DISTRIBUTION OF EIGENVALUES OF TOEPLITZ MATRICES WITH SMOOTH ENTRIES

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Abstract. We investigate distribution of eigenvalues of growing size Toeplitz matrices $[a_{n+k-j}]_{1\leq j,k\leq n}$ as $n\to\infty$, when the entries $\{a_j\}$ are "smooth" in the sense, for example, that for some $\alpha>0$,

$$\frac{a_{j-1}a_{j+1}}{a_j^2}=1-\frac{1}{\alpha j}\left(1+o\left(1\right)\right),\ j\to\infty.$$

Typically they are Maclaurin series coefficients of an entire function. We establish that when suitably scaled, the eigenvalue counting measures have limiting support on $[0,1]$, and under mild additional smoothness conditions, the universal scaled and weighted limit distribution is $\left|\pi\log t\right|^{-1/2}dt$ on $[0,1]$.

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1. Introduction and Results

The distribution of eigenvalues of Toeplitz matrices is a much studied topic. The archetypal result is Szegő's theorem on the eigenvalues of

$$\left[c_{k-j}\right]_{1\leq j,k\leq n},$$

where

$$c_j=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ijx}f\left(x\right)dx,\ \ j=0,\pm1,\pm2,...$$

are the trigonometric moments of some real valued function f [10, Chapter 5]. There are numerous extensions and refinements, notably the strong Szegő limit theorem, which continues to be investigated in the context of Fisher-Hartwig symbols, while Toeplitz operators are a vast subject on their own. See, for example, [2], [4], [5], [6], [12], [21], [31]. Eigenvalues of random Hankel and Toeplitz matrices have been studied in, for example, [7], [11], [15], [25], [26].

There is a classical connection to complex function theory: Polya [22] proved that if $f\left(z\right)=\sum_{j=0}^{\infty}a_j/z^j$ can be analytically continued to

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a function analytic in the complex plane outside a set of logarithmic capacity $\tau \geq 0$, then

$$\limsup_{n\to\infty} \left| \det \left[a_{n-j+k}\right]_{1\leq j,k\leq n} \right|^{1/n^2} \leq \tau.$$

Wilson [29] and Edrei [9] obtained asymptotic upper bounds for entire and meromorphic functions of finite order, and functions with finitely many essential singularities. For example if f is entire of order at most α with Maclaurin series coefficients $\{a_j\}$, then

$$\limsup_{n\to\infty} \left| \det \left[a_{n-j+k}\right]_{1\leq j,k\leq n} \right|^{1/\left(n^2 \log n\right)} \leq e^{-1/\alpha}.$$

Pommerenke [23] investigated refinements of Polya's result. Note that these authors considered Hankel matrices, but their results immediately apply to the corresponding Toeplitz matrices.

Toeplitz matrices also arise in studying Padé approximation and continued fraction expansions. Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series, and $m, n \geq 0$. The (m,n) Pade approximant to f is a rational function $[m/n] = P/Q$ where P has degree at most m, Q has degree at most n and is not identically 0, and

$$(fQ - P)(z) = O\left(z^{m+n+1}\right),$$

in the sense that the power series on the left-hand side has 0 as the coefficient of z^j, provided $0 \leq j \leq m+n$. Q, suitably normalized, admits the representation [1]

$$Q(z) = \det \begin{bmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ a_{m-n+2} & a_{m-n+3} & \cdots & a_{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \\ z^n & z^{n-1} & \cdots & 1 \end{bmatrix},$$

where we set $a_j = 0$ if $j < 0$, and we assume that the determinant does not vanish identically. In particular, the constant coefficient is the determinant of

$$A_{mn} = \left[a_{m-j+k}\right]_{1\leq j,k\leq n}. \tag{1.1}$$

These determinants also play a role in understanding convergence of continued fraction expansions, and sequences of Padé approximants.

For classical special functions, $\det A_{mn}$ can be evaluated explicitly, but of course not in general.

Numerical computation of Padé approximants suggested that they behave well when the coefficients are "smooth". When $a_j \neq 0$ for large enough j, the author attempted to quantify this using the double ratio

$$q_j = \frac{a_{j-1}a_{j+1}}{a_j^2}. \tag{1.2}$$

In particular if n is fixed, and

$$\lim_{j\to\infty} q_j = q, \tag{1.3}$$

it was shown [16, p. 308] that

$$\lim_{m\to\infty} \det\left(A_{mn}\right)/a_m^n = \prod_{j=1}^{n-1}\left(1-q^j\right)^{n-j}.$$

This is useful only if q is not a root of unity, so additional assumptions are required for that case: if there is a complete asymptotic expansion, so that for each $\ell \geq 1$,

$$q_j = q - \frac{c_1}{j} + \frac{c_2}{j^2} + ... + \frac{c_\ell}{j^\ell} + O\left(j^{-\ell-1}\right),$$

where $c_1 \neq 0$, then [16, p. 309] for each fixed $n \geq 1$,

$$\lim_{m\to\infty} \det\left(A_{mn}\right)/\left\{a_m^n \prod_{j=1}^{n-1}\left(1-q_m^j\right)^{n-j}\right\} = 1.$$

The case where n grows with m is more delicate. Rusak and Starovoitov [24] showed that one can handle the situation where $n = o\left(m^{1/3}\right)$, and that this last relation persists.

Undoubtedly the most interesting and challenging case is the "diagonal" one where $m = n \to \infty$. One situation where analysis is possible, is where (1.3) holds with $|q| < 1$, which holds for example, for

$$f(z) = \sum_{j=0}^{\infty} q^{j^2/2}\left(j!\right)^\beta z^j,$$

for any $\beta \in \mathbb{C}$. The author proved [17, p. 324] that in this case,

$$\lim_{n\to\infty}\left|\det\left(A_{nn}\right)/\left\{a_n^n \prod_{j=1}^{n-1}\left(1-q_n^j\right)^{n-j}\right\}\right|^{1/n} = 1.$$

This is sufficent to analyze convergence of diagonal Pade sequences and obtain asymptotics for errors of best rational approximation [14]. Unfortunately (1.3) holds with $|q| < 1$ only when f is a limited class of entire functions of order 0. For further orientation on convergence of Padé approximants, see for example [1], [8], [18], [20], [28].

Can one say anything about the Toeplitz matrices associated with entire functions of finite positive order beyond the asymptotics of Edrei, Polya, and Wilson? In the spirit of Szegő's early theorems on eigenvalue distribution, the focus of this paper is to investigate the distribution of eigenvalues of the matrices A_{mn}, when we have a relation such as (1.3). To this author's knowledge, these are the first results for Toeplitz matrices of these type and are new even for the exponential function $f(z) = e^z$.

