UNIFORM MEAN VALUE ESTIMATES AND DISCRETE HILBERT INEQUALITIES VIA ORTHOGONAL DIRICHLET SERIES

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ABSTRACT. Let $\{\lambda_j\}_{j=0}^{\infty}$ be a strictly increasing sequence of positive numbers with $\lambda_0 = 0$ and $\lambda_1 = 1$. We use orthogonal Dirichlet polynomials associated with the arctangent density, to observe that for $r > 0$,

$$\int_0^\infty \left| \sum_{n=1}^{\infty} (-1)^{n-1} a_n \lambda_n^{-irt} \right|^2 \frac{dt}{\pi (1+t^2)}$$
$$= \sum_{n=1}^{\infty} \left(\lambda_n^{2r} - \lambda_{n-1}^{2r} \right) \left| \sum_{k=n}^{\infty} (-1)^{k-1} \frac{a_k}{\lambda_k^r} \right|^2,$$

when the right-hand side converges. As a consequence, we obtain uniform mean value estimates, discrete Hilbert type inequalities, and asymptotics as $r \to \infty$ for classes of Dirichlet series.

1. Introduction

Throughout, let

$$\lambda_0 = 0 \text{ and } 1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \tag{1.1}$$

with

$$\lim_{k\to\infty} \lambda_k = \infty. \tag{1.2}$$

A *Dirichlet series* associated with this sequence of exponents has the form

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-it} = \sum_{n=1}^{\infty} a_n e^{-i(\log \lambda_n)t},$$

where all $a_j \in \mathbb{C}$. In particular, when $\lambda_j = j$, $j \geq 1$, we obtain the standard Dirichlet series, of which the Riemann zeta function is a special case. A *Dirichlet polynomial* has the form

$$P(t) = \sum_{n=1}^{m} a_n \lambda_n^{-it}.$$

Dirichlet series and polynomials are intimately connected with the theory of almost-periodic functions, developed by Besicovitch, Bohr, and Stepanoff, amongst

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others [1], [2]. For Dirichlet polynomials P, Q of the above form (but with possibly different exponents $\{\lambda_j\}$ in Q), one defines an inner product

$$(P, Q) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P(t) \overline{Q(t)} dt.$$

The norm of P with respect to this inner product is

$$(P, P)^{1/2} = \left(\sum_{n=1}^{m} |a_n|^2 \right)^{1/2}.$$

The closure of the space of all Dirichlet polynomials with respect to this norm defines a Hilbert space, that admits a Parseval identity and Riesz-Fisher theorem [1, pp. 109–110]. In particular, if

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

then

$$F(t) = \sum_{n=1}^{\infty} a_n \lambda_n^{-it}$$

is a well defined element of this Hilbert space, and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(t)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 . \tag{1.3}$$

Note that this a-priori yields a mean value estimate

$$\frac{1}{T} \int_0^T |F(t)|^2 dt = O\left(\sum_{n=1}^{\infty} |a_n|^2 \right), \quad T > 0.$$

However, the order term is not uniform in F, and of course, that uniformity is not possible in general.

In studies of the Riemann-zeta function, mean value theorems and estimates play an important role. Classical limits include [6, p. 28], [10, p. 30]

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma), \quad \sigma > \frac{1}{2},$$

and [6, p. 30], [10, p. 34] as $T \to \infty$,

$$\frac{1}{T} \int_1^T |\zeta(\sigma + it)|^2 dt = \log T (1 + o(1)).$$

The classical Montgomery-Vaughan mean value theorem for Dirichlet polynomials [6, p. 131] asserts that

$$\frac{1}{T} \int_0^T \left| \sum_{n=1}^{m} a_n n^{-it} \right|^2 dt = \sum_{n=1}^{m} |a_n|^2 + O\left(\frac{1}{T} \sum_{n=1}^{m} n |a_n|^2 \right), \tag{1.4}$$

where, crucially, the order term is uniform in $T > 0$, $m \geq 1$, and $\{a_n\}$. The proof of this uses Hilbert's inequality for the norm of the Hilbert matrix. There are many extensions and variations of these results – some recent references include [4], [8], [9], [12], [13], [14].

In this paper, we use orthogonal expansions involving Dirichlet orthogonal functions for the arctan density to prove related mean value asymptotics, uniform estimates, and discrete Hilbert type inequalities.

