Dirichlet Orthogonal Polynomials with Laguerre Weight

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Abstract

Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers. We find explicit formulae for the orthogonal Dirichlet polynomials $\{\psi_n\}$ formed from linear combinations of $\left\{\lambda_j^{-it}\right\}_{j=1}^{n}$, associated with the Laguerre weight. Thus

$$\int_0^\infty \psi_n(t)\,\overline{\psi_m(t)}e^{-t}dt = \delta_{mn}.$$

In addition, we estimate Christoffel functions and establish Markov-Bernstein inequalities.

Key words: Dirichlet polynomials, orthogonal polynomials
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1. Introduction

Throughout, let

$$\{\lambda_j\}_{j=1}^{\infty} \text{ be a sequence of distinct positive numbers.} \tag{1.1}$$

Given $m \geq 1$, a *Dirichlet polynomial of degree* $\leq n$ [7], [9] associated with this sequence of exponents has the form

$$\sum_{n=1}^{m} a_n \lambda_n^{-it} = \sum_{n=1}^{m} a_n e^{-i(\log \lambda_n)t},$$

where $\{a_n\} \subset \mathbb{C}$. We denote the set of all such polynomials by $\mathcal{L}_n$.

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The traditional orthogonal Dirichlet polynomials are just the "monomials" $\left\{\lambda_n^{-it}\right\}$ themselves. Indeed, in the theory of almost-periodic functions [1], [2], heavy use is made of orthogonality in the mean:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \lambda_j^{-it}\overline{\lambda_k^{-it}}dt=\delta_{jk}.$$

In the hope that a more standard orthogonality relation might have some advantages, the author [5], investigated Dirichlet orthogonal polynomials associated with the arctangent density. Thus $\phi_n\in\mathcal{L}_n$ has positive leading coefficient, and

$$\int_{-\infty}^{\infty}\phi_n(t)\,\overline{\phi_m(t)}\frac{dt}{\pi(1+t^2)}=\delta_{mn},\quad m,n\geq 1.$$

These Dirichlet orthogonal polynomials admit a very simple explicit expression, at least when $1=\lambda_1<\lambda_2<\cdots$:

Theorem A. *For $n=1$, $\phi_1=1$, and for $n\geq 2$,*

$$\phi_n(t)=\frac{\lambda_n^{1-it}-\lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2-\lambda_{n-1}^2}}.$$

In that paper, we also analyzed the associated Christoffel functions, established universality limits, and proved Markov–Bernstein inequalities. We note that Krein systems [4], which involve "continuous orthogonal polynomials", have been intensively studied, but do not seem to have much contact with the type of Dirichlet orthogonal polynomials considered in [5] or here.

In this paper, we study the Dirichlet orthogonal polynomials for the Laguerre weight. Thus $\psi_n\in\mathcal{L}_n$, has positive leading coefficient, and

$$(\psi_n,\psi_m)=\int_0^{\infty}\psi_n(t)\,\overline{\psi_m(t)}e^{-t}dt=\delta_{mn}.\tag{1.2}$$

We present explicit representations for ψ_n in Section 2, as well as related identities. In Section 3, we present estimates for Christoffel functions, and Markov-Bernstein inequalities. The results of Section 2 are proved in Section 4, and those of Section 3 in Section 5.

As far as we know, the results are new. There is a vast literature dealing with density of complex exponentials on a finite interval, and also on nonharmonic Fourier series (see e.g. [10]), but this does not seem to overlap the topic of this paper. We expect that a theory of orthogonal Dirichlet polynomials for various weights, will give new insight into properties of general Dirichlet polynomials.

