CHRISTOFFEL FUNCTIONS AND UNIVERSALITY LIMITS FOR ORTHOGONAL RATIONAL FUNCTIONS

KARL DECKERS AND DORON S. LUBINSKY

Abstract. We establish limits for Christoffel functions associated with orthogonal rational functions, whose poles remain a fixed distance away from the interval of orthogonality $[-1,1]$, and admit a suitable asymptotic distribution. The measure of orthogonality μ is assumed to be regular on $[-1,1]$, and to satisfy a local condition such as continuity of μ'. As a consequence, we deduce universality limits in the bulk for reproducing kernels associated with orthogonal rational functions.

Orthogonal Rational Functions, Universality Limits, Christoffel functions
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1. Introduction

Let μ be a finite positive Borel measure on $[-1,1]$, with infinitely many points in its support. Then we can define orthonormal polynomials $p_n(x) = p_n(d\mu, x) = \gamma_n x^n + ..., n \geq 0$, satisfying

$$\int_{-1}^{1} p_n p_m d\mu = \delta_{mn}.$$

We say the measure μ is *regular on* $[-1,1]$ *in the sense of Stahl, Totik, and Ullmann*, or just *regular* [17], if

$$\lim_{n\to\infty} \gamma_n^{1/n} = 2.$$

An equivalent definition involves norms of polynomials of degree $\leq n$:

$$\lim_{n\to\infty} \left[\sup_{\deg(P)\leq n} \frac{\|P\|^2_{L_\infty[-1,1]}}{\int_{-1}^{1} |P|^2 d\mu} \right]^{1/n} = 1. \tag{1.1}$$

Regularity of a measure is useful in studying asymptotics of orthogonal polynomials. One simple criterion for regularity is that $\mu' > 0$ a.e. on

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$[-1,1]$, the so-called Erdős-Turán condition. However, there are pure jump measures, and pure singularly continuous measures that are regular.

We denote the nth reproducing kernel by

$$K_n(d\mu,x,y)=\sum_{j=0}^{n-1}p_j(d\mu,x)\,p_j(d\mu,y)\,, \tag{1.2}$$

and its normalized cousin by

$$\tilde{K}_n(d\mu,x,y)=\mu'(x)^{1/2}\,\mu'(y)^{1/2}\,K_n(d\mu,x,y)\,. \tag{1.3}$$

When $y=x$, we obtain the *Christoffel function*

$$\lambda_n(d\mu,x)=1/K_n(d\mu,x,x)\,,$$

which satisfies the extremal property

$$\lambda_n(d\mu,x)=\inf_{\deg(P)\leq n-1}\frac{\int|P|^2\,d\mu}{|P(x)|^2}. \tag{1.4}$$

A classical result of Maté, Nevai, and Totik [13] (see also [18]) asserts that if μ is regular on $[-1,1]$, and in some subinterval $[a,b]$

$$\int_a^b\log\mu'>-\infty, \tag{1.5}$$

then for a.e. $x\in[a,b]$,

$$\lim_{n\to\infty}n\lambda_n(d\mu,x)=\pi\mu'(x)\sqrt{1-x^2}. \tag{1.6}$$

If instead we assume that μ is regular in $[-1,1]$, while μ is absolutely continuous in a neighborhood of some $x\in(-1,1)$, and μ' is continuous at x, then this last limit holds at x.

In the theory of random matrices [8], [14], universality limits describe local spacing of eigenvalues of random matrices. It is a remarkable fact that the universality limit in the bulk at a given point $\xi\in(-1,1)$ reduces to the technical assertion

$$\lim_{n\to\infty}\frac{\tilde{K}_n\left(d\mu,\xi+\frac{a}{\tilde{K}_n(d\mu,\xi,\xi)},\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)}{\tilde{K}_n(d\mu,\xi,\xi)}=\frac{\sin\pi(a-b)}{\pi(a-b)}, \tag{1.7}$$

uniformly for a,b in compact subsets of the real line. Sometimes, $\tilde{K}_n$ is replaced by K_n, and we can then allow a,b to be complex. There is a substantial literature on this limit. Amongst recent results, we note Totik's [9], [19] that if μ is compactly supported and regular, and (1.5) holds, then the universality limit (1.7) holds for a.e. $\xi\in(a,b)$. Barry Simon had a similar result for finitely many intervals [16]. It has also recently been shown [12] that without any local or global conditions on μ, universality holds in measure in $\{x:\mu'(x)>0\}$.

The aim of this paper is to establish limits for Christoffel functions, and universality limits associated with orthogonal rational functions. The latter have been studied and applied extensively for over thirty years, with many of the key results collected in the monograph [2]. Some other aspects of orthogonal rational functions, including asymptotics, are given in [1], [3], [4], [5], [6], [7], [20], [21].

We shall assume that we are given a sequence of extended complex numbers that will serve as our poles

$$A = \{\alpha_1, \alpha_2, \alpha_3, ...\} \subset \bar{\mathbb{C}} \backslash [-1, 1] .$$

Thus we are allowing some (or even all) of the $\alpha_j = \infty$. We let $\eta > 0$ and

$$A_\eta = \left\{ z \in \bar{\mathbb{C}} : \mathrm{dist}\,(z, [-1, 1]) \geq \eta \right\}$$

and assume that all $\alpha_j \in A_\eta$, so that for $j \geq 1$,

$$\mathrm{dist}\,(\alpha_j, [-1, 1]) \geq \eta. \tag{1.8}$$

We let $\pi_0(x) = 1$, and for $k \geq 1$,

$$\pi_k(x) = \prod_{j=1}^{k} (1 - x/\alpha_j) . \tag{1.9}$$

We let $\mathcal{P}_k$ denote the polynomials of degree $\leq k$, and define nested spaces of rational functions by $\mathcal{L}_{-1} = \{0\}$; $\mathcal{L}_0 = \mathbb{C}$; and for $k \geq 1$,

$$\mathcal{L}_k = \mathcal{L}_k \{\alpha_1, \alpha_2, ..., \alpha_k\} = \left\{ \frac{P}{\pi_k} : \deg(P) \leq k \right\} .$$

Note that if all $\alpha_j = \infty$, then $\mathcal{L}_k = \mathcal{P}_k$. Moreover, $\mathcal{L}_{k-1} \subset \mathcal{L}_k$ for $k \geq 1$. We shall assume that the poles have an asymptotic distribution ν with support in $\bar{\mathbb{C}} \backslash [-1, 1]$, so that

$$\lim_{k \to \infty} \log |\pi_{k-1}(x)|^{1/k} = \int \log |1 - x/t| \, d\nu(t) , \tag{1.10}$$

uniformly for $x \in [-1, 1]$. An alternative formulation is that the pole counting measures

$$\nu_n = \frac{1}{n} \left(\delta_\infty + \sum_{j=1}^{n-1} \delta_{\alpha_j} \right) , \tag{1.11}$$

converge weakly to ν as $n \to \infty$. Here δ_α denotes a point mass at α. The uniform convergence in (1.10) follows simply from weak convergence because of the fact that the poles are a distance at least η from $[-1, 1]$.