We can apply the Gram-Schmidt process to $\left\{\lambda_n^{-it}\right\}_{n\geq 1}$, yielding orthonormal Dirichlet polynomials $\{\phi_n\}_{n=1}^{\infty}$ with positive leading coefficient, and

$$\int_{-\infty}^{\infty} \phi_n(t)\,\overline{\phi_m(t)}\frac{dt}{\pi(1+t^2)} = \delta_{mn}, \quad m,n\geq 1.$$

There is a simple representation for these [7]: for $n\geq 1$,

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}. \tag{1.5}$$

We let $\mathcal{L}$ denote the (Hilbert) space of all measurable functions $f:\mathbb{R}\to\mathbb{C}$ with

$$\|f\|^2 = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{\pi(1+t^2)}dt < \infty. \tag{1.6}$$

By considering orthonormal expansions with respect to powers $\{\lambda_n^r\}_{n\geq 1}$ of $\{\lambda_n\}_{n\geq 1}$, we observe that:

Theorem 1.1. *Let $r>0, \{a_n\}_{n\geq 1}\subset\mathbb{C}$, $\{\lambda_n\}_{n\geq 0}$ be as above, and*

$$F(t) = \sum_{n=1}^{\infty} a_n\lambda_n^{-it}. \tag{1.7}$$

Then

$$\int_{-\infty}^{\infty} |F(rt)|^2 \frac{dt}{\pi(1+t^2)} = \sum_{k=1}^{\infty}\left(\lambda_k^{2r} - \lambda_{k-1}^{2r}\right)\left|\sum_{n=k}^{\infty}\frac{a_n}{\lambda_n^r}\right|^2, \tag{1.8}$$

provided the series on the right-hand side converges. In this case, $F\in\mathcal{L}$ and in $\mathcal{L}$, F is the limit of a suitable subsequence of its partial sums.

A simple consequence of this is the following uniform estimate:

Corollary 1.2. *Let $r>0$, $\{\lambda_n\}_{n\geq 0}$ be as above, and $\{a_k\}_{k\geq 1}$ be a sequence of non-negative numbers such that $\{a_k/\lambda_k^r\}_{k\geq 1}$ is a decreasing sequence. Then*

$$F(t) = \sum_{n=1}^{\infty}(-1)^{n-1} a_n\lambda_n^{-it} \tag{1.9}$$

has $F\in\mathcal{L}$, and

$$\int_{-\infty}^{\infty}|F(rt)|^2\frac{dt}{\pi(1+t^2)} \leq \sum_{n=1}^{\infty}|a_n|^2, \tag{1.10}$$

provided the series on the right-hand side converges. In particular, if $\{a_k\}_{k\geq 1}$ is decreasing, then for all $T>0$,

$$\frac{1}{T}\int_0^T\left(|F(t)|^2 + \left|F\left(\frac{T^2}{t}\right)\right|^2\right)dt \leq \pi\sum_{n=1}^{\infty}|a_n|^2. \tag{1.11}$$

In the special case where $\lambda_k = k$, we can obtain more:

Corollary 1.3. *Let $r \geq \frac{1}{2}$, and $\{a_k\}_{k\geq 1}$ be a sequence of non-negative numbers such that $\{a_k/k^r\}_{k\geq 1}$ is a decreasing sequence, and*

$$\sum_{n=1}^{\infty} \frac{a_n^2}{n} < \infty. \tag{1.12}$$

(a) *Then*

$$F(t) = \sum_{n=1}^{\infty} (-1)^{n-1} a_n n^{-it} \tag{1.13}$$

has $F \in \mathcal{L}$.

(b) *For any positive integer L,*

$$\int_{-\infty}^{\infty} |F(rt)|^2 \frac{dt}{\pi(1+t^2)} \leq \sum_{n=1}^{L} a_n^2 + 2r \sum_{n=L+1}^{\infty} \frac{a_n^2}{n} \tag{1.14}$$

and if $\{a_k\}_{k\geq 1}$ is decreasing,

$$\int_{-\infty}^{\infty} |F(rt)|^2 \frac{dt}{\pi(1+t^2)} \geq \left(1 - \left(1 + \frac{1}{L}\right)^{-r}\right)^3 \sum_{n=1}^{L} a_n^2. \tag{1.15}$$

(c) *If $\{a_k\}_{k\geq 1}$ is decreasing, then for all $T \geq \frac{1}{2}$,*

$$\frac{1}{T}\int_0^T \left(|F(t)|^2 + \left| F\left(\frac{T^2}{t}\right)\right|^2 \right) dt \leq \pi \left(\sum_{n=1}^{L} a_n^2 + 2T \sum_{n=L+1}^{\infty} \frac{a_n^2}{n} \right) \tag{1.16}$$

and

$$\frac{1}{T}\int_0^T \left(|F(t)|^2 + \left| F\left(\frac{T^2}{t}\right)\right|^2 \right) dt \geq \frac{\pi}{2} \left(1 - \left(1 + \frac{1}{L}\right)^{-T}\right)^3 \sum_{n=1}^{L} a_n^2. \tag{1.17}$$

As a motivating example, consider

$$F(t) = \left(1 - 2^{1-(\sigma+it)}\right) \zeta(\sigma+it) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-\sigma-it}.$$