Throughout, in addition to the inner product in (1.2), we use the induced norm

$$\|f\| = \left(\int_0^\infty |f(t)|^2 e^{-t} dt \right)^{1/2}.$$

2. Identities

For $n \geq 1$, we let

$$R_n(z) = \frac{1}{z - i \log \lambda_n} \prod_{j=1}^{n-1} \left(1 + \frac{1}{z - i \log \lambda_j} \right); \tag{2.1}$$

$$D_n = \prod_{k=1}^{n-1} \left(1 + [i \log (\lambda_k / \lambda_n)]^{-1} \right); \tag{2.2}$$

and

$$\Delta_n = D_n / |D_n|. \tag{2.3}$$

We begin with explicit representations for ψ_n:

Theorem 2.1. *Let $\{\lambda_j\}_{j=1}^\infty$ be a distinct sequence of positive numbers. Let $n \geq 1$.*
(a) Let Γ be a simple closed positively oriented contour in the half-plane $\mathrm{Re}\, z > -1$ that encloses $i \log \lambda_j$, $1 \leq j \leq n$. For $t \in \mathbb{C}$,

$$\psi_n(t) = \frac{\Delta_n}{2\pi i} \int_\Gamma e^{-tz} R_n(z)\, dz. \tag{2.4}$$

(b)

$$\psi_n(t) = \sum_{\ell=1}^{n} B_{n\ell} \lambda_\ell^{-it}, \tag{2.5}$$

where for $\ell < n$,

$$B_{n\ell} = \frac{\Delta_n}{i \log \lambda_\ell / \lambda_n} \prod_{j=1, j \neq \ell}^{n-1} \left(1 + \frac{1}{i \log \lambda_\ell / \lambda_j} \right)$$

and

$$B_{nn} = |D_n| = \left(\prod_{k=1}^{n-1} \left(1 + [\log (\lambda_n / \lambda_k)]^{-2} \right) \right)^{1/2}. \tag{2.6}$$

(c) Let $0<\alpha<1$ and Δ_n be as in (2.3). Then for $t\in(0,\infty)$,

$$\psi_n(t) = -\frac{\Delta_n}{2\pi}e^{t\alpha}\int_{-\infty}^{\infty} e^{-its}R_n(-\alpha+is)\,ds. \tag{2.7}$$

Remarks. (a) One consequence of the explicit formula for the leading coefficient B_{nn} is an expression for the distance from λ_n^{-it} to $\mathrm{Span}\left\{\lambda_j^{-it}\right\}_{j=1}^{n-1}$:

$$\inf_{c_1,c_2,\dots,c_{n-1}}\left\|\lambda_n^{-it}-\sum_{j=1}^{n-1}c_j\lambda_j^{-it}\right\| = B_{nn}^{-1} = \left(\prod_{k=1}^{n-1}\left(1+\left[\log\left(\lambda_n/\lambda_k\right)\right]^{-2}\right)\right)^{-1/2}.$$

Indeed, this follows directly from the minimality properties of "monic" orthogonal polynomials.
(b) Theorem 2.1(c) is an analogue of a Bromwich type integral for Müntz orthogonal polynomials given in [8].

Define the nth reproducing kernel

$$K_n(u,v) = \sum_{j=1}^{n}\psi_j(u)\overline{\psi_j(v)}.$$

Theorem 2.2. *(a)*

$$\psi_n(0) = \Delta_n. \tag{2.8}$$

(b)

$$\psi_n'(t) = (-i\log\lambda_n)\,\psi_n(t) - \psi_n(0)\,K_{n-1}(t,0). \tag{2.9}$$

(c)

$$\frac{\psi_n(t)}{\psi_n(0)} = \lambda_n^{-it}\left\{1+\int_0^t\lambda_n^{is}K_{n-1}(s,0)\,ds\right\}. \tag{2.10}$$

(d)

$$\int_0^{\infty}\left|\psi_n'(t)\right|^2 e^{-t}dt = (\log\lambda_n)^2+n-1. \tag{2.11}$$