Observe that if $\alpha > 0$, and

$$f(z) = \sum_{j=0}^{\infty} z^j / (j!)^{1/\alpha}, \tag{1.4}$$

an entire function of order α, then q_j of (1.2) satisfies

$$q_j = \exp\left(-\frac{1}{\alpha j} + O\left(\frac{1}{j^2}\right)\right).$$

This is also true for the Mittag-Leffler function

$$f(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j/\alpha + \beta)}, \tag{1.5}$$

any $\beta \in \mathbb{C} \backslash (-\infty, 0]$. For the hypergeometric function with parameters $\{c_i\}, \{d_i\}$ in $\mathbb{C} \backslash (-\infty, 0]$,

$$f(z) = \sum_{j=0}^{\infty} \frac{(c_1)_j (c_2)_j \dots (c_k)_j}{(d_1)_j (d_2)_j \dots (d_\ell)_j} z^j, \tag{1.6}$$

where $(c)_j = c(c+1)\dots(c+j-1)$ is the usual Pochhammer symbol, and $\ell \geq k+1$,

$$q_j = \exp\left(-\frac{(\ell - k)}{j} + O\left(\frac{1}{j^2}\right)\right).$$

It is interesting here that even though $\{c_j\}, \{d_j\}$ may be complex numbers, this has little effect on q_j.

In order to handle more general asymptotics, we need to replace multiples of j by more general sequences $\{\rho_j\}_{j \geq 1}$:

Definition 1.1
Let $\{\rho_j\}_{j \geq 1}$ be an increasing sequence of positive numbers, with limit

∞, *with*

$$\lim_{j\to\infty} \rho_j/j^2 = 0; \tag{1.7}$$

$$\limsup_{j\to\infty} \rho_{2j}/\rho_j < \infty; \tag{1.8}$$

and such that for each $D > 0$,

$$\lim_{k\to\infty} \left(\max_{|j|\le\sqrt{D\rho_k}} \left| 1 - \frac{\rho_{k+j}}{\rho_k} \right| \right) = 0. \tag{1.9}$$

Then we call $\left\{\rho_j\right\}_{j\ge1}$ *an* ***asymptotic comparison sequence****.*

It is clear that $\rho_j = \alpha j$, for $\alpha > 0$, satisfies the above hypotheses. Given a square matrix B, $\Lambda(B)$ denotes the collection of its eigenvalues, with repetition according to its multiplicity. In particular, if A_{mn} denotes the matrix in (1.1), $\Lambda(A_{mn}/a_m)$ denotes the set of eigenvalues of A_{mn}/a_m, with repetition according to multiplicity. Of course these are the eigenvalues of A_{mn} divided by a_m. Define for a given $\rho_m > 0$, the scaled counting measure

$$\mu_{mn} = \frac{1}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} \delta_{\lambda/\sqrt{2\pi\rho_m}}, \tag{1.10}$$

and for $\ell = 1, 2$, the weighted measures

$$\mu_{mn}^{[1]} = \frac{1}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} (\operatorname{Re}\lambda)\, \delta_{\lambda/\sqrt{2\pi\rho_m}}; \tag{1.11}$$

$$\mu_{mn}^{[2]} = \frac{1}{n\sqrt{\pi\rho_m}} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} \lambda^2 \delta_{\lambda/\sqrt{2\pi\rho_m}}. \tag{1.12}$$

Observe that while μ_{mn} is a probability measure, $\mu_{mn}^{[1]}$ is a possibly signed measure, and $\mu_{mn}^{[2]}$ may be complex.

Theorem 1.2

Assume that $\{a_j\}_{j\ge0}$ *is a sequence of non-zero complex numbers, such that for some asymptotic comparison sequence* $\left\{\rho_j\right\}_{j\ge1}$,

$$q_j = \frac{a_{j-1}a_{j+1}}{a_j^2} = \exp\left(-\frac{1}{\rho_j} \left(1 + \eta_j\right) \right), \tag{1.13}$$

where $\left\{\eta_j\right\}$ *are complex numbers satisfying*

$$\lim_{j\to\infty} \eta_j = 0. \tag{1.14}$$

Fix $R > 1$, and for $n \geq 1$, let $m = m(n)$ be an integer such that

$$\frac{1}{R} < \frac{m}{n} < R. \tag{1.15}$$

For $n \geq 1$, let $A_{mn} = [a_{m-j+k}]_{1 \leq j,k \leq n}$, with $a_j = 0$ for $j < 0$.
(I) As $n \to \infty$,

$$\max_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda| = \sqrt{2\pi\rho_m}\,(1 + o(1)). \tag{1.16}$$

(II) The set of limit points of the sets $\{\Lambda(A_{mn}/a_m)/\sqrt{2\pi\rho_m}\}_{n\geq 1}$ is $[0,1]$.
(III) As $n \to \infty$,

$$d\mu_{mn} \xrightarrow{*} d\delta_0$$

in the sense that for every real valued function f defined and continuous in some open subset of the plane containing $[0,1]$,

$$\lim_{n\to\infty} \int f \; d\mu_{mn} = f(0). \tag{1.17}$$

(IV) As $n \to \infty$,

$$d\mu_{mn}^{[2]} \xrightarrow{*} t\sqrt{\frac{2}{\pi\,|\log t|}}\;dt$$

in the sense that for every real valued function f defined and continuous in some open subset of the plane containing $[0,1]$,

$$\lim_{n\to\infty} \int f \; d\mu_{mn}^{[2]} = \int_0^1 f(t)\,t\sqrt{\frac{2}{\pi\,|\log t|}}dt. \tag{1.18}$$

The above result shows that while the eigenvalue of A_{mn} of maximal modulus grows like $|a_m|\sqrt{2\pi\rho_m}\,(1 + o(1))$, nevertheless, all but $o(n)$ eigenvalues have much smaller modulus, namely $o(|a_m|\sqrt{2\pi\rho_m})$. Under additional conditions, we can analyze the measures $\mu_{mn}^{[1]}$, but we need more assumptions on the $\{\rho_j\}$:

Definition 1.3
Let $\{\rho_j\}_{j\geq 1}$ be an asymptotic comparison sequence in the sense of Definition1.1. Assume in addition, that

$$\lim_{k\to\infty} \left(\max_{1\leq|j|\leq\sqrt{D\rho_k \log\rho_k}} \left|1 - \frac{\rho_{k+j}}{\rho_k}\right| \frac{\rho_k^{3/4}}{j} \right) = 0, \tag{1.19}$$

and

$$\lim_{k\to\infty}\left(\max_{1\le|j|\le\sqrt{D\rho_k\log\rho_k}}\left|\frac{1}{\rho_{k+j}}+\frac{1}{\rho_{k-j}}-\frac{2}{\rho_k}\right|\frac{\rho_k^2}{|j|}\right)=0. \tag{1.20}$$

Then we call $\{\rho_j\}_{j\ge1}$ *a* ***smooth symptotic comparison sequence****.*

Theorem 1.4
Assume that for some smooth asymptotic comparison sequence $\{\rho_j\}_{j\ge1}$*, (1.13) holds, with*

$$\eta_j = o\left(\rho_j^{-1/2}\right). \tag{1.21}$$

Then
(I)

$$\liminf_{n\to\infty}\left(\inf_{\lambda\in\Lambda(A_{mn}/a_m)}\operatorname{Re}\lambda\right)\ge 0. \tag{1.22}$$