For $\sigma > \frac{1}{2}$, we obtain, for all $T > 0$,

$$\frac{1}{T}\int_0^T \left(\left|\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\sigma-it}\right|^2 + \left|\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\sigma-iT^2/t}\right|^2 \right) dt \leq \pi\zeta(2\sigma). \tag{1.18}$$

When $\sigma = \frac{1}{2}$, choosing L in (1.16) to be the least integer $\geq T \geq \frac{1}{2}$, yields

$$\begin{aligned} &\frac{1}{T}\int_0^T \left(\left|\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{2}-it}\right|^2 + \left|\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{2}-iT^2/t}\right|^2 \right) dt \\ &\leq \pi(3 + \log(1+T)). \end{aligned} \tag{1.19}$$

A similar lower bound follows from (1.17). When $0<\sigma<\frac{1}{2}$, choosing L to be the least integer $\geq T\geq\frac{1}{2}$, yields

$$\frac{1}{T}\int_0^T\left(\left|\sum_{n=1}^{\infty}(-1)^{n-1}\,n^{-\sigma-it}\right|^2+\left|\sum_{n=1}^{\infty}(-1)^{n-1}\,n^{-\sigma-iT^2/t}\right|^2\right)dt$$

$$\leq\pi\,(T+1)^{1-2\sigma}\,\frac{(1-\sigma)}{\sigma\,(1-2\sigma)}. \tag{1.20}$$

For the case of alternating Dirichlet polynomials with decreasing coefficients $\{a_n\}_{n=1}^m$, Corollary 1.3 yields for all $T>0$,

$$\frac{1}{T}\int_0^T\left(\left|\sum_{n=1}^{m}(-1)^{n-1}\,a_n\lambda_n^{-it}\right|^2+\left|\sum_{n=1}^{m}(-1)^{n-1}\,a_n\lambda_n^{-iT^2/t}\right|^2\right)dt\leq\pi\sum_{n=1}^{m}|a_n|^2, \tag{1.21}$$

which can be better than (1.4) for $T\leq m$. Thus, for example, if we consider the alternating sum used by Turán in an equivalent formulation of the Lindelöf hypothesis [11],

$$\frac{1}{T}\int_0^T\left(\left|\sum_{n=1}^{m}(-1)^{n-1}\,n^{-it}\right|^2+\left|\sum_{n=1}^{m}(-1)^{n-1}\,n^{-iT^2/t}\right|^2\right)dt\leq\pi m. \tag{1.22}$$

In the case where $T<\frac{m}{2}$, choosing L in (1.16) to be the least integer $\geq 2T$ yields

$$\frac{1}{T}\int_0^T\left(\left|\sum_{n=1}^{m}(-1)^{n-1}\,n^{-it}\right|^2+\left|\sum_{n=1}^{m}(-1)^{n-1}\,n^{-iT^2/t}\right|^2\right)dt$$

$$\leq\pi\,(2T+1)\left(1+\log^{+}\frac{m}{2T+1}\right). \tag{1.23}$$

Here, $\log^{+}x=\max\{0,\log x\}$.

We can also deduce mean value asymptotics from Corollary 1.3. However, in many cases, an integration by parts and the asymptotic (1.3) of almost periodic functions yields as much. To illustrate, let $g:\mathbb{R}\to\mathbb{C}$ be continuous, with

$$M(r)=\frac{1}{2r}\int_{-r}^{r}|g(t)|^2\,dt=o(r)\,,\quad r\to\infty.$$

Then an integration by parts shows that

$$\int_{-\infty}^{\infty}|g(rt)|^2\,\frac{dt}{\pi\,(1+t^2)}=\frac{4}{\pi}\int_0^{\infty}M(rs)\,\frac{s^2}{(1+s^2)^2}ds.$$

Hence

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}|g(rt)|^2\,\frac{dt}{\pi\,(1+t^2)}=\lim_{r\to\infty}M(r)$$

if the limit on the right-hand side exists. Thus if F is given by (1.7), then from this last limit and (1.3),

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}|F(rt)|^2\,\frac{dt}{\pi\,(1+t^2)}=\sum_{n=1}^{\infty}|a_n|^2$$

provided the series on the right-hand side converges. In particular, for $\sigma > \frac{1}{2}$,

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \left|\zeta\left(\sigma + rt\right)\right|^2 \frac{dt}{\pi\left(1+t^2\right)} = \zeta\left(2\sigma\right).$$

The logarithmic growth for $\sigma = \frac{1}{2}$ also follows.