We define orthogonal rational functions φ_0, φ_1, φ_2, ... corresponding to the measure μ, such that $\varphi_k \in \mathcal{L}_k \backslash \mathcal{L}_{k-1}$, and

$$\int_{-1}^{1} \varphi_j \, \overline{\varphi_k} \, d\mu = \delta_{jk}. \tag{1.12}$$

These may be generated by applying the Gram-Schmidt process to $\left\{x^k/\pi_k(x)\right\}_{k=0}^{\infty}$. We also define the corresponding rational kernel functions

$$K_n^r(d\mu,x,y)=\sum_{j=0}^{n-1}\varphi_j(x)\;\overline{\varphi_j(y)}. \tag{1.13}$$

The normalized form is

$$\tilde{K}_n^r(d\mu,x,y)=\mu'(x)^{1/2}\,\mu'(y)^{1/2}\,K_n^r(d\mu,x,y),$$

and when clear from the context, we shall just write $K_n^r(x,y)$ and $\tilde{K}_n^r(x,y)$. Observe that for $R\in\mathcal{L}_{n-1}$,

$$R(x)=\int_{-1}^{1}R(t)\,K_n^r(d\mu,x,t)\,d\mu(t).$$

This and Cauchy-Schwarz' inequality, easily yield an extremal property for the rational Christoffel functions

$$\lambda_n^r(d\mu,x)=1/K_n^r(d\mu,x,x)=\sum_{j=0}^{n-1}\left|\varphi_j(x)\right|^2, \tag{1.14}$$

analogous to (1.4), namely

$$\lambda_n^r(d\mu,x)=\inf_{R\in\mathcal{L}_{n-1}}\frac{\int_{-1}^{1}|R|^2\,d\mu}{|R(x)|^2}. \tag{1.15}$$

We shall often use the abbreviation $\lambda_n^r(x)$, when it is clear that the measure involved is μ.

Our main result deals with asymptotics of rational Christoffel functions:

Theorem 1.1
Let μ be a regular measure on $[-1,1]$. Let I be an open subinterval of $(-1,1)$ in which μ is absolutely continuous. Assume that μ' is positive and continuous at a given $x\in I$. Assume that the poles $\{\alpha_j\}$ satisfy the distance restriction (1.8) and have the asymptotic distribution specified by (1.10). Let $r>0$. Then uniformly for $s\in[-r,r]$,

$$\lim_{n\to\infty}n\lambda_n^r\left(x+\frac{s}{n}\right)=\mu'(x)\,\pi\sqrt{1-x^2}/\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t). \tag{1.16}$$

Here the branch of the square root is chosen so that $\sqrt{t^2-1}>0$ for $t\in(1,\infty)$. If μ' is positive and continuous in I, then this last limit also holds uniformly for x in compact subsets of I.

Remarks
(a) Observe that if all poles are at ∞, then $\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t)=1$, and the theorem reduces to the familiar limit (1.6) for Christoffel functions associated with polynomials.
(b) Up to now this theorem has been known only when μ' is a Chebyshev weight such as in Theorem 3.1 below, but under additional restrictions on the

poles. Our proof heavily relies on a classical explicit formula for Christoffel functions for Szegő-Bernstein weights, and a comparison technique essentially due to Totik.

As a consequence, we can prove universality limits for rational reproducing kernels. In its formulation, we use the notation

$$e^{i\arg(z)} = \frac{z}{|z|},\ z \neq 0.$$

Theorem 1.2
Assume the hypotheses of Theorem 1.1 in the stronger form that μ' is positive and continuous in I. Then for $x \in I$ and uniformly for a, b in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n^r\left(x + \frac{a}{\tilde{K}_n^r(x,x)}, x + \frac{b}{\tilde{K}_n^r(x,x)}\right)}{K_n^r(x,x)} e^{i\left[\arg\left(\pi_{n-1}\left(x+\frac{a}{\tilde{K}_n^r(x,x)}\right)\right) - \arg\left(\pi_{n-1}\left(x+\frac{b}{\tilde{K}_n^r(x,x)}\right)\right)\right]}$$
$$= \frac{\sin \pi (a-b)}{\pi (a-b)}.$$

(1.17)

This paper is organized as follows. We present three elementary lemmas in Section 2. These are used to relate properties of orthogonal rational functions to orthogonal polynomials, and to extend to rational functions, some well known estimates for polynomials. In Section 3, we establish asymptotics of rational Christoffel functions for the Chebyshev weight of the second kind. In Section 4, we prove Theorem 1.1, and in Section 5, we prove Theorem 1.2.

2. Three Elementary Lemmas

In this section, we establish three elementary lemmas, which in some way relate properties of orthogonal rational functions to analogous properties for polynomials. The first lemma of this section relates rational and polynomial reproducing kernels. In its formulation, we let

$$d\mu_n(t) = d\mu(t) / |\pi_{n-1}(t)|^2 \tag{2.1}$$

and $\{p_{n,j}\}_{j\geq 0}$ denote the corresponding orthonormal polynomials, so that

$$\int p_{n,j} p_{n,k} d\mu_n = \delta_{jk}.$$

We also let

$$K_n(d\mu_n, x, t) = \sum_{j=0}^{n-1} p_{n,j}(x) p_{n,j}(t),$$

and

$$\tilde{K}_n(d\mu_n, x, t) = \mu_n'(x)^{1/2} \mu_n'(t)^{1/2} K_n(d\mu_n, x, t). \tag{2.2}$$

Recall that K_n^r is given by (1.13).

Lemma 2.1

$$K_n^r(x,t) = K_n(d\mu_n, x, t) \,/\, \left(\pi_{n-1}(x)\,\overline{\pi_{n-1}(t)}\right). \tag{2.3}$$

In particular, for real x,

$$\lambda_n^r(x) = \lambda_n(d\mu_n, x)\,|\pi_{n-1}(x)|^2. \tag{2.4}$$

Proof

Recall our notation (1.12). For $j \geq 0$, write

$$\varphi_j(x) = \frac{s_j(t)}{\pi_j(t)},$$

where $s_j \in \mathcal{P}_j$. Let

$$\Psi_n(x,t) = \pi_{n-1}(x)\,\overline{\pi_{n-1}(t)} K_n^r(x,t).$$

Then we see that for fixed complex x,

$$\overline{\Psi_n(x,t)} = \overline{\pi_{n-1}(x)} \sum_{j=0}^{n-1} \overline{\left(\frac{s_j(x)}{\pi_j(x)}\right)} \frac{s_j(t)\,\pi_{n-1}(t)}{\pi_j(t)}$$

is a polynomial of degree $\leq n-1$ in t. The reproducing kernel relation for K_n^r gives, for polynomials P of degree $\leq n-1$,

$$\begin{aligned} \frac{P(x)}{\pi_{n-1}(x)} &= \int K_n^r(x,t)\,\frac{P(t)}{\pi_{n-1}(t)} d\mu(t) \\ &= \frac{1}{\pi_{n-1}(x)} \int \Psi_n(x,t)\,P(t)\,d\mu_n(t). \end{aligned}$$

That is,

$$P(x) = \int \Psi_n(x,t)\,P(t)\,d\mu_n(t).$$

Since also

$$P(x) = \int K_n(d\mu_n, x, t)\,P(t)\,d\mu_n(t),$$

we obtain for all such P,

$$0 = \int P(t)\left[\Psi_n(x,t) - K_n(d\mu_n, x, t)\right] d\mu_n(t).$$

Let

$$P(t) = \overline{\Psi_n(x,t)} - K_n(d\mu_n, \bar{x}, t),$$

a polynomial of degree $\leq n-1$ in t. Since $K_n(d\mu_n, x, t)$ has real coefficients in x, t, we also have for real t,

$$P(t) = \overline{\Psi_n(x,t) - K_n(d\mu_n, x, t)}.$$

Thus,

$$0 = \int \left| \Psi_n(x,t) - K_n(d\mu_n, x, t) \right|^2 d\mu_n(t),$$

so for real t,

$$K_n(d\mu_n, x, t) = \Psi_n(x,t) = \pi_{n-1}(x) \overline{\pi_{n-1}(t)} K_n^r(x,t).$$

This extends to complex t, as both sides are polynomials in $x, \bar{t}$. ■

Our next lemma shows that a relationship similar to (1.1), holds for rational functions with poles in the $\{\alpha_k\}$.