(e)

$$\frac{\psi_n'(0)}{\psi_n(0)} = -\left(n-1+i\log\lambda_n\right). \tag{2.12}$$

(f)

$$\frac{\psi_n''(0)}{\psi_n(0)} = -(\log\lambda_n)^2 + i\sum_{j=1}^{n-1}\log(\lambda_j\lambda_n) + \frac{(n-1)(n-2)}{2}. \tag{2.13}$$

Following is the representation of monomials in terms of the orthonormal polynomials:

Theorem 2.3. *For $\ell \geq 1$,*

$$\lambda_\ell^{-it} = \sum_{j=1}^{\ell} c_{\ell j} \psi_j(t), \tag{2.14}$$

where

$$c_{\ell\ell} = B_{\ell\ell}^{-1} = \left[\prod_{k=1}^{\ell-1} \left(1 + \left[\log \frac{\lambda_\ell}{\lambda_k} \right]^{-2} \right) \right]^{-1/2} \tag{2.15}$$

and for $j < \ell$,

$$c_{\ell j} = -\Delta_j R_j (1 + i \log \lambda_\ell), \tag{2.16}$$

where Δ_j and R_j are given respectively by (2.3) *and* (2.1)*. Moreover,*

$$\sum_{j=1}^{\ell} |c_{\ell j}|^2 = 1. \tag{2.17}$$

3. Estimates and Inequalities

The nth Christoffel function is

$$\Lambda_n(x) = \inf_{P \in \mathcal{L}_n} \frac{\int_0^\infty |P(t)|^2 e^{-t} dt}{|P(x)|^2} = 1 / \sum_{j=1}^{n} |\psi_j(x)|^2 .$$

Christoffel functions play an important role in analysing orthogonal polynomials, and in approximation theory [6]. We show that $e^{-x}\Lambda_n(x)$, and some derivative cousins, are decreasing functions.

Theorem 3.1. *(a) Let $n \geq 1$ and $\ell \geq 0$. The function*

$$e^{-x} \sum_{j=1}^{n} \left| \psi_j^{(\ell)}(x) \right|^2 \tag{3.1}$$

is a decreasing function of $x \in [0, \infty)$.
(b) In particular, for $x \in [0, \infty)$,

$$e^{-x} \sum_{j=1}^{n} |\psi_j(x)|^2 \leq \sum_{j=1}^{n} |\psi_j(0)|^2 = n; \tag{3.2}$$

$$e^{-x} \sum_{j=1}^{n} |\psi_j'(x)|^2 \leq \sum_{j=1}^{n} |\psi_j'(0)|^2 = \frac{n(n-1)(2n-1)}{6} + \sum_{j=1}^{n} (\log \lambda_j)^2 . \tag{3.3}$$

The estimate can be compared to the results in [5], where the factors of n are absent, and the growth is governed by $\log \lambda_n$. The jump discontinuity at 0 in the Laguerre weight, is the reason for the powers of n. Next, we present Markov-Bernstein inequalities:

Theorem 3.2. *For $n \geq 1$ and $P \in \mathcal{L}_n$, we have*

$$\|P'\| \leq \left(\max_{1\leq j\leq n} |\log \lambda_j| + \sqrt{\frac{n(n-1)}{2}} \right) \|P\|. \tag{3.4}$$

Remark. Now (2.11) shows that

$$\|\psi_n'\| / \|\psi_n\| = \sqrt{(\log \lambda_n)^2 + n - 1},$$

so Theorem 3.2 is sharp with respect to the rate of growth in $\log \lambda_n$. To see that the right-hand side of (3.4) must include a constant multiple of n, one can use the polynomial $P_n(t) = K_n(t, 0)$. Here $\|P_n\|^2 = n$, and a straightforward calculation shows that

$$\|P_n'\| \geq \sqrt{\frac{n(n-1)(2n-1)}{6}} - \sqrt{n\left(\max_{1\leq j\leq n} |\log \lambda_j|\right)^2},$$