When asymptotics for $M\left(r\right)$ are not available, and (1.3) fails, we can instead apply Corollary 1.3. For example, choosing L in (1.14) to be the largest integer $\leq r$ gives

$$\int_{-\infty}^{\infty} \left|\sum_{n=1}^{\infty} \left(-1\right)^{n-1} n^{-\frac{1}{2}-irt}\right|^2 \frac{dt}{\pi\left(1+t^2\right)} \leq \log r + O\left(1\right).$$

In the other direction, we choose L to be the largest integer $\leq r/\log\log r$ in (1.15):

$$\int_{-\infty}^{\infty} \left|\sum_{n=1}^{\infty} \left(-1\right)^{n-1} n^{-\frac{1}{2}-irt}\right|^2 \frac{dt}{\pi\left(1+t^2\right)} \geq \log r + O\left(\log\log\log r\right).$$

Thus as $r \to \infty$,

$$\int_{-\infty}^{\infty} \left|\sum_{n=1}^{\infty} \left(-1\right)^{n-1} n^{-\frac{1}{2}-irt}\right|^2 \frac{dt}{\pi\left(1+t^2\right)} = \log r + O\left(\log\log\log r\right). \tag{1.24}$$

Finally, by integrating over a range of r, we can obtain discrete Hilbert type inequalities involving alternating coefficients. Some recent references in this active area include [3, p. 3529], [5], [15].

Theorem 1.4. *Let $\left\{\lambda_n\right\}_{n\geq 0}$ be as above, and $\left\{a_k\right\}_{1\leq k\leq m}$ be a finite decreasing sequence of non-negative numbers.*

(a) *Let $r \geq 0$. Then*

$$0 \geq \sum_{j\neq k} a_j a_k \left(-1\right)^{j+k} e^{-r\left|\log\lambda_j - \log\lambda_k\right|} \geq -\sum_{j=1}^{m} a_j^2. \tag{1.25}$$

(b) *For $\alpha, \beta \geq 0$,*

$$0 \geq \sum_{j\neq k} \frac{a_j a_k}{\left(\alpha\left|\log\lambda_j - \log\lambda_k\right| + 1\right)^{\beta}} \left(-1\right)^{j+k} \geq -\sum_{j=1}^{m} a_j^2. \tag{1.26}$$

Of course the coefficient restriction is severe, but there is no restriction on the spacing of the $\left\{\lambda_j\right\}$ as in most Hilbert type inequalities, and there is no growth factor in the sum in the right-hand side. By integrating (1.25), we can obtain inequalities involving general Laplace transforms, and that is how (b) is proved. For general complex coefficients, we obtain:

Theorem 1.5. *Let $\left\{\lambda_n\right\}_{n\geq 0}$ be as above, and $\left\{a_k\right\}_{1\leq k\leq m} \subset \mathbb{C}$. Let $\beta \geq 0, \alpha > 0$, and $\left\{\rho_k\right\}_{k=1}^{m} \subset (0, \infty)$.*

(a) *For $\alpha > 0, \beta \geq 0$,*

$$\left|\sum_{j\neq k} \frac{a_j \overline{a_k}}{\left(\alpha\left|\log\lambda_j - \log\lambda_k\right| + 1\right)^{\beta}}\right| \leq \sum_{n=1}^{m} \rho_n \left|a_n\right|^2 \left(\sum_{\ell=1}^{m} \frac{1}{\rho_\ell} \left[2\alpha \log^+ \frac{\lambda_\ell}{\lambda_n} + 1\right]^{-\beta}\right). \tag{1.27}$$

Here $\log^+ x = \max\left\{0, \log x\right\}$.

(b) *In particular, for* $\beta > 1$,

$$\left| \sum_{j \neq k} \frac{a_j \overline{a_k}}{(\alpha |j-k| + 1)^{\beta}} \right| \leq \sum_{n=1}^{m} |a_n|^2 \left\{ n + \frac{1}{(\beta - 1)(2\alpha)^{\beta}} \right\}. \tag{1.28}$$

2. Proofs

Throughout, we let $\{\lambda_n\}_{n \geq 0}$ be as at (1.1), (1.2). For $r > 0$, we let

$$\phi_{n,r}(t) = \frac{\lambda_n^{r(1-it)} - \lambda_{n-1}^{r(1-it)}}{\sqrt{\lambda_n^{2r} - \lambda_{n-1}^{2r}}}. \tag{2.1}$$

These are the orthonormal polynomials for the sequence $\{\lambda_n^r\}_{n \geq 1}$, for the arctan density. Although this follows from the results in [7], we include the proof for the reader's convenience:

Lemma 2.1. (a) *For* $0 < \lambda \leq \lambda_{n-1}^r$,

$$\int_{-\infty}^{\infty} \phi_{n,r}(t) \lambda^{it} \frac{dt}{\pi(1+t^2)} = 0. \tag{2.2}$$

(b)

$$\int_{-\infty}^{\infty} |\phi_{n,r}(t)|^2 \frac{dt}{\pi(1+t^2)} = 1. \tag{2.3}$$

(c) *For* $m \geq 1$,

$$\lambda_m^{r(1-it)} = \sum_{n=1}^{m} \sqrt{\lambda_n^{2r} - \lambda_{n-1}^{2r}} \phi_{n,r}(t). \tag{2.4}$$