(II)

$$\lim_{n\to\infty}\int d\left|\mu_{mn}^{[1]}\right| = \lim_{n\to\infty}\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}|\operatorname{Re}\lambda| = 1. \tag{1.23}$$

(III) As $n\to\infty$*,*

$$d\mu_{mn}^{[1]} \xrightarrow{*} |\pi\log t|^{-1/2}\,dt \tag{1.24}$$

in the sense that for each function f *defined and continuous in an open subset of the plane containing* $[0,1]$*,*

$$\lim_{n\to\infty}\int f\; d\mu_{mn}^{[1]} = \int_0^1 f(t)\,|\pi\log t|^{-1/2}\,dt. \tag{1.25}$$

Remarks and examples
(a) The hypotheses of Theorems 1.2, 1.4 are fulfilled for the examples in (1.4) to (1.6).
(b) Another class of functions to which Theorems 1.2 and 1.4 may be applied, is

$$f(z)=\sum_{j=0}^{\infty}e^{-\phi(j)}z^j, \tag{1.26}$$

where ϕ is a function on $[0,\infty)$ such that $\phi^{(4)}$ is continuous on $[A,\infty)$ for some $A>0$, and ϕ is strictly convex, so that $\phi''>0$. Then for large enough j,

$$q_j = \exp\left(-\phi''(j) + O\left(\left\|\phi^{(4)}\right\|_{L_\infty[j-1,j+1]}\right)\right) \tag{1.27}$$

so we can choose

$$\rho_j = \frac{1}{\phi''(j)} \tag{1.28}$$

provided it satisfies the technical conditions above. As a particular example, if $\alpha, \beta > 0$ and for $x \geq 2$,

$$\phi(x) = \alpha x (\log x)^\beta, \tag{1.29}$$

then f of (1.26) is of infinite order if $\beta < 1$, of order $\frac{1}{\alpha}$ if $\beta = 1$, and of zero order if $\beta > 1$. One can check that ρ_j of (1.28) yields a smooth asymptotic comparison sequence.
(c) Another example is

$$\phi(x) = x (\log\log x)^\gamma \tag{1.30}$$

where $\gamma > 0$ and x is large enough. Here f is of infinite order. One can again check that ρ_j of (1.28) yields a smooth asymptotic comparison sequence.
(d) Series with finite radius of convergence also fit into this framework. Let

$$f(z) = \sum_{j=0}^{\infty} e^{\psi(j)} z^j, \tag{1.31}$$

where ψ is a function on $[0, \infty)$ such that $\psi^{(4)}$ is continuous on $[A, \infty)$ for some $A > 0$, and ψ is strictly concave, so that $\psi'' < 0$. Then for large enough j,

$$q_j = \exp\left(\psi''(j) + O\left(\left\|\psi^{(4)}\right\|_{L_\infty[j-1,j+1]}\right)\right) \tag{1.32}$$

so we can choose

$$\rho_j = -\frac{1}{\psi''(j)} \tag{1.33}$$

provided it satisfies the technical conditions above. As examples, we can choose

$$\psi(x) = x^\beta,\ 0 < \beta < 1, \tag{1.34}$$

or

$$\psi(x) = (\log x)^\gamma, \gamma > 1. \tag{1.35}$$

For these examples, we can choose ρ_j by (1.33) and check that $\{\rho_j\}_{j\geq 1}$ is an asymptotic comparison sequence. It is a smooth asymptotic comparison sequence for ψ of (1.34) when $\beta > \frac{2}{3}$, but not for ψ of (1.35) for any γ.
(e) Under additional conditions, namely when $\{\rho_j\}$ satisfies

$$\limsup_{k\to\infty}\left(\max_{1\leq|j|\leq\sqrt{D\rho_k\log\rho_k}}\left|\frac{1}{\rho_{k+j}}-\frac{1}{\rho_{k-j}}\right|\frac{\rho_k^2}{|j|}\right)<\infty, \tag{1.36}$$

the assertions (II), (III) of Theorem 1.4 hold with $\mu_{mn}^{[1]}$ replaced by its complex analogue,

$$\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}\lambda\delta_{\lambda/\sqrt{2\pi\rho_m}}. \tag{1.37}$$

However, (1.36) is not satisfied by any of our examples where f is entire of infinite order, or has finite radius of convergence.

This paper is organised as follows: in Section 2, we present the similarity transformation that essentially reduces study of A_{mn}/a_m to $\left[q_m^{(j-k)^2/2}\right]_{1\leq j,k\leq n}$ as well as some technical estimates. In Section 3, we establish asymptotics for $Tr\left([A_{mn}/a_m]^k\right)$ for each $k\geq 1$. In Section 4, we estimate the location of eigenvalues using Gerschgorin's theorem, and classical inequalities of Schur and Bendixson-Hirsch. We prove Theorem 1.2 in Section 5 and Theorem 1.4 in Section 6. Throughout, $C, C_1, C_2, \ldots$ denote constants independent of n, x, z, t and possibly other specified parameters. The same symbol does not necessarily denote the same constant in different occurrences.

2. Technical Preliminaries

For a given m, we let $e_0 = 1$ and for non-zero integers t,

$$e_t = q_m^{|t|/2}\prod_{\ell=1}^{|t|-1} q_{m+\ell\,\mathrm{sign}(t)}^{|t|-\ell}, \text{ when } m+t\geq 0, t\neq 0 \tag{2.1}$$

and

$$e_t = 0,\ m+t<0. \tag{2.2}$$

Of course the $\{e_t\}$ depend on m, but we do not explicitly indicate this dependence. Also, let

$$E_{mn} = [e_{k-j}]_{1\leq j,k\leq n}. \tag{2.3}$$

The basic idea is that the matrices A_{mn}/a_m and E_{mn} are related by a similarity transformation:

Lemma 2.1
Let D be the $n \times n$ diagonal matrix whose (k,k) entry is $q_m^{k/2}\left(\frac{a_{m+1}}{a_m}\right)^{-k}$, $1 \leq k \leq n$.
(a) Then

$$D^{-1}A_{mn}D/a_m = E_{mn}. \tag{2.4}$$

(b) λ is an eigenvalue of A_{mn}/a_m iff λ is an eigenvalue of E_{mn}.
Proof
(a) We see that the (j,k) entry of $D^{-1}A_{mn}D/a_m$ is

$$g_{k-j} = q_m^{(k-j)/2}\left(\frac{a_{m+1}}{a_m}\right)^{j-k}\frac{a_{m-j+k}}{a_m}. \tag{2.5}$$

We claim that this last number equals e_{k-j}. Assume first $t = k-j > 0$. Then

$$\frac{a_{m-j+k}}{a_m} = \frac{a_{m+t}}{a_m} = \prod_{\ell=0}^{t-1}\frac{a_{m+\ell+1}}{a_{m+\ell}}.$$

