - E_n^*(z) E_n(v)}{z - v}. \tag{7.6}$$

This was proved in [23], but also follows easily from the theory of de Branges spaces. Indeed, as E_n is a polynomial of degree n, so $\mathcal{H}(E_n)$ is the set of polynomials of degree $\leq n - 1$. The right-hand side of (7.6) is the reproducing kernel for $\mathcal{H}(E_n)$ (apart from notational conventions such as conjugate variables). By uniqueness of reproducing kernels, it equals the left-hand side.

Next, by (7.5) and (7.2), uniformly for z, v in compact sets,

$$\lim_{n\to\infty} K_n(\Omega_n, z, v) = f(z, v). \tag{7.7}$$

In particular,

$$\lim_{n\to\infty} K_n(\Omega_n, 0, 0) = f(0,0) = 1, \tag{7.8}$$

and hence

$$\lim_{n\to\infty} \tilde{K}_n(\Omega_n, 0, 0) = \lim_{n\to\infty} \frac{K_n(\Omega_n, 0, 0)}{|E_n(0)|^2} = \frac{1}{|E(0)|^2}.$$

Then

$$\lim_{n\to\infty} \frac{K_n\left(\Omega_n, 0 + \frac{z}{\tilde{K}_n(\Omega_n,0,0)}, 0 + \frac{v}{\tilde{K}_n(\Omega_n,0,0)}\right)}{K_n(\Omega_n, 0, 0)} = f\left(|E(0)|^2 z, |E(0)|^2 v\right). \tag{7.9}$$

Step 3: Truncate the support of Ω_n: Choose $a_n > 0$ such that

$$\int_{|t|\geq a_n} K_n(\Omega_n, t, t)\, \Omega_n(t)\, dt \leq \frac{1}{n}.$$

Let P be a polynomial of degree $\leq n-1$, possibly with complex coefficients. Using the Christoffel function inequality,

$$|P(t)|^2 \leq K_n(\Omega_n, t, t) \int_{-\infty}^{\infty} |P|^2 \Omega_n, \quad t \in \mathbb{R},$$

we see then that

$$\int_{|t|\geq a_n} |P|^2 \Omega_n \leq \frac{1}{n} \int_{-\infty}^{\infty} |P|^2 \Omega_n.$$

Let

$$J_n = [-a_n, a_n].$$

From this last inequality, and the extremal properties of Christoffel functions, it follows easily that for real x,

$$1 \leq \lambda_n(\Omega_n, x) / \lambda_n\left(\Omega_{n|J_n}, x\right) \leq \left(1 - \frac{1}{n}\right)^{-1}. \tag{7.10}$$

More generally, for complex z, the extremal property

$$K_n(\Omega_n, z, \bar{z}) = \sup_{\deg(P)\leq n-1} \frac{|P(z)|^2}{\int |P|^2 \Omega_n}$$

gives

$$1 \geq K_n\left(\Omega_n, z, \bar{z}\right) / K_n\left(\Omega_{n|J_n}, z, \bar{z}\right) \geq 1 - \frac{1}{n}. \tag{7.11}$$

We use this to derive a complex analogue of an inequality that formed the basis of [21]. By the reproducing kernel property of K_n,

$$\begin{aligned}
&\int \left|K_n\left(\Omega_{n|J_n}, z, t\right) - K_n\left(\Omega_n, z, t\right)\right|^2 \omega_{n|J_n}(t)\, dt \\
&\quad = K_n\left(\Omega_{n|J_n}, z, \bar{z}\right) - 2K_n\left(\Omega_n, z, \bar{z}\right) + \int \left|K_n\left(\Omega_n, z, t\right)\right|^2 \Omega_{n|J_n}(t)\, dt \\
&\quad \leq K_n\left(\Omega_{n|J_n}, z, \bar{z}\right) - 2K_n\left(\Omega_n, z, \bar{z}\right) + \int \left|K_n\left(\Omega_n, z, t\right)\right|^2 \Omega_n(t)\, dt \\
&\quad = K_n\left(\Omega_{n|J_n}, z, \bar{z}\right) - K_n\left(\Omega_n, z, \bar{z}\right).
\end{aligned} \tag{7.12}$$