Proof. (a) By the residue theorem, for real μ,

$$\int_{-\infty}^{\infty} \frac{e^{i\mu t}}{\pi(1+t^2)} dt = e^{-|\mu|}. \tag{2.5}$$

Then for $0 \leq \mu \leq \log \lambda_{n-1}^r$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \phi_{n,r}(t) \frac{e^{i\mu t}}{\pi(1+t^2)} dt \\ &= \frac{1}{\sqrt{\lambda_n^{2r} - \lambda_{n-1}^{2r}}} \int_{-\infty}^{\infty} \left(\lambda_n^r e^{i(\mu - r\log\lambda_n)t} - \lambda_{n-1}^r e^{i(\mu - r\log\lambda_{n-1})t} \right) \frac{dt}{\pi(1+t^2)} \\ &= \frac{1}{\sqrt{\lambda_n^{2r} - \lambda_{n-1}^{2r}}} \left(\lambda_n^r e^{-|\mu - r\log\lambda_n|} - \lambda_{n-1}^r e^{-|\mu - r\log\lambda_{n-1}|} \right) \\ &= \frac{1}{\sqrt{\lambda_n^{2r} - \lambda_{n-1}^{2r}}} \left(e^{\mu} - e^{\mu} \right) = 0. \end{aligned}$$

(b) This follows similarly from (2.5).

(c) From (2.1), the right-hand side of (2.4) reduces to a telescopic series that sums to λ_m^{1-it}.

□

Proof of Theorem 1.1. For $k \geq 1$, let

$$c_k = \sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \sum_{n=k}^{\infty} \frac{a_n}{\lambda_n^r}.$$

First note that since $\mathcal{L}$ is complete, and since the coefficients $\{c_k\}$ are square summable by (1.8), $\mathcal{L}$ contains a function g with orthonormal expansion $\sum_{k=1}^{\infty} c_k \phi_{k,r}$. Moreover,

$$\int_{-\infty}^{\infty} |g(t)|^2 \frac{dt}{\pi(1+t^2)} = \sum_{k=1}^{\infty} |c_k|^2 .$$

Let

$$g_m(t) = \sum_{k=1}^{m} c_k \phi_{k,r}(t) = \sum_{k=1}^{m} \left(\sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \sum_{n=k}^{\infty} \frac{a_n}{\lambda_n^r} \right) \phi_{k,r}(t) .$$

denote the mth partial sum of g. For $m \geq 1$, let

$$S_m(rt) = \sum_{n=1}^{m} a_n \lambda_n^{-irt}$$

denote the mth partial sum of $F(rt)$. We'll show that for a suitable strictly increasing sequence of integers $\{\tau_k\}_{k\geq 1}$,

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} |S_{\tau_k - 1}(rt) - g_{\tau_k - 1}(t)|^2 \frac{dt}{\pi(1+t^2)} = 0, \tag{2.6}$$

and can then identify $g(t)$ with $F(rt)$ of (1.7). Then $F(r\cdot) \in \mathcal{L}$ and so also $F(\cdot) \in \mathcal{L}$, and is the limit in $\mathcal{L}$ of $\{S_{\tau_k - 1}\}_{k\geq 1}$. To this end, choose an increasing sequence of integers $\{\ell_k\}_{k\geq 1}$ such that

$$\lambda_{\ell_k}^r \geq 2\lambda_{\ell_{k-1}}^r .$$

Next, choose an integer $\tau_k \in (\ell_{k-1}, \ell_k]$ such that

$$\left| \sum_{n=\tau_k}^{\infty} \frac{a_n}{\lambda_n^r} \right| = \min \left\{ \left| \sum_{n=j}^{\infty} \frac{a_n}{\lambda_n^r} \right| : j \in (\ell_{k-1}, \ell_k] \right\} .$$

Observe that

$$\begin{aligned}
\frac{1}{2} \sum_{k=1}^{\infty} \lambda_{\tau_k}^{2r} \left| \sum_{n=\tau_k}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 &\leq \frac{1}{2} \sum_{k=1}^{\infty} \lambda_{\ell_k}^{2r} \left| \sum_{n=\tau_k}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 \\
&\leq \sum_{k=1}^{\infty} \left(\lambda_{\ell_k}^{2r} - \lambda_{\ell_{k-1}}^{2r} \right) \left| \sum_{n=\tau_k}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{j=\ell_{k-1}+1}^{\ell_k} \left(\lambda_j^{2r} - \lambda_{j-1}^{2r} \right) \left| \sum_{n=j}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 < \infty,
\end{aligned}$$

by (1.8). Consequently,

$$\lim_{k\to\infty} \lambda_{\tau_k}^{2r} \left| \sum_{n=\tau_k}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 = 0. \tag{2.7}$$