Here using

$$\frac{a_{j+1}}{a_j} = q_j\frac{a_j}{a_{j-1}},$$

we see that

$$\frac{a_{m+\ell+1}}{a_{m+\ell}} = q_{m+\ell}\frac{a_{m+\ell}}{a_{m+\ell-1}} = q_{m+\ell}q_{m+\ell-1}\cdots q_{m+1}\frac{a_{m+1}}{a_m},$$

so

$$\begin{aligned}\frac{a_{m-j+k}}{a_m} &= \left(\frac{a_{m+1}}{a_m}\right)^t\prod_{\ell=0}^{t-1}(q_{m+\ell}q_{m+\ell-1}\cdots q_{m+1}) \\ &= \left(\frac{a_{m+1}}{a_m}\right)^t q_{m+1}^{t-1}q_{m+2}^{t-2}\cdots q_{m+t-1}^1.\end{aligned}$$

Then

$$g_{k-j} = q_m^{t/2}q_{m+1}^{t-1}q_{m+2}^{t-2}\cdots q_{m+t-1}^1 = e_t = e_{k-j}.$$

Next, if $t = k-j < 0$, we use

$$\frac{a_{j-1}}{a_j} = q_j\frac{a_j}{a_{j+1}},$$

so that

$$\begin{aligned}\frac{a_{m-j+k}}{a_m} &= \frac{a_{m-|t|}}{a_m} = \prod_{\ell=1}^{|t|} \frac{a_{m-\ell}}{a_{m-\ell+1}} \\ &= \prod_{\ell=1}^{|t|} \left(q_{m-\ell+1} q_{m-\ell+2} \dots q_m \frac{a_m}{a_{m+1}} \right) \\ &= \left(\frac{a_m}{a_{m+1}} \right)^{|t|} q_m^{|t|} q_{m-1}^{|t|-1} \dots q_{m-|t|+1}^1 .\end{aligned}$$

Then from (2.5),

$$\begin{aligned}g_{k-j} &= q_m^{t/2} q_m^{|t|} q_{m-1}^{|t|-1} \dots q_{m-|t|+1}^1 \\ &= q_m^{|t|/2} \prod_{\ell=1}^{|t|-1} q_{m+\ell \operatorname{sign}(t)}^{|t|-\ell} = e_t = e_{k-j}.\end{aligned}$$

(b) This is an immediate consequence of (a). ■

In the sequel, we let

(2.6) $$\chi_m = \exp\left(-\frac{1}{\rho_m}\right), \ m \geq 1.$$

It has the advantage over q_m of being real and positive.

Lemma 2.2
Assume the hypotheses of Theorem 1.2.
(a) Let $D > 0$. Then

(2.7) $$\sup_{|t| \leq \sqrt{D\rho_m}} \left| e_t / \chi_m^{t^2/2} - 1 \right| \to 0 \ \text{as} \ m \to \infty.$$

(b)

(2.8) $$\sup_{|t| \leq \sqrt{D\rho_m}} \left| \frac{1}{2} \left[e_t + \overline{e_{-t}} \right] / \chi_m^{t^2/2} - 1 \right| \to 0 \ \text{as} \ m \to \infty$$

and

(2.9) $$\sup_{|t| \leq \sqrt{D\rho_m}} \left| \frac{1}{2} \left[e_t - \overline{e_{-t}} \right] \right| / \chi_m^{t^2/2} \to 0 \ \text{as} \ m \to \infty.$$

(c) There exists $C_1 > 0$ such that for all m and all $t \geq -m+1$,

(2.10) $$|e_t| \leq C_1 \exp\left(-\frac{|t|^2}{4\rho_{m+|t|-1}} \right).$$

Proof

(a) We see that there are $\frac{|t|^2}{2}$ factors in the product in (2.1) defining e_t. Then (1.13) shows that

$$\begin{aligned} e_t/\chi_m^{t^2/2} &= \left(\frac{q_m}{\chi_m}\right)^{|t|/2} \prod_{\ell=1}^{|t|-1} \left(\frac{q_{m+\ell\mathrm{sign}(t)}}{\chi_m}\right)^{|t|-\ell} \\ &= \exp\left(-\frac{|t|\,\eta_m}{2\rho_m} - \sum_{\ell=1}^{|t|-1} (|t|-\ell)\left[\frac{1+\eta_{m+\ell\mathrm{sign}(t)}}{\rho_{m+\ell\mathrm{sign}(t)}} - \frac{1}{\rho_m}\right]\right). \end{aligned}$$

(2.11)

Here

$$\begin{aligned} & -\sum_{\ell=1}^{|t|-1} (|t|-\ell)\left[\frac{1+\eta_{m+\ell\mathrm{sign}(t)}}{\rho_{m+\ell\mathrm{sign}(t)}} - \frac{1}{\rho_m}\right] \\ =\ & -\sum_{\ell=1}^{|t|-1} (|t|-\ell)\,\frac{\rho_m - \rho_{m+\ell\mathrm{sign}(t)}}{\rho_m\rho_{m+\ell\mathrm{sign}(t)}} - \sum_{\ell=1}^{|t|-1} (|t|-\ell)\,\frac{\eta_{m+\ell\mathrm{sign}(t)}}{\rho_{m+\ell\mathrm{sign}(t)}} \\ =\ & O\left(\frac{|t|^2}{\rho_{m-|t|+1}} \max_{|j|\le|t|-1} \left|1-\frac{\rho_{m+j}}{\rho_m}\right|\right) + O\left(\frac{|t|^2}{\rho_{m-|t|+1}} \sup_{j\ge m-|t|+1} \left|\eta_j\right|\right) = o(1), \end{aligned}$$

(2.12)

provided $|t| \le \sqrt{D\rho_m}$, in view of (1.9) and (1.8). Note that that relation also implies $\rho_{m-|t|+1}/\rho_m = 1 + o(1)$. Then from (2.11),

$$e_t/\chi_m^{t^2/2} = 1 + o(1). \tag{2.13}$$

(b) These follow directly from (a).

