On Asymptotics of Derivatives of Orthogonal Polynomials on the Unit Circle*

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Abstract

We show that ratio asymptotics of orthogonal polynomials on the circle imply ratio asymptotics for all their derivatives. Moreover, by reworking ideas of P. Nevai, we show that uniform asymptotics for orthogonal polynomials on an arc of the unit circle imply asymptotics for all their derivatives.

Let σ be a finite positive Borel measure on the unit circle (or $[0, 2\pi]$). Let $\{\varphi_n\}$ denote the orthonormal polynomials for σ, so that

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n\left(e^{i\theta}\right)\overline{\varphi_m}\left(e^{i\theta}\right) d\sigma\left(\theta\right) = \delta_{mn}.$$

Asymptotics for derivatives of orthogonal polynomials have been established under various hypotheses [1], [2], [5], [6], [8]. As far as the author is aware, the result that applies to the most general weights is due to P. Nevai [6]. Assuming Szegő's condition

$$\int_0^{2\pi} \log \sigma'\left(\theta\right) d\theta > -\infty,$$

and for some $\theta \in (0, 2\pi)$ and $\delta > 0$, σ is absolutely continuous in $(\theta - \delta, \theta + \delta)$, with

$$\int_{\theta-\delta}^{\theta+\delta} \left(\frac{\sigma'\left(\theta\right) - \sigma'\left(t\right)}{\theta - t}\right)^2 dt < \infty, \tag{1.1}$$

Nevai [6] proved that for each $m \geq 1$,

$$\lim_{n\to\infty} z^m \varphi_n^{(m)}\left(z\right) / \left(n^m \varphi_n\left(z\right)\right) = 1,$$

where $z = e^{i\theta}$. If the condition (1.1) holds uniformly for θ in an interval I, then the asymptotic holds uniformly for θ in I. Nevai's proof involved similar

* Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353.
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techniques to those for proving the asymptotics of $\{\varphi_n\}$ themselves, which are in turn equivalent to asymptotics for the reversed polynomials

$$\varphi_n^*(z) = z^n \overline{\varphi_n(1/\overline{z})}. \tag{1.2}$$

By reworking Nevai's ideas, we show that uniform asymptotics of $\{\varphi_n^*\}$ on an arc directly imply asymptotics for derivatives of φ_n.

Theorem A. *Let J be a subinterval of $[0, 2\pi]$, and assume that*

$$\lim_{n\to\infty} \varphi_n^*\left(e^{i\theta}\right) = g(\theta),$$

uniformly for $\theta \in J$, where $g(\theta) \neq 0$ for $\theta \in J$. Let $m \geq 1$ and $I \subset J^0$ be a closed interval. Then uniformly for $z = e^{i\theta}$, $\theta \in I$,

$$\lim_{n\to\infty} z^m \varphi_n^{(m)}(z) / (n^m \varphi_n(z)) = 1.$$

Proof. Note first that because of the uniform convergence, g is continuous, and hence $|g|$ is bounded away from 0 in any compact subinterval of J. Let I' be a compact subinterval of J such that $(I')^0 \supset I$. Because of uniform convergence, for some constant C independent of n,

$$\|\varphi_n^*\|_{L_\infty(I')} := \sup_{\theta\in I} \left|\varphi_n^*\left(e^{i\theta}\right)\right| \leq C, \qquad n \geq 1.$$

We now follow an idea of Nevai [6]. Let $[\sqrt{n}]$ denote the integer part of $\sqrt{n}$. By uniform convergence,

$$\eta_n = \left\|\varphi_n^* - \varphi_{[\sqrt{n}]}^*\right\|_{L_\infty(I')} \to 0,$$

$n \to \infty$. We shall apply Markov-Bernstein inequalities for trigonometric polynomials on arcs I' and their proper subarcs I. Let $D = \frac{d}{d\theta}$. If R is a trigonometric polynomial of degree $\leq n$, and $\ell \geq 1$ [3, pp. 242–243]

$$\left\|D^\ell R\right\|_{L_\infty(I)} \leq C_1 n^\ell \|R\|_{L_\infty(I')},$$

where C_1 depends only ℓ, I, I'. This and the above bounds give

$$\begin{aligned}\left\|D^\ell \varphi_n^*\right\|_{L_\infty(I)} &\leq \left\|D^\ell\left(\varphi_n^* - \varphi_{[\sqrt{n}]}^*\right)\right\|_{L_\infty(I)} + \left\|D^\ell \varphi_{[\sqrt{n}]}^*\right\|_{L_\infty(I)} \\ &\leq C_1 n^\ell \eta_n + C_1\left(\sqrt{n}\right)^\ell C = o\left(n^\ell\right).\end{aligned} \tag{1.3}$$