Next, using Lemma 2.1(c),

$$S_m(rt) = \sum_{n=1}^{m} \frac{a_n}{\lambda_n^r} \sum_{k=1}^{n} \sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \phi_{k,r}(t) = \sum_{k=1}^{m} \left(\sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \sum_{n=k}^{m} \frac{a_n}{\lambda_n^r} \right) \phi_{k,r}(t),$$

so

$$S_m(rt) - g_m(t) = -\sum_{k=1}^{m} \left(\sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \sum_{n=m+1}^{\infty} \frac{a_n}{\lambda_n^r} \right) \phi_{k,r}(t)$$

Then orthonormality gives

$$\begin{aligned} \int_{-\infty}^{\infty} |S_m(rt) - g_m(t)|^2 \frac{dt}{\pi(1+t^2)} &= \sum_{k=1}^{m} \left(\lambda_k^{2r} - \lambda_{k-1}^{2r} \right) \left| \sum_{n=m+1}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 \\ &= \lambda_m^{2r} \left| \sum_{n=m+1}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2 \le \lambda_{m+1}^{2r} \left| \sum_{n=m+1}^{\infty} \frac{a_n}{\lambda_n^r} \right|^2. \end{aligned}$$

Choosing $m = \tau_k - 1$, for $k \ge 1$, and then using (2.7), gives the desired relation (2.6). We thus have completed the proof of Theorem 1.1. □

Note that we have proved that $F(rt)$ has the convergent in $\mathcal{L}$ orthonormal expansion

$$F(rt) = \sum_{k=1}^{\infty} \left(\sqrt{\lambda_k^{2r} - \lambda_{k-1}^{2r}} \sum_{n=k}^{\infty} \frac{a_n}{\lambda_n^r} \right) \phi_{k,r}(t).$$

Proof of Corollary 1.2. As $\sum_{n=k}^{\infty} \frac{(-1)^{n-1} a_n}{\lambda_n^r}$ is an alternating series with decreasing non-negative terms $\frac{a_n}{\lambda_n^r}$, we have

$$\left| \sum_{n=k}^{\infty} \frac{(-1)^{n-1} a_n}{\lambda_n^r} \right| \le \frac{a_k}{\lambda_k^r}.$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} |F(rt)|^2 \frac{dt}{\pi(1+t^2)} &= \sum_{k=1}^{\infty} \left(\lambda_k^{2r} - \lambda_{k-1}^{2r} \right) \left| \sum_{n=k}^{\infty} \frac{(-1)^{n-1} a_n}{\lambda_n^r} \right|^2 \\ &\le \sum_{k=1}^{\infty} \left(\lambda_k^{2r} - \lambda_{k-1}^{2r} \right) \left(\frac{a_k}{\lambda_k^r} \right)^2 \le \sum_{k=1}^{\infty} a_k^2. \end{aligned} \tag{2.8}$$

Consequently, recalling that the $\{a_k\}$ are real numbers,

$$\begin{aligned}
&\frac{1}{\pi T}\int_0^T\left(|F(t)|^2+\left|F\left(\frac{T^2}{t}\right)\right|^2\right)dt \\
&\quad=\frac{1}{2\pi T}\int_{-T}^T\left(|F(t)|^2+\left|F\left(\frac{T^2}{t}\right)\right|^2\right)dt \\
&\quad=\frac{1}{2\pi}\int_{-1}^1\left(|F(Ts)|^2+\left|F\left(\frac{T}{s}\right)\right|^2\right)ds \\
&\quad\leq\int_{-1}^1|F(Ts)|^2\frac{ds}{\pi(1+s^2)}+\int_{-1}^1\left|F\left(\frac{T}{s}\right)\right|^2\frac{ds}{\pi(1+s^2)} \\
&\quad=\int_{-\infty}^\infty|F(Ts)|^2\frac{ds}{\pi(1+s^2)}\leq\sum_{k=1}^\infty a_k^2, \qquad (2.9)
\end{aligned}$$

where we made the substitution $s\to 1/s$ in the second last line. □

Proof of Corollary 1.3. (a) In the case $\lambda_k=k$ for all k, (2.8) gives

$$\begin{aligned}
\int_{-\infty}^\infty|F(rt)|^2\frac{dt}{\pi(1+t^2)}&\leq\sum_{k=1}^\infty\left(1-\left(1-\frac{1}{k}\right)^{2r}\right)a_k^2 \\
&\leq\sum_{k=1}^\infty\min\left\{1,\frac{2r}{k}\right\}a_k^2,
\end{aligned}$$

by the inequality $1-(1-x)^{2r}\leq 2rx$ for $x\in[0,1]\,,r\geq\frac{1}{2}$. So the right-hand side converges, by (1.12).