(c) From (1.13-1.14), we see that there exists J such for $j \ge J$,

$$|q_j| \le \exp\left(-\frac{1}{2\rho_j}\right).$$

Let

$$C = \sup_{j\ge1} |q_j| \exp\left(\frac{1}{2\rho_j}\right).$$

We can assume that $C \ge 1$. Then for all $j \ge 1$,

$$|q_j| \le \exp\left(-\frac{1}{2\rho_j}\right) \times \begin{cases} C, & 1 \le j < J, \\ 1, & j \ge J. \end{cases} \tag{2.14}$$

Observe that in the product (2.1) defining e_t, for $m \geq J$, we can have $m + \ell\text{sign}(t) < J$ only when $t < 0$ and $\ell > m - J$. Since $|t| \leq m$, we also have $|t| - \ell < m - (m - J) = J$, so that

$$\prod_{1\leq\ell\leq|t|-1, m+\ell\text{sign}(t)<J} q_{m+\ell\text{sign}(t)}^{|t|-\ell} \leq C^{J^2} \prod_{1\leq\ell\leq|t|-1, m+\ell\text{sign}(t)<J} \exp\left(-\frac{1}{2\rho_{m+\ell\text{sign}(t)}}\right)^{|t|-\ell}.$$

Then for all t, $m \geq J$, and with $C_1 = C^{J^2}$, (2.1) and (2.14) show that

$$\begin{aligned} |e_t| &\leq C_1 \exp\left(-\frac{1}{2}\left\{\frac{|t|}{\rho_m} + \sum_{\ell=1}^{|t|-1} \frac{|t|-\ell}{\rho_{m+\ell\text{sign}(t)}}\right\}\right) \\ &\leq C_1 \exp\left(-\frac{|t|\left(|t|+1\right)}{4\rho_{m+|t|-1}}\right). \end{aligned}$$

■

Using Lemma 2.2, we can estimate some sums:

Lemma 2.3

Assume the hypotheses of Theorem 1.2.

(a) Let $B > 0$. There exist C_2, C_3 such that for $n \geq L \geq 1$,

(2.15)

$$\sum_{\ell=L}^{n} \left(|e_\ell|^B + |e_{-\ell}|^B + \left|q_m^{B\ell^2/2}\right| + \chi_m^{B\ell^2/2}\right) \leq C_3\sqrt{\rho_m} \exp\left(-C_4 \frac{L^2}{\rho_m}\right).$$

(b) Let $A, B > 0$. There exist C_2, C_3 such that for $n \geq 1$,

$$\sum_{\ell=1}^{n} \left(|e_\ell|^B + |e_{-\ell}|^B + \left|q_m^{B\ell^2/2}\right| + \chi_m^{B\ell^2/2}\right) \ell^A \leq C_3 \rho_m^{(A+1)/2}. \tag{2.16}$$

(c)

$$\sum_{\ell=-n+1}^{n-1} e_\ell = \sqrt{2\pi\rho_m}\left(1 + o\left(1\right)\right). \tag{2.17}$$

The same asymptotic holds for $\sum_{\ell=-n+1}^{n-1} |e_\ell|$.

Proof

(a) Using (2.6), (2.10), (2.14), (1.8), and the fact that (recall (1.15))

$m > n/R$,

$$\begin{aligned}
&\sum_{\ell=L}^{n}\left(|e_\ell|^B + |e_{-\ell}|^B + \left|q_m^{B\ell^2/2}\right| + \chi_m^{B\ell^2/2}\right) \\
\leq\ & C_3 \sum_{\ell=L}^{n} \exp\left(-C_2 \frac{\ell^2}{\rho_m}\right) \\
\leq\ & C_3 \int_{L-1}^{\infty} \exp\left(-C_2 \frac{x^2}{\rho_m}\right) dx \\
\leq\ & C_3 \sqrt{\rho_m} \int_{\frac{L-1}{\sqrt{\rho_m}}}^{\infty} \exp\left(-C_2 t^2\right) dt \\
\leq\ & C_3 \sqrt{\rho_m} \exp\left(-C_2 \frac{L^2}{\rho_m}\right).
\end{aligned}$$

(To see this, consider separately the case $(L-1)/\sqrt{\rho_m} \geq 1$ or < 1). So we have (2.15).
(b) Here we use the fact that the function $x \to x^A \exp\left(-Cx^2\right)$ increases up to a certain point, after which it decreases:

$$\begin{aligned}
&\sum_{\ell=1}^{n}\left(|e_\ell|^B + |e_{-\ell}|^B + \left|q_m^{B\ell^2/2}\right| + \chi_m^{B\ell^2/2}\right) \ell^A \\
\leq\ & C_4 \sum_{\ell=1}^{n} \exp\left(-C_2 \frac{\ell^2}{\rho_m}\right) \ell^A \\
\leq\ & 2C_4 \int_{0}^{n+1} \exp\left(-C_2 \frac{x^2}{\rho_m}\right) x^A dx \\
\leq\ & 2C_4 \rho_m^{(A+1)/2} \int_{0}^{\infty} \exp\left(-C_2 t^2\right) t^A dt.
\end{aligned}$$

(c) Fix $D > 0$. From (2.7),

$$\sum_{|\ell| \leq \sqrt{D\rho_m}} e_\ell = \sum_{|\ell| \leq \sqrt{D\rho_m}} \chi_m^{\ell^2/2}\left(1 + o(1)\right).$$

Here, using the inequality

$$|e^u - e^v| \leq |e^u|\,|u-v|\,e^{|u-v|}, u, v \in \mathbb{C}, \tag{2.18}$$

we see that

$$\begin{aligned}
&\left|\chi_m^{\ell^2/2} - \int_\ell^{\ell+1} \chi_m^{x^2/2} dx\right| \\
\leq\ & \chi_m^{\ell^2/2} \left|\log \chi_m\right| \sup_{x\in[\ell,\ell+1]} \left|\ell^2/2 - x^2/2\right| \exp\left(\left|\log \chi_m\right| \left|\ell^2/2 - x^2/2\right|\right) \\
\leq\ & \chi_m^{\ell^2/2} \frac{2\ell+1}{2\rho_m} \exp\left(\frac{2\ell+1}{2\rho_m}\right) \\
\leq\ & C\chi_m^{\ell^2/2}/\sqrt{\rho_m},
\end{aligned}$$

uniformly in $\ell \leq \sqrt{D\rho_m}$. Thus using (b),

$$\begin{aligned}
\sum_{|\ell|\leq\sqrt{D\rho_m}} e_\ell &= \int_{-\sqrt{D\rho_m}}^{\sqrt{D\rho_m}} \chi_m^{x^2/2} dx + O\left(1\right) \\
&= \sqrt{2\rho_m} \int_{-\sqrt{D/2}}^{\sqrt{D/2}} e^{-t^2} dt + O\left(1\right).
\end{aligned}$$

Moreover, by (a) of this lemma,

$$\sum_{|\ell|>\sqrt{D\rho_m}} |e_\ell| \leq C_3 \sqrt{\rho_m} \exp\left(-C_4 D\right).$$

Thus as $m \to \infty$,

$$\frac{1}{\sqrt{2\rho_m}} \sum_{\ell=-n+1}^{n-1} e_\ell = \int_{-\sqrt{D/2}}^{\sqrt{D/2}} e^{-t^2} dt + O\left(\frac{1}{\rho_m}\right) + O\left(e^{-C_4 D}\right).$$

Here the constant in the order term $O\left(e^{-C_4 D}\right)$ is independent of D. So D may be chosen as large as we please and we deduce that

$$\frac{1}{\sqrt{2\rho_m}} \sum_{\ell=-n+1}^{n-1} e_\ell = \int_{-\infty}^{\infty} e^{-t^2} dt \left(1 + o\left(1\right)\right) = \sqrt{\pi} \left(1 + o\left(1\right)\right).$$