so at least when $n \gg \log \lambda_n$,

$$\|P_n'\| / \|P_n\| \geq \frac{n}{\sqrt{3}}(1 + o(1)).$$

4. Proofs of Theorem 2.1-2.3

Proof of (.a), (b) Assume that ψ_n is defined by the integral in (2.4). Then

$$\begin{aligned} I_j &= \int_0^\infty \psi_n(t) \lambda_j^{it} e^{-t} dt \\ &= \frac{\Delta_n}{2\pi i} \int_0^\infty \left[\int_\Gamma e^{-tz} R_n(z) dz \right] \lambda_j^{it} e^{-t} dt \\ &= \frac{\Delta_n}{2\pi i} \int_\Gamma R_n(z) \left[\int_0^\infty e^{-t[z - i\log\lambda_j + 1]} dt \right] dz \\ &= \frac{\Delta_n}{2\pi i} \int_\Gamma R_n(z) \frac{1}{z - i\log\lambda_j + 1} dz. \end{aligned} \tag{4.1}$$

Here the interchange and integration are justified as Γ lies in $\mathrm{Re}\, z > -1$. The integrand in the right-hand side of (4.1) is analytic outside Γ except for a simple pole at $-1 + i \log \lambda_j$, and is $O\left(z^{-2}\right)$ as $z \to \infty$, recall (2.1). So we can deform Γ into a negatively oriented circle C center $-1 + i \log \lambda_j$ and radius $\frac{1}{2}$. Thus

$$I_j = \frac{\Delta_n}{2\pi i} \int_C R_n(z) \frac{1}{z - i \log \lambda_j + 1} dz = -\Delta_n R_n(-1 + i \log \lambda_j).$$

Then from (2.1),

$$I_j = 0, \quad 1 \leq j \leq n-1,$$

so ψ_n is an orthogonal polynomial. For $j = n$, instead

$$\begin{aligned} I_n &= -\Delta_n R_n(-1 + i \log \lambda_n) \\ &= \Delta_n \prod_{j=1}^{n-1} \frac{1}{-(i \log \lambda_n/\lambda_j)^{-1} + 1} \\ &= \frac{D_n}{|D_n|} \prod_{j=1}^{n-1} \left(1 + (i \log \lambda_j/\lambda_n)^{-1}\right)^{-1} = \frac{1}{|D_n|}. \end{aligned} \tag{4.2}$$

Next, as R_n has simple poles inside Γ at $i \log \lambda_j$, $1 \leq j \leq n-1$, the residue theorem and the partial fraction decomposition or $R_n(z)$ show that

$$\psi_n(t) = \sum_{\ell=1}^{n} B_{n\ell} \lambda_\ell^{-it},$$

so $\psi_n \in \mathcal{L}_n$, where

$$B_{nn} = \Delta_n \prod_{j=1}^{n-1} \left(1 + \frac{1}{i \log \lambda_n/\lambda_j}\right) = \frac{D_n}{|D_n|} \overline{D_n} = |D_n|, \tag{4.3}$$

recall (2.2). Moreover, for $\ell < n$,

$$B_{n\ell} = \frac{\Delta_n}{i \log \lambda_\ell/\lambda_n} \prod_{j=1, j\neq\ell}^{n-1} \left(1 + \frac{1}{i \log \lambda_\ell/\lambda_j}\right).$$