Using the Christoffel function inequality

$$|P(v)|^2 \leq K_n\left(\Omega_{n|J_n}, v, \bar{v}\right) \int |P|^2\, \Omega_{n|J_n}$$

on the polynomial $P(t) = K_n\left(\Omega_{n|J_n}, z, t\right) - K_n\left(\Omega_n, z, t\right)$, and using (7.12), we obtain for all complex z, v,

$$\begin{aligned}
&\left|K_n\left(\Omega_{n|J_n}, z, v\right) - K_n\left(\Omega_n, z, v\right)\right|^2 \\
&\quad \leq K_n\left(\Omega_{n|J_n}, v, \bar{v}\right)\left(K_n\left(\Omega_{n|J_n}, z, \bar{z}\right) - K_n\left(\Omega_n, z, \bar{z}\right)\right).
\end{aligned}$$

Using (7.11), we continue this as

$$\leq \frac{C}{n} K_n\left(\Omega_n, v, \bar{v}\right) K_n\left(\Omega_n, z, \bar{z}\right).$$

The constant is independent of z, v, n. From this, (7.9), and (7.11), it follows easily that uniformly for z, v in compact sets,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{K_n\left(\Omega_{n|J_n}, 0 + \frac{z}{\tilde{K}_n\left(\Omega_{n|J_n}, 0, 0\right)}, 0 + \frac{v}{\tilde{K}_n\left(\Omega_{n|J_n}, 0, 0\right)}\right)}{K_n\left(\Omega_{n|J_n}, 0, 0\right)} \\
&\quad = f\left(|E(0)|^2 z, |E(0)|^2 v\right).
\end{aligned} \tag{7.13}$$

Step 4: Scale $\Omega_{n|J_n}$ to obtain μ_n: Define a measure μ_n on $[-1, 1]$ by

$$\mu_n'(x) = \frac{\Omega_n(a_n x)}{\Omega_n(0)}, \quad x \in [-1, 1],$$

and set $\mu_n' = 0$ outside $[-1,1]$. As E_n has no real zeros, Ω_n is infinitely differentiable on the real line, so the same is true of μ_n' on $(-1,1)$. A substitution in the orthonormality relations shows that $p_k(\mu_n, x) = p_k\left(\Omega_{n|J_n}, a_n x\right)[a_n \Omega_n(0)]^{1/2}$, and hence

$$K_n(\mu_n, z, v) = K_n\left(\Omega_{n|J_n}, a_n z, a_n v\right) a_n \Omega_n(0),$$

and recalling $\mu_n'(0) = 1$,

$$\tilde{K}_n(\mu_n, 0, 0) = K_n\left(\Omega_{n|J_n}, 0, 0\right) a_n \Omega_n(0) = a_n \tilde{K}_n\left(\Omega_{n|J_n}, 0, 0\right).$$

Then

$$\begin{aligned} & \frac{K_n\left(\mu_n, 0 + \frac{a}{\tilde{K}_n(\mu_n,0,0)}, 0 + \frac{b}{\tilde{K}_n(\mu_n,0,0)}\right)}{K_n(\mu_n, 0, 0)} \\ &= \frac{K_n\left(\Omega_{n|J_n}, 0 + \frac{a}{\tilde{K}_n\left(\Omega_{n|J_n},0,0\right)}, 0 + \frac{b}{\tilde{K}_n\left(\Omega_{n|J_n},0,0\right)}\right)}{K_n\left(\Omega_{n|J_n}, 0, 0\right)}, \end{aligned}$$

so (7.13) gives

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, 0 + \frac{a}{\tilde{K}_n(\mu_n,0,0)}, 0 + \frac{b}{\tilde{K}_n(\mu_n,0,0)}\right)}{K_n(\mu_n, 0, 0)} = f\left(|E(0)|^2 a, |E(0)|^2 b\right).$$

Then (1.29) follows if we assume $|E(0)| = 1$. Next, the upper bound (1.26) follows easily from the uniform convergence and the fact that $f(a, \cdot)$ is of exponential type. The lower bound (1.27) follows easily from (7.1) and the uniform convergence. Finally, for each real t,

$$\begin{aligned} \frac{\mu_n'\left(0 + \frac{t}{\tilde{K}_n(\mu_n,0,0)}\right)}{\mu_n'(0)} &= \Omega_n\left(\frac{t}{\tilde{K}_n\left(\Omega_{n|J_n}, 0, 0\right)}\right) / \Omega_n(0) \\ &= \left(\frac{|E_n(0)|}{\left|E_n\left(|E(0)|^2 t(1 + o(1))\right)\right|}\right)^2 \\ &\to \left(\frac{|E(0)|}{\left|E\left(|E(0)|^2 t\right)\right|}\right)^2, \end{aligned}$$

as $n \to \infty$. The condition (1.28) then follows for t in a given finite interval. Of course if $|E|$ is bounded above and below in the real line, it holds throughout the real line. $\square$

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