Lemma 2.2

Assume that the poles $\{\alpha_j\}$ have asymptotic distribution ν, as in (1.10). Assume that the measure μ is regular on $[-1,1]$. Then

$$\lim_{n\to\infty} \left[\sup_{R \in \mathcal{L}_{n-1}} \frac{\|R\|^2_{L_\infty[-1,1]}}{\int_{-1}^1 |R|^2 d\mu} \right]^{1/n} = 1. \tag{2.5}$$

Proof

Each $R \in \mathcal{L}_{n-1}$ has the form $R(x) = P(x)/\pi_{n-1}(x)$. Let

$$g_n(x) = 1/\left|\pi_{n-1}(x)\right|^2.$$

By our hypothesis (1.10), we have

$$\lim_{n\to\infty} g_n(x)^{1/n} = \exp\left(-2\int \log|1 - x/t|\, d\nu(t)\right) =: g(x),$$

uniformly for $x \in [-1,1]$. Here g is positive and continuous on $[-1,1]$. Then

$$\lim_{n\to\infty} \left[\sup_{R \in \mathcal{L}_{n-1}} \frac{\|R\|^2_{L_\infty[-1,1]}}{\int_{-1}^1 |R|^2 d\mu} \right]^{1/n} = \lim_{n\to\infty} \left[\sup_{P \in P_{n-1}} \frac{\left\|P^2 g_n\right\|_{L_\infty[-1,1]}}{\int_{-1}^1 |P|^2 g_n d\mu} \right]^{1/n} = 1,$$

by a result of Stahl and Totik [17, Thm. 3.2.3 (vi), p. 68]. ■

Our final lemma shows that we can construct rational functions with any given poles a distance at least η from $[-1,1]$, that decay as we recede from a given point $x \in [-1,1]$:

Lemma 2.3

Let $\eta \in (0,1)$ and $A_\eta = \{z : \operatorname{dist}(z, [-1,1]) \geq \eta\}$. There exists $\tau > 0$ with the following property: given any $x \in [-1,1]$ and any 3 points $\alpha, \beta, \Delta \in A_\eta$, there exists a rational function $R \in \mathcal{L}_3(\alpha, \beta, \Delta)$ such that $R(x) = 1$ and

$$|R(t)|^2 \leq 1 - \tau (t-x)^2, \; t \in [-1,1]. \tag{2.6}$$

Remark

We emphasize that τ is independent of x and α, β, Δ, depending only on η. R will have numerator and denominator degree at most 2.

Proof

Choose $\eta_1 \in (0,1)$ so small that if $z \in A_\eta$,

$$|z| \geq \eta_1; \tag{2.7}$$

$$\left|1 - \frac{t}{z}\right| \geq \eta_1 \text{ for } t \in [-1,1]; \tag{2.8}$$

We shall consider three configurations of poles:
(I) At least one pole α satisfies $|\alpha| \leq 4$ and $|\operatorname{Im}\alpha| \geq \eta_1^3/4$.
If none of the given three poles satisfies this, then either
(II) At least two of the poles α satisfy $|\alpha| > 4$.
or
(III) Two poles α satisfy both $|\alpha| \leq 4$ and $|\operatorname{Im}\alpha| < \eta_1^3/4$.
We turn to

Case (I)

Let α have the specified property, and

$$R(t) = \frac{1}{2}\left(1 + \frac{t-\bar{\alpha}}{t-\alpha} / \frac{x-\bar{\alpha}}{x-\alpha}\right).$$

Clearly $R(x) = 1$, R is a rational function of denominator degree 1, with pole at α, and straightforward calculations show that

$$|R(t)|^2 = \frac{1}{2}\left(1 + \operatorname{Re}\left(\frac{t-\alpha}{t-\bar{\alpha}}\frac{x-\bar{\alpha}}{x-\alpha}\right)\right)$$

and hence

$$\begin{aligned} 1 - |R(t)|^2 &= \frac{1}{2}\left(1 - \operatorname{Re}\left(\frac{t-\alpha}{t-\bar{\alpha}}\frac{x-\bar{\alpha}}{x-\alpha}\right)\right) \\ &= \frac{(\operatorname{Im}\alpha)^2 (x-t)^2}{|(x-\alpha)(t-\bar{\alpha})|^2}. \\ &\geq \frac{(\eta_1^6/16)(t-x)^2}{5^2}, \end{aligned}$$

for $t \in [-1,1]$, and by our assumptions on $|\alpha|$ and $|\operatorname{Im}\alpha|$, namely $|\alpha| \leq 4$ and $|\operatorname{Im}\alpha| \geq \eta_1^3/4$. Thus for $t \in [-1,1]$,

$$|R(t)|^2 \leq 1 - \frac{\eta_1^6}{400}(t-x)^2. \tag{2.9}$$

Case II

Here we choose α, β with $|\alpha|, |\beta| \geq 4$, and let

$$R(t) = 1 - \rho \frac{(t-x)^2}{(1-t/\alpha)(1-t/\beta)}, \tag{2.10}$$

where

$$\rho = \frac{1}{16}.$$

Now

$$|R(t)|^2 = 1 + \frac{\rho^2 (t-x)^4}{|(1-t/\alpha)(1-t/\beta)|^2} - 2\,\mathrm{Re}\left(\rho \frac{(t-x)^2}{(1-t/\alpha)(1-t/\beta)}\right),$$

so

$$\begin{aligned} & 1 - |R(t)|^2 \\ = \quad & \frac{\rho (t-x)^2}{|(1-t/\alpha)(1-t/\beta)|^2} \left(2\,\mathrm{Re}\left((1-t/\bar{\alpha})\left(1-t/\bar{\beta}\right)\right) - \rho (t-x)^2\right). \end{aligned}$$

(2.11)

Here

$$\begin{aligned} & \mathrm{Re}\left((1-t/\bar{\alpha})\left(1-t/\bar{\beta}\right)\right) \\ = \quad & 1 - t\,\mathrm{Re}\left(\frac{1}{\bar{\alpha}} + \frac{1}{\bar{\beta}}\right) + t^2\,\mathrm{Re}\left(\frac{1}{\overline{\alpha\beta}}\right) \\ \geq \quad & 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{16} \geq \frac{1}{4}, \end{aligned}$$

so

$$\begin{aligned} & 1 - |R(t)|^2 \\ \geq \quad & \frac{\rho (t-x)^2}{|(1-t/\alpha)(1-t/\beta)|^2} \left(\frac{1}{2} - 4\rho\right) \\ \geq \quad & \frac{(t-x)^2}{16 (5/4)^2\, 4}, \end{aligned}$$

recall $\rho = \frac{1}{16}$. Thus for $t \in [-1,1]$,

$$|R(t)|^2 \leq 1 - \frac{1}{100}(t-x)^2. \tag{2.12}$$

Case III
Here we again choose R by (2.10), but with

$$\rho = \pm \eta_1^2/32, \tag{2.13}$$

and with α, β having the properties specified in Case III. The sign of ρ is chosen to be the same as the sign of $\mathrm{Re}\left((1-t/\bar{\alpha})\left(1-t/\bar{\beta}\right)\right)$, which we shall show is constant in $[-1,1]$. Indeed, for $t \in [-1,1]$,

$$|\mathrm{Im}(1-t/\bar{\alpha})| = |t| \frac{|\mathrm{Im}\,\alpha|}{|\alpha|^2} \leq \frac{\eta_1^3/4}{\eta_1^2} = \eta_1/4,$$

by (2.7). Then inasmuch as $|1-t/\bar{\alpha}| \geq \eta_1$, we have

$$|\mathrm{Re}(1-t/\bar{\alpha})| \geq \eta_1 - \eta_1/4 \geq \eta_1/2,$$

with similar inequalities for β. Then

$$\begin{aligned} & \left|\operatorname{Re}\left((1-t/\bar{\alpha})\left(1-t/\bar{\beta}\right)\right)\right| \\ = & \left|\operatorname{Re}\left(1-t/\bar{\alpha}\right)\operatorname{Re}\left(1-t/\bar{\beta}\right)-\operatorname{Im}\left(1-t/\bar{\alpha}\right)\operatorname{Im}\left(1-t/\bar{\beta}\right)\right| \\ \geq & \left(\eta_1/2\right)^2-\left(\eta_1/4\right)^2 \geq \eta_1^2/8. \end{aligned}$$