It remains to show that ψ_n is orthonormal. Orthogonality gives

$$\int_0^\infty |\psi_n(t)|^2 e^{-t} dt = \int_0^\infty \psi_n(t) B_{nn} \lambda_n^{it} e^{-t} dt = |D_n| I_n = 1,$$

by (4.2).
(c) Let L be a large positive number, and $0<\alpha<1<\beta$. Let $t\in(0,\infty)$. We can take Γ in (a) to be a rectangular contour, consisting of vertical line segments $\Gamma_1=\{\beta+is : s\in[-L,L]\}$ and $-\Gamma_3=\{-\alpha+is : s\in[-L,L]\}$, and horizontal line segments $-\Gamma_2=\{s+iL : s\in[-\alpha,\beta]\}$, $\Gamma_4=\{s-iL : s\in[-\alpha,\beta]\}$. Here as $R_n(z)=O\left(z^{-1}\right), z\to\infty$, we see that for large enough L,

$$\left|\int_{\Gamma_4} e^{-tz}R_n(z)\,dz\right|\leq(\alpha+\beta)\,e^{|t|\max\{\alpha,\beta\}}\frac{C}{L},$$

where C is independent of L. A similar estimate holds for $\int_{\Gamma_2}$. We now let $L\to\infty$, to deduce that

$$\psi_n(t)=\frac{\Delta_n}{2\pi}\int_{-\infty}^{\infty}\left[e^{-t(\beta+is)}R_n(\beta+is)-e^{-t(-\alpha+is)}R_n(-\alpha+is)\right]ds. \tag{4.4}$$

We next let $\beta\to\infty$, in the first integral and show that it has limit 0. Since the integral is not absolutely convergent, this requires some care. We integrate by parts, and use $R_n(z)=O\left(\frac{1}{z}\right)$ at ∞ to obtain

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{-t(\beta+is)}R_n(\beta+is)\,ds\right| &= \left|\frac{e^{-t\beta}}{t}\int_{-\infty}^{\infty} e^{-its}R_n'(\beta+is)\,ds\right| \\ &\leq \frac{e^{-t\beta}}{t}\int_{-\infty}^{\infty}\left|R_n'(\beta+is)\right|ds \\ &\leq C\frac{e^{-t\beta}}{t}\int_{-\infty}^{\infty}\frac{ds}{\left(\beta^2+s^2\right)},\end{aligned}$$

where C is independent of β. We can now let $\beta\to\infty$ and obtain the result. □

We note that it is also possible to derive the explicit formula for the coefficients from the determinantal representation for ψ_n involving moments. Indeed,

$$\left(\lambda_j^{-it},\lambda_k^{-it}\right)=\frac{1}{1+i\left(\log\lambda_j-\log\lambda_k\right)},$$

and one can then use Cauchy's determinant formula to evaluate the coefficients $B_{n\ell}$. This was the author's first method, but the contour integral approach is shorter. □

Proof of (.a) From (2.4),

$$\psi_n(0)=\Delta_n\frac{1}{2\pi i}\int_\Gamma R_n(z)\,dz.$$

Here R_n is analytic outside Γ and $R_n(z) = \frac{1}{z}(1 + o(1))$ as $z \to \infty$. Hence, deforming the contour into ∞ gives $\psi_n(0) = \Delta_n$.
(b) Since

$$\frac{d}{dt}\left(\lambda_n^{-it}\right) = (-i \log \lambda_n)\, \lambda_n^{-it},$$

we can write, for some $\{c_j\}$,

$$\psi_n'(t) = (-i \log \lambda_n)\, \psi_n(t) + \sum_{j=1}^{n-1} c_j \psi_j(t).$$

An integration by parts gives, for $j \leq n-1$,

$$\begin{aligned} c_j &= \int_0^\infty \psi_n'(t)\, \overline{\psi_j(t)} e^{-t} dt \\ &= -\psi_n(0)\, \overline{\psi_j(0)} - \int_0^\infty \psi_n(t)\, \overline{\psi_j'(t)} e^{-t} dt + \int_0^\infty \psi_n(t)\, \overline{\psi_j(t)} e^{-t} dt \\ &= -\psi_n(0)\, \overline{\psi_j(0)}. \end{aligned}$$

So

$$\begin{aligned} \psi_n'(t) &= (-i \log \lambda_n)\, \psi_n(t) - \psi_n(0) \sum_{j=1}^{n-1} \overline{\psi_j(0)} \psi_j(t) \\ &= (-i \log \lambda_n)\, \psi_n(t) - \psi_n(0)\, K_{n-1}(t, 0). \end{aligned}$$