Next, we apply Leibniz's formula to the identity

$$\varphi_n\left(e^{i\theta}\right) = e^{in\theta}\overline{\varphi_n^*\left(e^{i\theta}\right)}, \tag{1.4}$$

which follows directly from (1.2), giving

$$D^m\left[\varphi_n\left(e^{i\theta}\right)\right]=\sum_{k=0}^{m}\binom{m}{k}(in)^k e^{in\theta}D^{m-k}\left[\overline{\varphi_n^*\left(e^{i\theta}\right)}\right].$$

Using (1.4) again, we obtain

$$\frac{D^m\left[\varphi_n\left(e^{i\theta}\right)\right]}{(in)^m\varphi_n\left(e^{i\theta}\right)}=1+\sum_{k=0}^{m-1}\binom{m}{k}(in)^{k-m}\frac{D^{m-k}\left[\overline{\varphi_n^*\left(e^{i\theta}\right)}\right]}{\overline{\varphi_n^*\left(e^{i\theta}\right)}}. \tag{1.5}$$

Now as g is bounded away from 0 in I, we also obtain for large enough n,

$$\inf_{\theta\in I}\left|\varphi_n^*\left(e^{i\theta}\right)\right|\geq C_2.$$

Then (1.3) and (1.5) give

$$\lim_{n\to\infty}\frac{D^m\left[\varphi_n\left(e^{i\theta}\right)\right]}{(in)^m\varphi_n\left(e^{i\theta}\right)}=1 \tag{1.6}$$

uniformly for $\theta\in I$. Next, Faa di Bruno's formula for derivatives of a composition of functions [4, p. 19], gives

$$D^m\left[\varphi_n\left(e^{i\theta}\right)\right]=\sum\frac{m!}{j_1!j_2!\cdots j_r!}\varphi_n^{(\ell)}\left(e^{i\theta}\right)e^{i\ell\theta}i^m\left(\frac{1}{1!}\right)^{j_1}\left(\frac{1}{2!}\right)^{j_2}\cdots\left(\frac{1}{r!}\right)^{j_r}, \tag{1.7}$$

where the sum is over all $r\geq 1$ and $r-$tuples $(j_1,j_2,\ldots,j_r)$ of positive integers with $j_1+2j_2+\cdots+rj_r=m$, while $\ell=j_1+j_2+\cdots+j_r$. From this, we see that $\varphi_n^{(m)}$ arises only when $r=1$, $j_1=m$. Thus

$$D^m\left[\varphi_n\left(e^{i\theta}\right)\right]=\varphi_n^{(m)}\left(e^{i\theta}\right)\left(ie^{i\theta}\right)^m+\Sigma,$$

where Σ is a linear combination of $\varphi_n^{(k)}\left(e^{i\theta}\right)i^m$, $0\leq k\leq m-1$, with coefficients independent of n. Dividing by $(in)^m\varphi_n\left(e^{i\theta}\right)$, using the lower bounds for $|\varphi_n|$, and an easy induction on m, gives the result. $\square$

Remarks.
(a) Note that we do not assume Szegö's condition above.
(b) Let $\theta\in[0,2\pi]$. From the identity (1.5), and then (1.7), it is easy to see that the following are equivalent:
(I) For all $m\geq 1$,

$$\lim_{n\to\infty}\frac{D^m\left[\varphi_n\left(e^{i\theta}\right)\right]}{(in)^m\varphi_n\left(e^{i\theta}\right)}=1.$$

(II) For all $m\geq 1$,

$$\lim_{n\to\infty}\frac{D^m\varphi_n^*\left(e^{i\theta}\right)}{\varphi_n^*\left(e^{i\theta}\right)}=0.$$

(III) For all $m \geq 1$,

$$\lim_{n\to\infty} z^m \varphi_n^{(m)}(z) / (n^m \varphi_n(z)) = 1,$$

where $z = e^{i\theta}$.
(c) The above argument is easily modified to deal with orthogonal polynomials with varying weights.

Our second result deals with ratio asymptotics, which are often stated in terms of the monic orthogonal polynomials

$$\Phi_n(z) = \varphi_n(z) / \kappa_n = z^n + \cdots .$$

Here κ_n is the leading coefficient of φ_n. These polynomials satisfy the recurrence relation [7, p. 56]

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z), \tag{1.8}$$

where $\alpha_n = \overline{\Phi_{n+1}(0)}$ is a Verblunsky coefficient, and

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}.$$

The ratio asymptotic takes the form

$$\lim_{n\to\infty} \frac{\Phi_{n+1}(z)}{\Phi_n(z)} = z.$$

There is a major theory for asymptotics of this type, with key initial advances due to Máté, Nevai, Rakhmanov, and Totik, and many later works. See [7]. In particular, Rakhmanov' theorem asserts that if $\sigma' > 0$ a.e. on $[0, 2\pi]$, then we have this ratio asymptotic. We prove:

Theorem B. *Assume that*

$$\lim_{n\to\infty} \frac{\Phi_{n+1}(z)}{z\Phi_n(z)} = 1, \tag{1.9}$$

holds at some z with $|z| = 1$. Let $m \geq 1$. Then uniformly in $\{z : |z| \geq 1\}$,

$$\lim_{n\to\infty} \frac{\Phi_{n+1}^{(m)}(z)}{z\Phi_n^{(m)}(z)} = 1. \tag{1.10}$$

Proof. We note that $|\Phi_n^*(z)| = |\Phi_n(z)|$ for $|z| = 1$. Then the recurrence relation (1.8) gives

$$\left| \frac{\Phi_{n+1}(z)}{\Phi_n(z)} - z \right| = |\alpha_n|,$$

and hence our hypothesis gives

$$\lim_{n\to\infty} \alpha_n = 0. \tag{1.11}$$

It then also follows that (1.9) holds uniformly on the unit circle. Differentiating (1.8) m times gives

$$\Phi_{n+1}^{(m)}(z) = z\Phi_n^{(m)}(z) + m\Phi_n^{(m-1)}(z) - \overline{\alpha_n}\Phi_n^{*(m)}(z),$$

and hence

$$\left|\frac{\Phi_{n+1}^{(m)}(z)}{z\Phi_n^{(m)}(z)} - 1\right| \leq m\left|\frac{\Phi_n^{(m-1)}(z)}{\Phi_n^{(m)}(z)}\right| + |\alpha_n|\left|\frac{\Phi_n^{*(m)}(z)}{\Phi_n^{(m)}(z)}\right|. \tag{1.12}$$

Now Φ_n has all its zeros in the open unit ball, so the same is true for its derivatives (by Lucas' theorem [3, p. 18]). In particular, $\Phi_n^{(m-1)}$ has all its zeros there. A remarkable inequality of Turan [2, Lemma 1, p. 775] asserts that if P is a polynomial of degree n with all zeros inside the unit circle, then for all $|z| = 1$,

$$|P'(z)| \geq \frac{n}{2}|P(z)|.$$

Hence for any given z on the unit circle, and $n \geq m+1$,

$$\left|\Phi_n^{(m)}(z)\right| \geq \frac{n-m}{2}\left|\Phi_n^{(m-1)}(z)\right|. \tag{1.13}$$

Next, let $\overline{\Phi}_n$ denote the polynomial whose coefficients are the complex conjugates of those of Φ_n. Observe that for $|z| = 1$,

$$\Phi_n^*(z) = z^n\overline{\Phi}_n(1/z)$$

and also for each $j \geq 0$,

$$\left|\overline{\Phi}_n^{(j)}(1/z)\right| = \left|\Phi_n^{(j)}(z)\right|.$$

Then for each $\ell \geq 0$, Faa di Bruno's formula for the derivatives of a composition of functions [4, p. 19] shows that

$$\left|\left(\frac{d}{dz}\right)^{\ell}\left(\overline{\Phi}_n(1/z)\right)\right| \leq C_\ell \sum_{j=0}^{\ell}\left|\Phi_n^{(j)}(z)\right|,$$

where C_ℓ depends only on ℓ (not on n, Φ_n or z). Then Leibniz' formula gives

$$\begin{aligned}\left|\Phi_n^{*(m)}(z)\right| &= \left|\sum_{\ell=0}^{m}\binom{m}{\ell}n(n-1)\cdots(n-m+\ell+1)z^{n-m+\ell}\left[\left(\frac{d}{dz}\right)^{\ell}\left(\overline{\Phi}_n(1/z)\right)\right]\right| \\ &\leq C_m\sum_{\ell=0}^{m}n^{m-\ell}\sum_{j=0}^{\ell}\left|\Phi_n^{(j)}(z)\right|,\end{aligned}$$

where C_m depends on m, but not on n, Φ_n or z. On applying (1.13) repeatedly, we obtain

$$\left|\frac{\Phi_n^{*(m)}(z)}{\Phi_n^{(m)}(z)}\right| \leq C,$$

where C depends only on m, not on n, z, or the particular system of orthogonal polynomials. Substituting this and (1.13) into (1.12) gives the quantitative estimate

$$\left|\frac{\Phi_{n+1}^{(m)}(z)}{z\Phi_n^{(m)}(z)} - 1\right| \leq \frac{2m}{n-m} + C\left|\alpha_n\right|.$$

This estimate holds uniformly for $|z| = 1$, all orthogonal systems, and $n \geq m+1$. Since the function $\frac{\Phi_{n+1}^{(m)}(z)}{z\Phi_n^{(m)}(z)} - 1$ is analytic in $\{z : |z| \geq 1\}$ and takes the finite value $\frac{m}{n-m+1}$ at ∞, the maximum-modulus principle gives the result. $\square$

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