The same proof works for $\sum_{\ell=-n+1}^{n-1} |e_\ell|$. ■

Our final lemma in this section involves the smoother hypotheses of Theorem 1.4:

Lemma 2.4
Assume the hypotheses of Theorem 1.4. Let $D > 0$.
(a) For $|t| \leq \sqrt{D\rho_m \log \rho_m}$,

$$(2.19)\qquad \frac{1}{2}\left(e_t+\overline{e_{-t}}\right)/\chi_m^{t^2/2}=1+o\left(\frac{|t|^3}{\rho_m^2}\right)+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right).$$

(b) If in addition (1.36) holds, then

$$(2.20)\qquad \frac{1}{2}\left(e_t-\overline{e_{-t}}\right)/\chi_m^{t^2/2}=O\left(\frac{|t|^3}{\rho_m^2}\right)+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right).$$

Proof

(a) From (2.11) and (1.21),

(2.21)

$$e_t/\chi_m^{t^2/2}=\exp\left(o\left(\frac{|t|}{\rho_m^{3/2}}\right)-\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\left[\frac{1}{\rho_{m+\ell\mathrm{sign}(t)}}-\frac{1}{\rho_m}\right]+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)\right).$$

Here from (1.19),

$$\begin{aligned}&\left|\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\left[\frac{1}{\rho_{m+\ell\mathrm{sign}(t)}}-\frac{1}{\rho_m}\right]\right|\\ =\ & o\left(\frac{1}{\rho_m}\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\frac{\ell}{\rho_m^{3/4}}\right)=o\left(\frac{|t^3|}{\rho_m^{7/4}}\right)=o\left(1\right),\end{aligned}$$

if $|t|\leq\sqrt{D\rho_m\log\rho_m}$. Then

(2.22)

$$e_t/\chi_m^{t^2/2}=1-\exp\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\left[\frac{1}{\rho_{m+\ell\mathrm{sign}(t)}}-\frac{1}{\rho_m}\right]+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right).$$

Next from (1.20),

$$\begin{aligned}\frac{1}{2}\left(e_t+\overline{e_{-t}}\right)/\chi_m^{t^2/2} &= 1-\frac{1}{2}\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\left[\frac{1}{\rho_{m+\ell}}+\frac{1}{\rho_{m-\ell}}-\frac{2}{\rho_m}\right]+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right)\\ &= 1-\frac{1}{2}\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)o\left(\frac{\ell}{\rho_m^2}\right)+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right)\\ &= 1+o\left(\frac{|t|^3}{\rho_m^2}\right)+o\left(\frac{|t|^2}{\rho_m^{3/2}}\right)+o\left(\frac{|t^6|}{\rho_m^{7/2}}\right).\end{aligned}$$

(b) Similarly, using (1.36), we have for $t \geq 0$,

$$\begin{aligned}
\frac{1}{2}\left(e_t - \overline{e_{-t}}\right)/\chi_m^{t^2/2} &= -\frac{1}{2}\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right)\left[\frac{1}{\rho_{m+\ell}} - \frac{1}{\rho_{m-\ell}}\right] + o\left(\frac{|t|^2}{\rho_m^{3/2}}\right) + o\left(\frac{|t^6|}{\rho_m^{7/2}}\right) \\
&= -\frac{1}{2}\sum_{\ell=1}^{|t|-1}\left(|t|-\ell\right) O\left(\frac{\ell}{\rho_m^2}\right) + o\left(\frac{|t|^2}{\rho_m^{3/2}}\right) \\
&= O\left(\frac{|t|^3}{\rho_m^2}\right) + o\left(\frac{|t|^2}{\rho_m^{3/2}}\right) + o\left(\frac{|t^6|}{\rho_m^{7/2}}\right).
\end{aligned}$$

■

3. Asymptotics of the Moments

The result of this section is:

Theorem 3.1
Assume the hypotheses of Theorem 1.2. Fix $k \geq 1$. Then as $n \to \infty$,

(3.1) $$Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right] = n\left(2\pi\rho_m\right)^{(k-1)/2}\frac{1+o(1)}{\sqrt{k}}.$$

First we need:

Lemma 3.2
Fix $k \geq 1$ and

$$\Psi\left(s_1, s_2, ..., s_k\right) = \frac{1}{2}\left[\left(s_2 - s_1\right)^2 + \left(s_3 - s_2\right)^2 + ... + \left(s_k - s_{k-1}\right)^2 + \left(s_1 - s_k\right)^2\right].$$

(3.2)

Given $1 \leq \ell_1, \ell_2, ..., \ell_k \leq n$, let

$$I\left(\ell_1, \ell_2, ..., \ell_k\right) = \left[\ell_1, \ell_1 + 1\right] \times \left[\ell_2, \ell_2 + 1\right] \times ... \times \left[\ell_k, \ell_k + 1\right].$$

(a) Then for $q \in \mathbb{C}$,

$$\begin{aligned}
&\left|q^{\Psi(\ell_1,\ell_2,...,\ell_k)} - \int_{I(\ell_1,\ell_2,...,\ell_k)} q^{\Psi(s_1,s_2,...,s_k)} ds_1 ds_2 ... ds_k\right| \\
\leq\ & R\left|\log q\right| e^{R|\log q|}\left|\int_{I(\ell_1,\ell_2,...,\ell_k)} |q|^{\Psi(s_1,s_2,...,s_k)}\, ds_1 ds_2 ... ds_k\right|,
\end{aligned}$$

where

$$R=2\sum_{j=1}^{k}\left(\left|\ell_{j+1}-\ell_j\right|+1\right) \tag{3.3}$$

and we set $\ell_{k+1}=\ell_1$.
(b) Let $D>0$ *and* χ_m *be given by (2.6). If* $\left|\ell_{j+1}-\ell_j\right|\leq\sqrt{D\rho_m}$, *for* $1\leq j\leq k-1$, *then*

$$\begin{aligned}&\left|\chi_m^{\Psi(\ell_1,\ell_2,...,\ell_k)}-\int_{I(\ell_1,\ell_2,...,\ell_k)}e^{-\frac{\Psi(s_1,s_2,...,s_k)}{\rho_m}}ds_1ds_2...ds_k\right|\\ \leq\;& C_3\frac{1}{\sqrt{\rho_m}}\int_{I(\ell_1,\ell_2,...,\ell_k)}e^{-\frac{\Psi(s_1,s_2,...,s_k)}{\rho_m}}ds_1ds_2...ds_k.\end{aligned} \tag{3.4}$$

The constant C_3 *depends only on* D *and* k, *not on* $n,m,\{\ell_j\}$.