(b) Given any $L\geq 1$, we can continue the above as

$$\int_{-\infty}^\infty|F(rt)|^2\frac{dt}{\pi(1+t^2)}\leq\sum_{k=1}^L a_k^2+2r\sum_{k=L+1}^\infty\frac{a_k^2}{k}.$$

For the lower bound, we first observe that as the series is alternating with decreasing terms, and as $\{a_n\}$ is also decreasing,

$$\begin{aligned}
\left|\sum_{n=k}^\infty\frac{(-1)^{n-1}a_n}{n^r}\right|&\geq\frac{a_k}{k^r}-\frac{a_{k+1}}{(k+1)^r}\geq\frac{a_k}{k^r}-\frac{a_k}{(k+1)^r} \\
&=\frac{a_k}{k^r}\left(1-\left(1+\frac{1}{k}\right)^{-r}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
k^{2r}-(k-1)^{2r}&=k^{2r}\left(1-\left(1-\frac{1}{k}\right)^{2r}\right) \\
&\geq k^{2r}\left(1-\left(1-\frac{1}{k}\right)^{r}\right)\geq k^{2r}\left(1-\left(1+\frac{1}{k}\right)^{-r}\right),
\end{aligned}$$

by the inequality $1-x\le(1+x)^{-1}$, $x\in[0,1]$. Hence, from (1.8),

$$\int_{-\infty}^{\infty}|F(rt)|^2\frac{dt}{\pi(1+t^2)}\ge\sum_{k=1}^{L}\left(1-\left(1+\frac{1}{k}\right)^{-r}\right)^3 a_k^2$$
$$\ge\left(1-\left(1+\frac{1}{L}\right)^{-r}\right)^3\sum_{k=1}^{L}a_k^2.$$

(c) For the upper bound (1.16), we can use (2.9) and (b) above. For the lower bound, as in the proof of (2.9),

$$\frac{2}{\pi T}\int_0^T\left(|F(t)|^2+\left|F\left(\frac{T^2}{t}\right)\right|^2\right)dt$$
$$\ge\int_{-1}^{1}|F(Ts)|^2\frac{ds}{\pi(1+s^2)}+\int_{-1}^{1}\left|F\left(\frac{T}{s}\right)\right|^2\frac{ds}{\pi(1+s^2)}$$
$$=\int_{-\infty}^{\infty}|F(Ts)|^2\frac{ds}{\pi(1+s^2)},$$

and we can now apply (b). □

Proof of Theorem 1.4. (a) Applying (2.5) gives

$$\int_{-\infty}^{\infty}\left|\sum_{j=1}^{m}a_j(-1)^{j-1}\lambda_j^{-irx}\right|^2\frac{dx}{\pi(1+x^2)}$$
$$=\sum_{j=1}^{m}|a_j|^2+\sum_{j\ne k}a_ja_k(-1)^{j+k}e^{-r|\log\lambda_j/\lambda_k|}.$$

By (1.10), an upper bound for this last right-hand side is $\sum_{j=1}^{m}|a_j|^2$. It follows that for all $r>0$,

$$0\ge\sum_{j\ne k}a_ja_k(-1)^{j+k}e^{-r|\log\lambda_j/\lambda_k|}\ge-\sum_{j=1}^{m}|a_j|^2. \tag{2.10}$$

(b) First let $\alpha,\beta>0$. We replace r by $r^{1/\beta}$ in (2.10), multiply by $e^{-r^{1/\beta}/\alpha}$ and integrate:

$$0\ge\sum_{j\ne k}a_ja_k(-1)^{j+k}\int_0^{\infty}e^{-r^{1/\beta}(|\log\lambda_j/\lambda_k|+1/\alpha)}dr\ge-\left(\int_0^{\infty}e^{-r^{1/\beta}/\alpha}dr\right)\sum_{j=1}^{m}|a_j|^2.$$

As

$$\int_0^{\infty}e^{-r^{1/\beta}A}dr=\frac{\beta\Gamma(\beta)}{A^{\beta}}$$

for $A>0$, we obtain (1.25) for $\alpha,\beta>0$. Finally, we can use continuity to let α and/or $\beta\to0+$. □

Proof of Theorem 1.5. (a) In this case, Theorem 1.1 followed by (2.5) gives

$$\sum_{k=1}^{m}\left(\lambda_k^{2r}-\lambda_{k-1}^{2r}\right)\left|\sum_{n=k}^{m}\frac{a_n}{\lambda_n^r}\right|^2 = \int_{-\infty}^{\infty}\left|\sum_{j=1}^{m}a_j\lambda_j^{-irx}\right|^2\frac{dx}{\pi\left(1+x^2\right)}$$
$$= \sum_{j=1}^{m}\left|a_j\right|^2+\sum_{j\neq k}a_j\overline{a_k}e^{-r\left|\log\lambda_j/\lambda_k\right|}.$$