Inasmuch as $\operatorname{Re}\left((1-t/\bar{\alpha})\left(1-t/\bar{\beta}\right)\right)$ is continuous, it will have a constant sign for t in $[-1,1]$, and it is that that we choose as the sign of ρ. Then (2.11) gives

$$\begin{aligned} & 1-\left|R\left(t\right)\right|^2 \\ \geq & \frac{\left|\rho\right|\left(t-x\right)^2}{\left|\left(1-t/\alpha\right)\left(1-t/\beta\right)\right|^2}\left(\eta_1^2/4-4\left|\rho\right|\right) \\ \geq & \frac{\left|\rho\right|\left(t-x\right)^2}{\left(1+1/\eta_1\right)^4}\left(\eta_1^2/8\right)=\frac{1}{256}\frac{\eta_1^8}{\left(1+\eta_1\right)^4}\left(t-x\right)^2, \end{aligned}$$

recall (2.13) and (2.7). Considering this, (2.9), and (2.12), in the statement of the lemma, we can choose

$$\tau=\min\left\{\frac{\eta_1^6}{400},\frac{1}{256}\frac{\eta_1^8}{\left(1+\eta_1\right)^4}\right\}.$$

■

3. Christoffel Functions for Chebyshev Weights

In this section, we state a special case of Theorem 1.1 for the Chebyshev weight of the second kind:

Theorem 3.1
Assume that μ is the Chebyshev measure of the second kind, so that

$$\mu'\left(x\right)=\sqrt{1-x^2},\ x\in\left(-1,1\right). \tag{3.1}$$

Assume that the sequence of poles $A=\{\alpha_1,\alpha_2,\ldots\}$ satisfies the hypotheses of Theorem 1.1. Then uniformly for x in compact subsets of $\left(-1,1\right)$,

$$\lim_{n\to\infty} n\lambda_n^r\left(x\right)=\mu'\left(x\right)\pi\sqrt{1-x^2}/\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t). \tag{3.2}$$

We note that with purely notational adjustments to the proof, we can allow varying poles in Theorem 3.1. That is, we can consider at the nth stage poles $\{\alpha_{n,j}\}_{j=1}^{n-1}$ in A_η. We would need to assume that

$$\frac{1}{n}\left\{\delta_\infty+\sum_{j=1}^{n-1}\delta_{\alpha_{n,j}}\right\}$$

converges weakly to ν as $n \to \infty$. However, we cannot prove such an extension in Theorem 1.1 because of the difficulty of establishing (4.1) below for varying weights.

We shall use a classical representation for the Christoffel function for Bernstein-Szegő weights:

Lemma 3.2
Let

$$d\mu_n(t) = \frac{\sqrt{1-t^2}}{|\pi_{n-1}(t)|^2} dt,\ t \in (-1,1),$$

where π_n is given by (1.9). Let $x = \cos\theta$, where $\theta \in [0,\pi]$. Then

$$\begin{aligned} & \pi\lambda_n^{-1}(d\mu_n, x)\,\mu_n'(x)\sqrt{1-x^2} \\ = \; & n - \frac{1}{2} + \Gamma_n'(\theta) + \frac{1}{2\sqrt{1-x^2}} \sin\left((2n-1)\theta + 2\Gamma_n(\theta)\right), \end{aligned} \tag{3.3}$$

where

$$\Gamma_n(\theta) = \frac{\sqrt{1-x^2}}{2\pi} PV \int_{-1}^{1} \frac{\log g_n(t)}{t-x} \frac{dt}{\sqrt{1-t^2}}, \tag{3.4}$$

and

$$\Gamma_n'(\theta) = -\frac{1}{2\pi} PV \int_{-1}^{1} \frac{g_n'(t)}{g_n(t)} \frac{\sqrt{1-t^2}}{t-x} dt, \tag{3.5}$$

and PV stands for Cauchy Principal Value Integral, while

$$g_n(t) = \frac{1-t^2}{|\pi_{n-1}(t)|^2}. \tag{3.6}$$

Proof
This is the special case of Theorem B.4(b) in [10, p. 440], where $S(t) = |\pi_{n-1}(t)|^2$, and $q = n-1$. There $\Gamma_n(\theta)$ is denoted $\Gamma(f;\theta)$, with

$$f(\theta) = \frac{\sin^2\theta}{|\pi_{n-1}(\cos\theta)|^2} = \frac{1-x^2}{|\pi_{n-1}(x)|^2} = g_n(x).$$

The representations (3.4) and (3.5) for Γ_n and Γ_n' are given in Lemma B.5 of [10, pp.440-441]. ■

We can now deduce:

Lemma 3.3
Assume the hypotheses of Lemma 3.2. Let ν_n denote the pole counting measure as in (1.11). Let $[a,b] \subset (-1,1)$. Then uniformly for x in $[a,b]$, as $n \to \infty$,

$$\frac{\pi}{n}\lambda_n^{-1}(d\mu_n, x)\,\mu_n'(x)\sqrt{1-x^2} = \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu_n(t) + O\left(\frac{1}{n}\right). \tag{3.7}$$

Remark
We note that this lemma does not require the poles to be a fixed distance away from $[-1,1]$, nor does it require weak convergence of $\{\nu_n\}$. Moreover, the order term does not depend on the particular choice of $\{\pi_n\}$. It depends only on the size of $\frac{1}{\sqrt{1-x^2}}$.
Proof
We first recall some standard integrals [15, Example I.3.5, pp.45-46 and p. 225]:

$$\frac{1}{\pi}PV\int_{-1}^{1}\frac{1}{s-x}\frac{ds}{\sqrt{1-s^2}}=0,\ x\in(-1,1)\,; \tag{3.8}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{s-u}\frac{ds}{\sqrt{1-s^2}}=-\frac{1}{\sqrt{u^2-1}},\ u\in\mathbb{C}\backslash\,[-1,1]\,. \tag{3.9}$$

From these, we readily derive (by writing $\sqrt{1-s^2}=\frac{1-x^2+x^2-s^2}{\sqrt{1-s^2}}$, etc.)

$$\frac{1}{\pi}PV\int_{-1}^{1}\frac{\sqrt{1-s^2}}{s-x}ds=-x,\ x\in(-1,1)\,; \tag{3.10}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{s-u}ds=\sqrt{u^2-1}-u,\ u\in\mathbb{C}\backslash\,[-1,1]\,. \tag{3.11}$$

Then we see that for $x\in(-1,1)$ and $u\in\mathbb{C}\backslash\,[-1,1]$,

$$\begin{aligned}&\frac{1}{\pi}PV\int_{-1}^{1}\frac{1}{u-s}\frac{\sqrt{1-s^2}}{s-x}ds\\ =\ &\frac{1}{u-x}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{u-s}ds+\frac{1}{\pi}PV\int_{-1}^{1}\frac{\sqrt{1-s^2}}{s-x}ds\right]\\ =\ &1-\frac{\sqrt{u^2-1}}{u-x}.\end{aligned} \tag{3.12}$$