Proof

(a) We again use the inequality (2.18) to deduce that for $(s_1,s_2,...,s_k)\in I(\ell_1,\ell_2,...,\ell_k)$,

$$\left|q^{\Psi(\ell_1,\ell_2,...,\ell_k)}-q^{\Psi(s_1,s_2,...,s_k)}\right|\leq|q|^{\Psi(s_1,s_2,...,s_k)}re^r,$$

where

$$r=|\log q|\max_{(s_1,s_2,...,s_k)\in I(\ell_1,\ell_2,...,\ell_k)}\left|\Psi(\ell_1,\ell_2,...,\ell_k)-\Psi(s_1,s_2,...,s_k)\right|.$$

Here as each $|s_j-\ell_j|\leq 1$, so

$$\begin{aligned}&\left|\Psi(\ell_1,\ell_2,...,\ell_k)-\Psi(s_1,s_2,...,s_k)\right|\\ =\;&\frac{1}{2}\left|\sum_{j=1}^{k}\left(\ell_{j+1}-\ell_j-s_{j+1}+s_j\right)\left(\ell_{j+1}-\ell_j+s_{j+1}-s_j\right)\right|\\ \leq\;&\left|\sum_{j=1}^{k}\left(2\left(\ell_{j+1}-\ell_j\right)+\left(s_{j+1}-\ell_{j+1}\right)+\left(\ell_j-s_j\right)\right)\right|\leq\sum_{j=1}^{k}\left(2\left|\ell_{j+1}-\ell_j\right|+2\right)=R.\end{aligned}$$

Substituting this above, we obtain

$$\left|q^{\Psi(\ell_1,\ell_2,...,\ell_k)}-q^{\Psi(s_1,s_2,...,s_k)}\right|\leq\left|q^{\Psi(s_1,s_2,...,s_k)}\right||\log q|\,Re^{|\log q|R}.$$

Integrating over $I(\ell_1,\ell_2,...,\ell_k)$ gives the desired result.
(b) Here we use (a) with $q=\chi_m$. Our hypothesis gives $|\ell_k-\ell_1|\leq(k-1)\sqrt{D\rho_m}$ so that R of (3.3) is $O\left(\sqrt{\rho_m}\right)$ and we can apply (a). ■

Proof of Theorem 3.1
Because of Lemma 2.1,

$$Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right] = Tr\left[E_{mn}^k\right].$$

When $k = 1$, $Tr\left[E_{mn}\right] = n$, so the result is immediate. So fix $k \geq 2$. We see that

$$\begin{aligned} & Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right] \\ = & \sum_{\ell_1=1}^{n}\sum_{\ell_2=1}^{n}\sum_{\ell_3=1}^{n}...\sum_{\ell_k=1}^{n} e_{\ell_2-\ell_1}e_{\ell_3-\ell_2}e_{\ell_4-\ell_3}...e_{\ell_k-\ell_{k-1}}e_{\ell_1-\ell_k}. \end{aligned}$$

Let $D > 0$. We shall split this sum into a central term in which all differences $\left|\ell_j - \ell_{j-1}\right|, 2 \leq j \leq k$, are $\leq \sqrt{D\rho_m}$, and a tail term in which one of these is larger than $\sqrt{D\rho_m}$. We first handle the central part of the sum. So let $\Sigma_{central}$ denote that part of the sum with $\left|\ell_j - \ell_{j-1}\right| \leq \sqrt{D\rho_m}, 2 \leq j \leq k$. Then $\left|\ell_1 - \ell_k\right| \leq (k-1)\sqrt{D\rho_m}$. The only "free" index is ℓ_1, which may range from 1 to n. By (2.7) and (3.4), and recalling (3.2), and that $\{\rho_m\}$ increase to ∞,

$$\begin{aligned} \Sigma_{central} &= (1+o(1))\sum \chi_m^{\Psi(\ell_1,\ell_2,...,\ell_k)} \\ &= (1+o(1))\int ... \int_I e^{-\frac{\Psi(s_1,s_2,...,s_k)}{\rho_m}} ds_1 ds_2 ... ds_k, \end{aligned}$$

where

$$I = \bigcup I\left(\ell_1, \ell_2, ..., \ell_k\right)$$

is the union over all the indices $\left(\ell_1, \ell_2, ..., \ell_k\right)$ satisfying the inequalities above. Then also

$$\Sigma_{central} = (1+o(1))\int\int ... \int_{\mathcal{S}} e^{-\frac{\Psi(s_1,s_2,...,s_k)}{\rho_m}} ds_1 ds_2 ... ds_k + O\left(\left(\sqrt{\rho_m}\right)^k\right),$$

where $\mathcal{S}$ is the range $(s_1, s_2, ..., s_k)$ with $\left|s_{j+1} - s_j\right| \leq \sqrt{D\rho_m}$ for all $1 \leq j \leq k-1$, while $s_1 \in [0, n]$. Note that then $\left|s_k - s_1\right| \leq (k-1)\sqrt{D\rho_m}$. Here we are also using the fact that the integrand is bounded, so we may allow $s_1 \in [0, n]$, and other s_j to be possibly negative, while incurring an error $O\left(\left(\sqrt{\rho_m}\right)^k\right)$. We now make the substitution

$$\begin{aligned} t_1 &= \frac{s_1}{\sqrt{\rho_m}}; \\ t_j &= \frac{s_j - s_{j-1}}{\sqrt{\rho_m}}, \ 2 \leq j \leq k. \end{aligned}$$

A simple calculation shows that

$$\frac{\partial\left(t_1, t_2, \ldots, t_k\right)}{\partial\left(s_1, s_2, \ldots, s_k\right)}=\left(\frac{1}{\sqrt{\rho_m}}\right)^k$$

and hence also that

$$\frac{\partial\left(s_1, s_2, \ldots, s_k\right)}{\partial\left(t_1, t_2, \ldots, t_k\right)}=\left(\sqrt{\rho_m}\right)^k .$$

The region S corresponds to a region $J \times \mathcal{T}$ in which t_1 runs through an interval J of length $\frac{n}{\sqrt{\rho_m}}(1+o(1))$, while $\mathcal{T}$ is the set of $\left(t_2, t_3, \ldots, t_k\right)$ with $\left|t_j\right| \leq \sqrt{D}$ for all $2 \leq j \leq k$. Moreover,

$$s_k-s_1=\sqrt{\rho_m}\left(t_k+t_{k-1}+\ldots+t_2\right) .$$

Thus from (3.2),

$$\begin{aligned} \frac{\Psi\left(s_1, s_2, \ldots, s_k\right)}{\rho_m} & =\frac{1}{2}\left(t_2^2+t_3^2+\ldots+t_k^2+\left(t_k+t_{k-1}+\ldots+t_2\right)^2\right) \\ & =\Phi\left(t_2, \ldots, t_k\right), \end{aligned}$$

say. Then

$$\Sigma_{central}=(1+o(1))\, n\left(\sqrt{\rho_m}\right)^{k-1} \int_{-\sqrt{D}}^{\sqrt{D}} \int_{-\sqrt{D}}^{\sqrt{D}} \ldots \int_{-\sqrt{D}}^{\sqrt{D}} e^{-\Phi\left(t_2, \ldots, t_k\right)} dt_2 \ldots dt_k+O\left(\left(\sqrt{\rho_m}\right)^k\right) .$$