(c) This follows by solving the first order linear differential equation in the last line.
(d), (e) These follow directly from (a), (b), from orthonormality, and the fact that $|\Delta_n| = 1$.
(f) From (b),

$$\begin{aligned} \psi_n''(t) &= (-i \log \lambda_n)\, \psi_n'(t) - \psi_n(0) \sum_{j=1}^{n-1} \overline{\psi_j(0)} \psi_j'(t) \\ &= (-i \log \lambda_n) \left[(-i \log \lambda_n)\, \psi_n(t) - \psi_n(0)\, K_{n-1}(t, 0) \right] \\ &\quad - \psi_n(0) \sum_{j=1}^{n-1} \overline{\psi_j(0)} \left\{ (-i \log \lambda_j)\, \psi_j(t) - \psi_j(0)\, K_{j-1}(t, 0) \right\}. \end{aligned}$$

Then

$$\begin{aligned}\psi_n''(0)/\psi_n(0) &= -(\log\lambda_n)^2 + (i\log\lambda_n)(n-1) \\ &\quad + \sum_{j=1}^{n-1}\{i\log\lambda_j + j - 1\} \\ &= -(\log\lambda_n)^2 + i\sum_{j=1}^{n-1}\log(\lambda_j\lambda_n) + \frac{(n-1)(n-2)}{2}.\end{aligned}$$

□

Remark. One can also use integration by parts in

$$1 = \int_0^\infty |\psi_n(t)|^2 e^{-t}dt$$

to derive $|\psi_n(0)| = 1$.

Proof of C.omparing leading coefficients in (2.5) and (2.14), we see that

$$c_{\ell\ell} = B_{\ell\ell}^{-1} = \left(\prod_{k=1}^{\ell-1}\left(1 + [\log(\lambda_\ell/\lambda_k)]^{-2}\right)\right)^{-1/2}.$$

Also, for $j < \ell$, our contour integral representation gives

$$\begin{aligned}c_{\ell j} &= \int_0^\infty \psi_j(t)\lambda_\ell^{it}e^{-t}dt \\ &= \frac{\Delta_j}{2\pi i}\int_\Gamma\left[\int_0^\infty e^{-tz-t}\lambda_\ell^{it}dt\right]R_j(z)\,dz \\ &= \frac{\Delta_j}{2\pi i}\int_\Gamma\frac{1}{1+z-i\log\lambda_\ell}R_j(z)\,dz.\end{aligned}$$

As in the proof of orthogonality via the contour integral representation, we can deform the contour into a negatively oriented circle Γ_0 of small positive radius, and center $-1 + i\log\lambda_\ell$. Then the residue theorem gives

$$c_{\ell j} = -\Delta_j R_j(-1 + i\log\lambda_\ell).$$

Finally, orthonormality gives

$$1 = \int_0^\infty \left|\lambda_\ell^{-it}\right|^2 e^{-t}dt = \sum_{j=1}^{\ell}|c_{\ell j}|^2.$$

□

5. Proof of Theorems 3.1 and 3.2

Proof of (.a) We use the standard fact that

$$\Lambda_n^{\ell}(x) := \inf_{P \in \mathcal{L}_n} \frac{\int_0^\infty |P(t)|^2 e^{-t} dt}{\left|P^{(\ell)}(x)\right|^2} = 1 / \sum_{j=1}^{n} \left|\psi_j^{(\ell)}(x)\right|^2. \tag{5.1}$$