We now apply this to evaluate $1+\frac{1}{n}\Gamma_n'(\theta)$. Observe that g_n of (3.6) satisfies

$$\log g_n(t)=\log\left(1-t^2\right)-2\sum_{j=1}^{n-1}\log\left|1-\frac{t}{\alpha_j}\right|,$$

so

$$\frac{g_n'(t)}{g_n(t)}=\frac{-2t}{1-t^2}+2\sum_{j=1}^{n-1}\operatorname{Re}\left\{\frac{1}{\alpha_j-t}\right\}.$$

Thus, recalling (3.5),

$$\begin{aligned}
&1+\frac{1}{n}\Gamma_n'(\theta)\\
=\ &1-\frac{1}{2n\pi}PV\int_{-1}^{1}\left\{\frac{-2t}{1-t^2}+2\sum_{j=1}^{n-1}\operatorname{Re}\left\{\frac{1}{\alpha_j-t}\right\}\right\}\frac{\sqrt{1-t^2}}{t-x}dt\\
=\ &1+\frac{1}{n\pi}PV\int_{-1}^{1}\frac{t}{t-x}\frac{dt}{\sqrt{1-t^2}}-\frac{1}{n}\sum_{j=1}^{n-1}\left(1-\frac{\sqrt{\alpha_j^2-1}}{\alpha_j-x}\right)\\
=\ &\frac{1}{n}+\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu_n(t),
\end{aligned}$$

where we have used (3.12) and (3.8), and the fact that ν_n has a point mass of size $\frac{1}{n}$ at infinity. We now substitute this into (3.3), and observe that the remaining terms are $O\left(1/\sqrt{1-x^2}\right)$, independently of n and the choice of $\{\pi_n\}$. ■

We can now give the

Proof of Theorem 3.1

By hypothesis, ν_n converges weakly to ν as $n\to\infty$. Moreover, the function $\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}$ is uniformly continuous for t in A_η, including at ∞, and for $x\in[-1,1]$. Thus for fixed $x\in(-1,1)$,

$$\lim_{n\to\infty}\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu_n(t)=\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t).$$

The previous lemma now gives pointwise convergence of the Christoffel functions. Indeed, we have shown

$$\frac{\pi}{n}\lambda_n^{-1}(d\mu_n,x)\frac{\mu'(x)}{|\pi_{n-1}(x)|^2}\sqrt{1-x^2}=\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t)+o(1),$$

which in view of Lemma 2.1 can be restated as

$$\frac{\pi}{n}\lambda_n^r(x)^{-1}\mu'(x)\sqrt{1-x^2}=\int\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}d\nu(t)+o(1). \tag{3.13}$$

To prove the uniform convergence for x in a compact subset of $(-1,1)$, we use the just stated uniform continuity of $\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}$. Let $\varepsilon>0$. Then we can find $L\geq 1$ and $\{x_j\}_{j=1}^L$, such that for all $x\in[-1,1]$,

$$\min_{1\leq j\leq L}\sup_{t\in A_\eta}\left|\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}-\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x_j}\right\}\right|\leq\varepsilon.$$

Note that L and $\{x_j\}$ are independent of n. Then for all $x \in [-1,1]$, and appropriate $1 \leq j \leq L$,

$$\begin{aligned} &\left| \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu_n(t) - \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} d\nu(t) \right| \\ \leq\ & \int \left| \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x} \right\} - \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x_j} \right\} \right| (d\nu_n(t) + d\nu(t)) \\ &+ \left| \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x_j} \right\} d\nu_n(t) - \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x_j} \right\} d\nu(t) \right| \\ \leq\ & 2\varepsilon + \max_{1\leq j\leq L} \left| \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x_j} \right\} d\nu_n(t) - \int \operatorname{Re}\left\{ \frac{\sqrt{t^2-1}}{t-x_j} \right\} d\nu(t) \right|. \end{aligned}$$

The right-hand side is independent of $x \in [-1,1]$, and approaches 2ε as $n \to \infty$. This easily yields the stated uniform convergence. Of course the $1/\sqrt{1-x^2}$ term implicit in the order term in (3.7) prevents proving uniform convergence throughout $[-1,1]$. ∎

4. Proof of Theorem 1.1

We first prove a comparison result for Christoffel functions:

Lemma 4.1

Let μ, ω be regular measures on $[-1,1]$, and $J = [a,b]$ be a subinterval of $(-1,1)$ such that for some positive constant c, $\mu = c\omega$ in J. Assume that the sequence of poles $A = \{\alpha_1, \alpha_2, \ldots\}$ satisfies the hypotheses of Theorem 1.1. Assume that for $x \in (a,b)$,

$$\lim_{\varepsilon\to 0+} \left(\limsup_{n\to\infty} \left| \frac{\lambda_n^r(d\mu, x)}{\lambda_{n\pm[\varepsilon n]}^r(d\mu, x)} - 1 \right| \right) = 0. \tag{4.1}$$

Then for $x \in (a,b)$,

$$\lim_{n\to\infty} \frac{\lambda_n^r(d\omega, x)}{\lambda_n^r(d\mu, x)} = c. \tag{4.2}$$

If (4.1) holds uniformly in (a,b), then (4.2) holds uniformly for x in compact subsets of (a,b).

Proof

Let $\varepsilon > 0$ and $x \in (a,b)$. By hypothesis, there exists $\eta > 0$ such that all our poles lie in the set A_η of Lemma 2.3. Let τ be the number from that lemma. From the $[\varepsilon n]$ poles $\alpha_{n-[\varepsilon n]}, \alpha_{n-[\varepsilon n]+1}, ..., \alpha_{n-1}$, we can construct at least $[[\varepsilon n]/3]$ rational functions with denominator degree at most 2 and with the properties specified in the Lemma 2.3. By multiplying these together, we obtain a rational function $S_{[\varepsilon n]} \in \mathcal{L}_{[\varepsilon n]}\left\{\alpha_{n-[\varepsilon n]}, \alpha_{n-[\varepsilon n]+1}, ..., \alpha_{n-1}\right\}$, such that $S_{[\varepsilon n]}(x) = 1$ and

$$\left|S_{[\varepsilon n]}(t)\right|^2 \leq \left(1 - \tau (t-x)^2\right)^{[[\varepsilon n]/3]}, \ t \in [-1,1].$$

Then there exists $\kappa \in (0,1)$, depending only on the distance from x to $[-1,1] \setminus (a,b)$, such that for $t \in [-1,1] \setminus J$,

$$\left|S_{[\varepsilon n]}(t)\right| \leq \kappa^n.$$

(In addition, if we restrict x to a compact subinterval of (a,b), then we may choose κ independent of x.) Then with $\mathcal{L}_{n-1} = \mathcal{L}_{n-1}\{\alpha_1, \alpha_2, ..., \alpha_{n-1}\}$, and $\mathcal{L}_{n-[\varepsilon n]-1} = \mathcal{L}_{n-[\varepsilon n]-1}\left\{\alpha_1, \alpha_2, ..., \alpha_{n-[\varepsilon n]-1}\right\}$,

$$\begin{aligned}
\lambda_n^r(d\omega, x) &= \inf_{R \in \mathcal{L}_{n-1}} \frac{\int |R|^2 d\omega}{|R(x)|^2} \\
&\leq \inf_{R_1 \in \mathcal{L}_{n-[\varepsilon n]-1}} \frac{\int |R_1|^2 \left|S_{[\varepsilon n]}\right|^2 d\omega}{|R_1(x)|^2 \left|S_{[\varepsilon n]}(x)\right|^2} \\
&\leq \inf_{R_1 \in \mathcal{L}_{n-[\varepsilon n]-1}} \left(c \frac{\int_J |R_1|^2 d\mu}{|R_1(x)|^2} + \kappa^{2n} \frac{\|R_1\|^2_{L_\infty[-1,1]}}{|R_1(x)|^2} \int_{[-1,1]\setminus J} d\omega \right).
\end{aligned}$$