Applying Cauchy-Schwarz on the second sum in the first line gives

$$\sum_{j\neq k}a_j\overline{a_k}e^{-r\left|\log\lambda_j/\lambda_k\right|} \leq \sum_{k=1}^{m}\left(\lambda_k^{2r}-\lambda_{k-1}^{2r}\right)\left(\sum_{n=k}^{m}\rho_n\left|a_n\right|^2\right)\left(\sum_{n=k}^{m}\rho_n^{-1}\lambda_n^{-2r}\right)$$
$$= \sum_{n=1}^{m}\rho_n\left|a_n\right|^2S_n\left(r\right), \tag{2.11}$$

where

$$S_n\left(r\right)=\sum_{k=1}^{n}\left(\lambda_k^{2r}-\lambda_{k-1}^{2r}\right)\left(\sum_{\ell=k}^{m}\rho_\ell^{-1}\lambda_\ell^{-2r}\right).$$

As above, we replace r by $r^{1/\beta}$ in (2.11), multiply by $e^{-r^{1/\beta}/\alpha}$ and integrate. We obtain

$$\beta\Gamma\left(\beta\right)\sum_{j\neq k}\frac{a_j\overline{a_k}}{\left(\left|\log\lambda_j-\log\lambda_k\right|+\alpha^{-1}\right)^{\beta}} \leq \sum_{n=1}^{m}\rho_n\left|a_n\right|^2\int_0^{\infty}S_n\left(r^{1/\beta}\right)e^{-r^{1/\beta}/\alpha}dr. \tag{2.12}$$

Here

$$\int_0^{\infty}S_n\left(r^{1/\beta}\right)e^{-r^{1/\beta}/\alpha}dr$$
$$=\beta\int_0^{\infty}S_n\left(t\right)e^{-t/\alpha}t^{\beta-1}dt$$
$$=\beta\sum_{k=1}^{n}\sum_{\ell=k}^{m}\rho_\ell^{-1}\int_0^{\infty}\left(\lambda_k^{2t}-\lambda_{k-1}^{2t}\right)\lambda_\ell^{-2t}e^{-t/\alpha}t^{\beta-1}dt$$
$$=\beta\sum_{\ell=1}^{m}\rho_\ell^{-1}\sum_{k=1}^{\min\{\ell,n\}}\left\{\int_0^{\infty}\lambda_k^{2t}\lambda_\ell^{-2t}e^{-t/\alpha}t^{\beta-1}dt-\int_0^{\infty}\lambda_{k-1}^{2t}\lambda_\ell^{-2t}e^{-t/\alpha}t^{\beta-1}dt\right\}$$
$$=\beta\sum_{\ell=1}^{m}\rho_\ell^{-1}\int_0^{\infty}\lambda_{\min\{\ell,n\}}^{2t}\lambda_\ell^{-2t}e^{-t/\alpha}t^{\beta-1}dt$$
$$=\beta\Gamma\left(\beta\right)\sum_{\ell=1}^{m}\rho_\ell^{-1}\left(2\left|\log\lambda_{\min\{\ell,n\}}-\log\lambda_\ell\right|+\alpha^{-1}\right)^{-\beta}$$
$$=\beta\Gamma\left(\beta\right)\sum_{\ell=1}^{m}\rho_\ell^{-1}\left(2\log^{+}\lambda_\ell/\lambda_n+\alpha^{-1}\right)^{-\beta}.$$

Substituting in (2.12), multiplying by $\alpha^{-\beta}$, and cancelling $\beta\Gamma(\beta)$, gives the result.
(b) We choose $\beta > 1$; $\rho_k = 1$ for $k \geq 1$;$\lambda_k = 2^{k-1}$, $k \geq 1$, so

$$\begin{aligned}
&\sum_{\ell=1}^{m} \rho_\ell^{-1} \left(2\alpha \log^+ \lambda_\ell/\lambda_n + 1\right)^{-\beta} \\
&\quad = n + \sum_{\ell=n+1}^{m} \left(\left(2\alpha \log 2\right)(\ell - n) + 1\right)^{-\beta} \\
&\quad \leq n + \int_1^\infty \left(\left(2\alpha \log 2\right) t + 1\right)^{-\beta} dt \\
&\quad = n + \frac{1}{(\beta - 1)\, 2\alpha \log 2} \left(2\alpha \log 2 + 1\right)^{1-\beta} \\
&\quad \leq n + \frac{1}{\beta - 1} \left(2\alpha \log 2\right)^{-\beta}.
\end{aligned}$$

Substituting in (a) gives

$$\left| \sum_{j \neq k} \frac{a_j \overline{a_k}}{\left(\alpha \log 2 \left|j - k\right| + 1\right)^\beta} \right| \leq \sum_{n=1}^{m} |a_n|^2 \left(n + \frac{1}{\beta - 1} \left(2\alpha \log 2\right)^{-\beta} \right).$$

Finally, replace $\alpha \log 2$ by α. $\square$

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