This is an easy consequence of the Cauchy-Schwarz inequality, and the Parseval identity in $\mathcal{L}_n$; the inf is attained for

$$P(t) = \sum_{j=1}^{n} \overline{\psi_j^{(\ell)}(x)} \psi_j(t).$$

Now $\mathcal{L}_n$ is closed under translations, that is, if $P(\cdot) \in \mathcal{L}_n$, $P(\cdot - x) \in \mathcal{L}_n$. Let $x > y \geq 0$. Then we see that

$$\begin{aligned}
\Lambda_n^{\ell}(x) &\geq \inf_{P \in \mathcal{L}_n} \frac{\int_{x-y}^\infty |P(t)|^2 e^{-t} dt}{\left|P^{(\ell)}(x)\right|^2} \\
&= e^{-(x-y)} \inf_{P \in \mathcal{L}_n} \frac{\int_0^\infty |P(u+x-y)|^2 e^{-u} du}{\left|P^{(\ell)}(y+x-y)\right|^2} \\
&= e^{-(x-y)} \inf_{P \in \mathcal{L}_n} \frac{\int_0^\infty |P(u)|^2 e^{-u} du}{\left|P^{(\ell)}(y)\right|^2} \\
&= e^{-(x-y)} \Lambda_n^{\ell}(y).
\end{aligned}$$

Thus,

$$e^x \Lambda_n^{\ell}(x) \geq e^y \Lambda_n^{\ell}(y),$$

or equivalently,

$$e^{-x} \sum_{j=1}^{n} \left|\psi_j^{(\ell)}(x)\right|^2 \leq e^{-y} \sum_{j=1}^{n} \left|\psi_j^{(\ell)}(y)\right|^2.$$

(b) For $\ell = 0$, we know from (2.8) that

$$\sum_{j=1}^{n} \left|\psi_j(0)\right|^2 = n;$$

For $\ell = 1$, we know from (2.12) that

$$\sum_{j=1}^{n} \left|\psi_j'(0)\right|^2 = \sum_{j=1}^{n} \left((j-1)^2 + (\log \lambda_j)^2\right)$$
$$= \frac{n(n-1)(2n-1)}{6} + \sum_{j=1}^{n} (\log \lambda_j)^2 .$$

□

Proof of W.rite

$$P(t) = \sum_{j=1}^{n} c_j \psi_j(t) .$$

Using (2.9), we see that

$$P'(t) = \sum_{j=2}^{n} c_j \left\{-i(\log \lambda_j)\psi_j(t) - \psi_j(0) K_{j-1}(t,0)\right\}$$
$$= R(t) + S(t), \tag{5.2}$$

where

$$R(t) = -i \sum_{j=2}^{n} c_j \log \lambda_j \psi_j(t) \tag{5.3}$$

and

$$S(t) = -\sum_{j=2}^{n} c_j \psi_j(0) K_{j-1}(t,0) .$$

Cauchy-Schwarz' inequality gives

$$\int_0^\infty |S(t)|^2 e^{-t} dt \le \int_0^\infty \left(\sum_{j=2}^{n} |c_j|^2\right) \left(\sum_{j=2}^{n} |K_{j-1}(t,0)|^2\right) e^{-t} dt$$
$$= \left(\sum_{j=2}^{n} |c_j|^2\right) \sum_{j=2}^{n} |K_{j-1}(0,0)|$$
$$\le \left(\int_0^\infty |P(t)|^2 e^{-t} dt\right) \sum_{j=2}^{n} (j-1)$$
$$= \|P\|^2 \frac{n(n-1)}{2}.$$

Also,

$$\int_0^\infty |R(t)|^2 e^{-t} dt = \sum_{j=2}^{n} |c_j|^2 (\log \lambda_j)^2$$
$$\leq \|P\|^2 \left(\max_{1 \leq j \leq n} |\log \lambda_j| \right)^2 .$$

Finally, the triangle inequality gives

$$\|P'\| \leq \|R\| + \|S\|$$
$$\leq \|P\| \left(\max_{1 \leq j \leq n} |\log \lambda_j| + \sqrt{\frac{n(n-1)}{2}} \right) .$$

□

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