Here we have used the hypothesis that $\mu = c\omega$ in J. Now because of the regularity of the measure μ, Lemma 2.2 gives

$$\|R_1\|^2_{L_\infty[-1,1]} \leq (1 + o(1))^{n-[\varepsilon n]-1} \int |R_1|^2 d\mu,$$

where the $o(1)$ term is independent of R_1, and decays to 0 as $n \to \infty$. Thus

$$\begin{aligned}
\lambda_n^r(d\omega, x) &\leq \left[c + \kappa^{2n}(1 + o(1))^n\right] \inf_{R_1 \in \mathcal{L}_{n-[\varepsilon n]-1}} \left(\frac{\int |R_1|^2 d\mu}{|R_1(x)|^2} \right) \\
&= \left[c + \kappa^{2n}(1 + o(1))^n\right] \lambda^r_{n-[\varepsilon n]}(d\mu, x).
\end{aligned}$$

Inasmuch as $\kappa < 1$, this gives

$$\limsup_{n\to\infty} \frac{\lambda_n^r(d\omega, x)}{\lambda^r_{n-[\varepsilon n]}(d\mu, x)} \leq c. \tag{4.3}$$

Note that if we restrict x to a compact subinterval of (a,b), then this holds uniformly for x in that compact subinterval, since κ and the $o(1)$ term are independent of x. The other direction is similar. Using the regularity of ω, we obtain as above

$$\begin{aligned}
\lambda^r_{n+[\varepsilon n]}(d\mu, x) &\leq \left[c^{-1} + \kappa^{2n}(1 + o(1))^n\right] \inf_{R_1 \in \mathcal{L}_{n-1}} \left(\frac{\int |R_1|^2 d\omega}{|R_1(x)|^2} \right) \\
&= \left[c^{-1} + \kappa^{2n}(1 + o(1))^n\right] \lambda_n(d\omega, x),
\end{aligned}$$

and so

$$\limsup_{n\to\infty} \frac{\lambda^r_{n+[\varepsilon n]}(d\mu, x)}{\lambda_n^r(d\omega, x)} \leq c^{-1}.$$

This, (4.3), and our hypothesis (4.1), easily yield the result. ■

Now we deduce:

Theorem 4.2
Let μ and ω be regular measures on $[-1,1]$. Let I be an open subinterval of $(-1,1)$ in which ω is absolutely continuous with respect to μ. Assume that $x \in I$ is such that the Radon-Nikodym derivative $\frac{d\omega}{d\mu}$ is positive and continuous at x. Assume, moreover, that (4.1) holds uniformly in some neighborhood of x. Let $r > 0$. Then uniformly for $s \in [-r,r]$,

$$\lim_{n\to\infty} \frac{\lambda_n^r\left(d\omega, x+\frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x+\frac{s}{n}\right)} = \frac{d\omega}{d\mu}(x). \tag{4.4}$$

If $\frac{d\omega}{d\mu}$ is positive and continuous in I and (4.1) holds uniformly in I, then this last limit is also uniform for x in any compact subset of I.

Proof
Let $\varepsilon > 0$ and

$$A = \frac{d\omega}{d\mu}(x) + \varepsilon.$$

Choose an interval $J \subset I$ containing x in its interior, such that

$$\frac{d\omega}{d\mu}(t) \leq A,\ t \in J.$$

Let ω_1 be the measure such that $d\omega_1 = d\omega$ in $[-1,1]\setminus J$, and $d\omega_1 = A\ d\mu$ in J. Then in $[-1,1]$,

$$d\omega \leq d\omega_1$$

so for all t,

$$\lambda_n^r(d\omega, t) \leq \lambda_n^r(d\omega_1, t). \tag{4.5}$$

Next, ω_1 is regular on $[-1,1]$ by a localization Theorem of Stahl and Totik [17, Thm. 5.3.3, p. 148]. Indeed, ω_1 is regular when restricted to J (where it is a positive multiple of a regular measure) and is the restiction of a regular measure in $[-1,1]\setminus J$, so is regular. Thus ω_1 and μ are regular, and $d\omega_1 = A$ $d\mu$ in J, so Lemma 4.1 gives uniformly for $s \in [-r,r]$,

$$\lim_{n\to\infty} \frac{\lambda_n^r\left(d\omega_1, x+\frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x+\frac{s}{n}\right)} = A.$$

Combining this and (4.5) gives, uniformly for $s \in [-r,r]$,

$$\limsup_{n\to\infty} \frac{\lambda_n^r\left(d\omega, x+\frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x+\frac{s}{n}\right)} \leq A = \frac{d\omega}{d\mu}(x) + \varepsilon.$$

Here the left-hand side is independent of ε, and ε is arbitrary, so uniformly for $s \in [-r,r]$,

$$\limsup_{n\to\infty} \frac{\lambda_n^r\left(d\omega, x+\frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x+\frac{s}{n}\right)} \leq \frac{d\omega}{d\mu}(x). \tag{4.6}$$

In exactly the same way, given $\varepsilon \in \left(0, \frac{d\omega}{d\mu}(x)\right)$, we can let $B = \frac{d\omega}{d\mu}(x) - \varepsilon$, and choose an interval J containing x in its interior, such that

$$\frac{d\omega}{d\mu}(t) \geq B, \; t \in J.$$

Let ω_2 be the measure such that $d\omega_2 = d\omega$ in $[-1,1] \backslash J$, and $d\omega_2 = B \; d\mu$ in J. Then in $[-1,1]$,

$$d\omega_2 \geq d\omega$$

so

$$\lambda_n^r(d\omega_2, x) \geq \lambda_n^r(d\omega, x). \tag{4.7}$$

But ω_2 and μ are regular, and $d\omega_2 = B \; d\mu$ in J, so Lemma 4.1 gives uniformly for $s \in [-r, r]$,

$$\lim_{n\to\infty} \frac{\lambda_n^r\left(d\omega_2, x + \frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x + \frac{s}{n}\right)} = B.$$

Combining this and (4.7) gives

$$\liminf_{n\to\infty} \frac{\lambda_n^r\left(d\omega, x + \frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x + \frac{s}{n}\right)} \geq B = \frac{d\omega}{d\mu}(x) - \varepsilon.$$

Here the left-hand side is independent of ε, and ε is arbitrary, so

$$\liminf_{n\to\infty} \frac{\lambda_n^r\left(d\omega, x + \frac{s}{n}\right)}{\lambda_n^r\left(d\mu, x + \frac{s}{n}\right)} \geq \frac{d\omega}{d\mu}(x).$$

Together with (4.6), this gives the result at x. The uniformity in x follows easily with simple adjustments, when $\frac{d\omega}{d\mu}$ is positive and continuous in I. ■

We can now turn to the

Proof of Theorem 1.1

We swap the roles of μ and ω in Theorem 4.2. Let ω denote the Chebyshev measure of the second kind on $[-1,1]$, so that ω is absolutely continuous, and

$$\omega'(x) = \sqrt{1-x^2}, \; x \in (-1,1).$$

Let $r > 0$. It follows from Theorem 3.1 that uniformly for x in compact subsets of $(-1,1)$ and $s \in [-r, r]$,

$$\lim_{n\to\infty} n\lambda_n^r\left(d\omega, x + \frac{s}{n}\right) = \pi\omega'(x)\sqrt{1-x^2} \Big/ \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu(t). \tag{4.8}$$

Moreover, ω will satisfy (4.1) with ω replacing μ. Our given measure μ will have Radon-Nikodym derivative

$$\frac{d\mu}{d\omega}(x) = \frac{\mu'(x)}{\omega'(x)} = \frac{\mu'(x)}{\sqrt{1-x^2}}$$

that exists a.e. in I. We now just apply Theorem 4.2 and (4.8) to deduce the result. ■

5. Universality Limits

We shall base our universality result on one from [11], but first need some concepts from potential theory for external fields [15]. Let Σ be a closed set on the real line, and

$$W(x) = \exp(-Q(x))$$

be a continuous function on Σ. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0.$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\omega}\left(\int\int \log\frac{1}{|x-t|}d\omega(x)\,d\omega(t) + 2\int Q\,d\omega\right),$$

where the inf is taken over all positive Borel measures ω with support in Σ and $\omega(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ω_Q, characterized by the following conditions: let

$$V^{\omega_Q}(z) = \int \log\frac{1}{|z-t|}d\omega_Q(t)$$

denote the potential for ω_Q. Then

$$\begin{aligned} V^{\omega_Q} + Q &\geq F_W \text{ on } \Sigma; \\ V^{\omega_Q} + Q &= F_W \text{ in } \operatorname{supp}[\omega_Q]. \end{aligned}$$

Here the number F_W is a constant. Usually ω_Q is denoted by μ_W, ν_W, μ_Q, or ν_Q, but we use a different symbol to avoid confusion with our measures of orthogonality μ and $\{\mu_n\}$, and the measure ν that describes our distribution of poles. Following is one of the main results from [11]. We emphasize that the measures $\left\{\mu_n^{\#}\right\}$ in its statement are not initially the same as $\{\mu_n\}$ in (2.1).

Lemma 5.1
For $n \geq 1$, let $\mu_n^{\#}$ be a positive Borel measure on the real line, with at least the first $2n+1$ power moments finite. Let I be a compact interval in which each $\mu_n^{\#}$ is absolutely continuous. Assume moreover that in I,

$$d\mu_n^{\#}(x) = h(x)\,W_n^{2n}(x)\,dx, \tag{5.1}$$

where

$$W_n = e^{-Q_n} \tag{5.2}$$

is continuous on I, and h is a bounded positive continuous function on I. Let ω_{Q_n} denote the equilibrium measure for the restriction of W_n to I. Let J be a compact subinterval of I^o. Assume that
(a) $\left\{\omega'_{Q_n}\right\}_{n=1}^{\infty}$ are positive and uniformly bounded in some open interval containing J;

(b) $\{Q_n'\}_{n=1}^{\infty}$ *are equicontinuous and uniformly bounded in some open interval containing* J;
(c) For some $C_1, C_2 > 0$, *and for* $n \geq 1$ *and* $\xi \in I$, *the Christoffel functions* $\lambda_n\left(d\mu_n^{\#}, \cdot\right)$ *satisfy*

$$C_1 \leq \lambda_n^{-1}\left(d\mu_n^{\#}, \xi\right) W_n^{2n}(\xi)/n \leq C_2. \tag{5.3}$$

(d) Uniformly for $\xi \in J$ *and* a *in compact subsets of the real line,*

$$\lim_{n\to\infty} \frac{\lambda_n\left(d\mu_n^{\#}, \xi + \frac{a}{n}\right)}{\lambda_n\left(d\mu_n^{\#}, \xi\right)} \frac{W_n^{2n}(\xi)}{W_n^{2n}\left(\xi + \frac{a}{n}\right)} = 1. \tag{5.4}$$

Then uniformly for $\xi \in J$, *and* a, b *in compact subsets of the real line, we have*

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(d\mu_n^{\#}, \xi + \frac{a}{\widetilde{K}_n\left(d\mu_n^{\#},\xi,\xi\right)}, \xi + \frac{b}{\widetilde{K}_n\left(d\mu_n^{\#},\xi,\xi\right)}\right)}{\widetilde{K}_n\left(d\mu_n^{\#}, \xi, \xi\right)} = \frac{\sin \pi (a-b)}{\pi (a-b)}. \tag{5.5}$$

Proof
See Theorem 1.2 in [11, p. 748]. ■

We now let

$$d\mu_n(t) = d\mu(t) / \left|\pi_{n-1}(t)\right|^2 \tag{5.6}$$

and as in Section 2, let $K_n(d\mu_n, x, t)$ denote the corresponding reproducing kernel, with normalized cousin

$$\tilde{K}_n(d\mu_n, x, t) = \mu_n'(x)^{1/2} \mu_n'(t)^{1/2} K_n(d\mu_n, x, t). \tag{5.7}$$

In order to apply Lemma 5.1, we choose

$$W_n(x) = e^{-Q_n(x)} = \frac{1}{\left|\pi_{n-1}(x)\right|^{1/n}}, \; x \in [-1, 1], \tag{5.8}$$

so that

$$Q_n(x) = \frac{1}{n} \log \left|\pi_{n-1}(x)\right| = \frac{1}{n} \sum_{j=1}^{n-1} \log \left|1 - \frac{x}{\alpha_j}\right|.$$

We shall need the equilibrium density for this external field. It is known [7], but we provide a proof, as there are additional restrictions there.

Lemma 5.2
The equilibrium measure ρ_n *for the external field* Q_n *on* $[-1, 1]$, *is given by*

$$\rho_n'(x) = \frac{1}{\pi\sqrt{1-x^2}} \int \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\} d\nu_n(t), \; x \in (-1, 1), \tag{5.9}$$

Proof
Define ρ_n' by (5.9). We have to prove that there is a constant C such that for $y \in [-1,1]$,

$$\int_{-1}^{1} \log \frac{1}{|y-x|} \rho_n'(x)\,dx + Q_n(y) = C,$$

for this property characterizes the equilibrium density [15]. It suffices, in turn, to establish the differentiated form of this, namely

$$-PV \int_{-1}^{1} \frac{1}{y-x} \rho_n'(x)\,dx + Q_n'(y) = 0,$$

$y \in (-1,1)$, where PV denotes Cauchy principal value. Integration of this latter relation, with the appropriate justification [15], then yields what we need. Since

$$Q_n'(y) = \frac{d}{dy} \log |\pi_{n-1}(y)|^{1/n} = -\frac{1}{n} \sum_{j=1}^{n-1} \operatorname{Re}\left(\frac{1}{\alpha_j - y}\right),$$

while ρ_n' is also a sum, we see that it actually suffices to prove for $\alpha \notin [-1,1]$, (allowing $\alpha = \infty$) that

$$-PV \int_{-1}^{1} \frac{1}{y-x} \rho_\alpha'(x)\,dx - \operatorname{Re}\left(\frac{1}{\alpha_j - y}\right) = 0,\ y \in (-1,1), \tag{5.10}$$

where

$$\rho_\alpha'(x) = \frac{1}{\pi\sqrt{1-x^2}} \operatorname{Re}\left\{\frac{\sqrt{\alpha^2-1}}{\alpha - x}\right\}.$$

When $\alpha = \infty$, then $\rho_\alpha'(x) = 1/\left(\pi\sqrt{1-x^2}\right)$, and this last relation follows from (3.8). Suppose now that α is finite. We see that

$$\begin{aligned}
& PV \int_{-1}^{1} \frac{1}{y-x} \rho_\alpha'(x)\,dx \\
= \; & \operatorname{Re}\left\{\sqrt{\alpha^2-1}\frac{1}{\pi} PV \int_{-1}^{1} \frac{1}{y-x}\frac{1}{\alpha-x}\frac{dx}{\sqrt{1-x^2}}\right\} \\
= \; & \operatorname{Re}\left\{\frac{\sqrt{\alpha^2-1}}{\alpha-y}\frac{1}{\pi} PV \int_{-1}^{1} \left[\frac{1}{y-x} - \frac{1}{\alpha-x}\right]\frac{dx}{\sqrt{1-x^2}}\right\} \\
= \; & \operatorname{Re}\left\{\frac{\sqrt{\alpha^2-1}}{\alpha-y}[0 - \frac{1}{\sqrt{\alpha^2-1}}]\right\} = -\operatorname{Re}\left\{\frac{1}{\alpha-y}\right\},
\end{aligned}$$

by (3.8) and (3.9). So we have (5.10). ∎

The proof of Theorem 1.2
In the sequel, we let I be a closed subinterval of $(-1,1)$ in which μ is absolutely continuous, and in which μ' is positive and continuous. There is a slight notational conflict with the statement of Theorem 1.2 where I is open,

but we can just take the I here to be a compact subinterval of the I there. Let us recall that we defined μ_n by (5.6). We choose $W_n(x)$, $x \in [-1,1]$ by (5.8). We let ω_{Q_n} denote the equilibrium measure for W_n restricted to I. The reason we work on the interval I, rather than $[-1,1]$, is that we need the bound (5.3) on the Christoffel functions to hold uniformly on I, and we don't have that bound throughout $[-1,1]$. This complicates issues, as we have a simple formula for the the equilibrium measure for W_n on $[-1,1]$, but not such a simple one on I.

Let ρ_n denote the equilibrium measure for W_n on $[-1,1]$, as in the lemma above. It is also known that we can then obtain a representation for ω_{Q_n} via balayage of $\rho_{n|[-1,1]\backslash I}$ onto I [15, Thm. IV.1.6(e), p. 196]. Thus if $I=[a,b]$, a representation for the balayage measure [15, Corollary II.4.12, p. 122] gives

(5.11)

$$\omega_{Q_n}'(x) = \rho_n'(x)_{|I} + \frac{1}{\pi}\int_{[-1,1]\backslash I} \frac{\left|\sqrt{(t-a)(t-b)}\right|}{|x-t|\sqrt{(x-a)(b-x)}}\rho_n'(t)\,dt,\ x \in I.$$

In order to apply Lemma 5.1, we choose

$$h(x) = \mu'(x),\ x \in I$$

and $d\mu_n^{\#}$ of Lemma 5.1, by

$$d\mu_n^{\#}(t) = h(t)\,W_n^{2n}(t)\,dt = \frac{1}{|\pi_{n-1}(t)|^2}\mu'(t)\,dt,\ t \in I.$$

We define $h=1$ and $d\mu_n^{\#}(t) = d\mu(t)\,/\,|\pi_{n-1}(t)|^2$ in $[-1,1]\backslash I$. Thus, with μ_n defined by (5.6),

$$\mu_n^{\#} = \mu_n.$$

We can now show that under our hypotheses on the poles, all the hypotheses of Lemma 5.1 are satisfied.

(a) Let J be a compact subinterval of I^o. We must show that $\left\{\omega_{Q_n}'\right\}_{n=1}^{\infty}$ are positive and uniformly bounded for t in some open interval containing J. As all $\{\nu_n\}$ have support in A_η, some $\eta>0$, $\mathrm{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}$ is uniformly bounded for $x \in [-1,1]$, and $|t| \leq 2$ with $t \in \bigcup_{n=1}^{\infty}\mathrm{supp}[\nu_n]$. For $x \in [-1,1]$, and $|t| \geq 2$, we have the trivial bound

$$\left|\frac{\sqrt{t^2-1}}{t-x}\right| \leq \frac{2|t|}{|t|/2} = 4.$$

It follows that ρ_n' of (5.9) admits the bound

$$\rho_n'(x) \leq \frac{C}{\sqrt{1-x^2}},\ x \in (-1,1),$$

where C is independent of n and x. This and (5.11) easily yield the uniform boundedness $\left\{\omega'_{Q_n}(x)\right\}_n$ in a suitable open interval containing J.
(b) We see that

$$Q_n(x) = \int \log|1 - x/t|\, d\nu_n(t)$$

so

$$Q'_n(x) = -\int \operatorname{Re}\left\{\frac{1}{t-x}\right\} d\nu_n(t).$$

Then

$$\begin{aligned}\left|Q'_n(y) - Q'_n(x)\right| &\leq \int \left|\operatorname{Re}\left\{\frac{1}{t-x} - \frac{1}{t-y}\right\}\right| d\nu_n(t) \\ &\leq |y-x| \int \frac{1}{|t-x|\,|t-y|} d\nu_n(t) \leq \frac{|y-x|}{\eta^2}.\end{aligned}$$

Thus $\{Q'_n\}$ even satisfy a uniform Lipschitz condition in $[-1,1]$, so are certainly equicontinuous on an open interval containing J.
(c) Lemma 2.1 gives

$$\lambda_n^{-1}(d\mu_n, x) = \lambda_n^r(x)^{-1} |\pi_{n-1}(x)|^2.$$

Thus, with our choice (5.8) and as $\mu_n^{\#} = \mu_n$,

$$\lambda_n^{-1}\left(d\mu_n^{\#}, x\right) W_n^{2n}(x)/n = \lambda_n^r(x)^{-1}/n.$$

We can now apply the uniform convergence in (1.16) in Theorem 1.1, for x in an open interval containing I, to obtain (5.3). Note that $\operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}$ is bounded above and below by positive constants for $x \in [-1,1]$ and $t \in \bigcup_{n=1}^{\infty} \operatorname{supp}[\nu_n]$. One way to prove this is to note that $\frac{1}{\pi\sqrt{1-x^2}} \operatorname{Re}\left\{\frac{\sqrt{t^2-1}}{t-x}\right\}$ is the Poisson kernel for the exterior of $[-1,1]$ and hence has to be bounded above and below by positive constants for t in each compact subset of $\bar{\mathbb{C}} \setminus [-1,1]$, and for $x \in [-1,1]$.
(d) This also follows from the previous considerations and Theorem 1.1.

Then, by Lemma 5.1, we have

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(d\mu_n, \xi + \frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)}{\tilde{K}_n(d\mu_n, \xi, \xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)},$$

uniformly for a, b in compact subsets of the real line. Using Lemma 2.1, we can recast this as

$$\lim_{n\to\infty}\frac{K_n^r\left(\xi+\frac{a}{\tilde{K}_n^r(\xi,\xi)},\xi+\frac{b}{\tilde{K}_n^r(\xi,\xi)}\right)}{K_n^r\left(\xi,\xi\right)}\frac{\pi_{n-1}\left(\xi+\frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)\overline{\pi_{n-1}\left(\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)}}{\left|\pi_{n-1}\left(\xi+\frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)\pi_{n-1}\left(\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)\right|}$$

$$=\lim_{n\to\infty}\frac{K_n\left(d\mu_n,\xi+\frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)},\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)}{K_n\left(d\mu_n,\xi,\xi\right)}\frac{\left|\pi_{n-1}\left(\xi\right)\right|^2}{\left|\pi_{n-1}\left(\xi+\frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)\pi_{n-1}\left(\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)\right|}$$

$$=\lim_{n\to\infty}\frac{\tilde{K}_n\left(d\mu_n,\xi+\frac{a}{\tilde{K}_n(d\mu_n,\xi,\xi)},\xi+\frac{b}{\tilde{K}_n(d\mu_n,\xi,\xi)}\right)}{\tilde{K}_n\left(d\mu_n,\xi,\xi\right)}=\frac{\sin\pi\left(a-b\right)}{\pi\left(a-b\right)},$$

uniformly for a, b in compact subsets of the real line. Here we have used the continuity of μ' at ξ. The limit above is easily reformulated as (1.17).■

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Department of Computer Science, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3001 Heverlee (Leuven), Belgium, karl.deckers@cs.kuleuven.be

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA, lubinsky@math.gatech.edu