(3.5)

For D large enough, this last integral is close to

$$I_{\infty}=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-\Phi\left(t_2, \ldots, t_k\right)} dt_2 \ldots dt_k. \tag{3.6}$$

To evaluate this, we use a classical identity for integrals of exponential of positive quadratic forms. We see that

$$\Phi\left(t_2, \ldots, t_k\right)=\sum_{j=2}^k t_j^2+\frac{1}{2} \sum_{j \neq k} t_j t_k=T^T B T,$$

where B is the positive definite $(k-1) \times(k-1)$ matrix with all diagonal entries equal 1 and all off diagonal entries equal $\frac{1}{2}$, while $T= \left[t_2\ t_3\ \ldots\ t_k\right]^T$. Thus

$$B=\left[\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{array}\right] .$$

Then [3, Thm. 3, p. 61]

$$I_\infty = \frac{\pi^{(k-1)/2}}{|\det B|^{1/2}}. \tag{3.7}$$

To evaluate $\det B$, we subtract the first row of B from the remaining rows. This leaves a matrix with off diagonal entries 0 in the 2nd, 3rd, ..., $(k-1)$th row, except in the first column where the entries are $-\frac{1}{2}$. In the diagonal, the entries are $\frac{1}{2}$ except in the first row. So

$$\det B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}.$$

Next, we subtract the jth row from the first row, for $j = k-1, k-2, ..., 2$. We obtain

$$\det B = \begin{bmatrix} 1+\frac{k-2}{2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix} = \frac{k}{2^{k-1}}.$$

Combining this and (3.5-3.7), we see that

$$\Sigma_{central} = (1+o(1))\, n \left((2\pi\rho_m)^{\frac{k-1}{2}} \frac{1}{\sqrt{k}} + O(\varepsilon_D) \right), \tag{3.8}$$

where ε_D is a term independent of m, n, but depends on D and approaches 0 as $D \to \infty$.

Now we have to handle the "tail terms". Let $\mathcal{I}_1$ denote the set of $(\ell_1, \ell_2, ..., \ell_k)$ such that all $1 \le \ell_j \le n$, and for at least one $2 \le j \le k$, $|\ell_j - \ell_{j-1}| > \sqrt{D\rho_m}$. From (2.10), (1.8), and (1.15),

$$\begin{aligned} |\Sigma_{Tail}| &= \left| \sum_{(\ell_1,\ell_2,...,\ell_k)\in\mathcal{I}_1} e_{\ell_2-\ell_1} e_{\ell_3-\ell_2} e_{\ell_4-\ell_3} ... e_{\ell_k-\ell_{k-1}} e_{\ell_1-\ell_k} \right| \\ &\le C_1^k \sum_{(\ell_1,\ell_2,...,\ell_k)\in\mathcal{I}_1} \exp\left(-\frac{C_4}{\rho_m} \left[\sum_{j=2}^{k} |\ell_j-\ell_{j-1}|^2 + |\ell_1-\ell_k|^2 \right] \right). \end{aligned}$$

Here $\ell_1 - \ell_k$ is contained in the set $\{-n+1, -n+2, ..., n-1\}$. Because of symmetry, we may assume that $|\ell_k - \ell_{k-1}| \geq \sqrt{D\rho_m}$. Once $\ell_1 - \ell_k$ is determined, we may set

$$i_j = \ell_j - \ell_{j-1}, 2 \leq j \leq k$$

and see that while $|i_k| > \sqrt{D\rho_m}$, we can allow the remaining $\{i_k\}$ to take any integer value. Thus

$$\begin{aligned}
|\Sigma_{Tail}| &\leq Cn \sum_{i_2=-\infty}^{\infty} \sum_{i_3=-\infty}^{\infty} ... \sum_{i_{k-1}=-\infty}^{\infty} \sum_{i_k \geq \sqrt{D\rho_m}} \exp\left(-\frac{C_4}{\rho_m} \sum_{j=2}^{k} i_j^2\right) \\
&= Cn \left(\sum_{i=-\infty}^{\infty} \exp\left(-\frac{C_4}{\rho_m} i^2\right)\right)^{k-2} \left(\sum_{i \geq \sqrt{D\rho_m}} \exp\left(-\frac{C_4}{\rho_m} i^2\right)\right) \\
&\leq Cn \left(2\int_{-\infty}^{\infty} \exp\left(-\frac{C_4}{\rho_m} x^2\right) dx\right)^{k-2} \left(2\int_{\sqrt{D\rho_m}}^{\infty} \exp\left(-\frac{C_4}{\rho_m} x^2\right) dx\right) \\
&\leq Cn \left(\sqrt{\rho_m}\right)^{k-1} e^{-C_4 D}.
\end{aligned}$$

The constants C and C_4 do not depend on D. Thus given $\eta > 0$, we can find D so large that

$$|\Sigma_{Tail}| \leq \eta n \left(\sqrt{\rho_m}\right)^{k-1}.$$

This and (3.8) give the result. ■

We note that we used $\rho_m = o(m^2)$ in the above proof to ensure that $\sqrt{D\rho_m} = o(n)$ for each $D > 0$, so that there are enough central terms to give the exponential integral.

4. Location of the Eigenvalues

In this section, we use classical tools such as Gershgorin's theorem, Schur's inequalities, and the Bendixson-Hirsch inequalities to estimate the location of the eigenvalues. We begin with

Lemma 4.1
Assume the hypotheses of Theorem 1.2. Then

$$\max_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda| \leq \sqrt{2\pi\rho_m} (1 + o(1)).$$

Proof
Now each such λ is an eigenvalue of E_{mn}. By Gershgorin's bounds [13,

p. 371, Theorem 1], [19, p. 146], they lie in disks centered on the diagonal entries 1 of E_{mn}, and with radius

$$\max_{1\leq j\leq n}\sum_{k=1,k\neq j}^{n}\left|e_{j-k}\right|\leq\sum_{j=-n+1,j\neq 0}^{n-1}\left|e_{j}\right|=\sqrt{2\pi\rho_{m}}\left(1+o\left(1\right)\right),$$

by Lemma 2.3(c). ■

Next, we use Schur's inequalities. For our matrix E_{mn}, let

$$F_{mn}=\frac{1}{2}\left(E_{mn}+E_{mn}^{H}\right)\text{ and }G_{mn}=\frac{1}{2i}\left(E_{mn}-E_{mn}^{H}\right). \tag{4.1}$$

Here the superscript H refers to the conjugate transpose. Thus F_{mn} and G_{mn} are Hermitian matrices, and in particular, have real eigenvalues. Schur's inequalities assert that [13, p. 385, Theorem 1],

(4.2)
$$\sum_{\lambda\in\Lambda\left(E_{mn}\right)}\left(\operatorname{Re}\lambda\right)^{2}\leq Tr\left(F_{mn}^{2}\right)\text{ and }\sum_{\lambda\in\Lambda\left(E_{mn}\right)}\left(\operatorname{Im}\lambda\right)^{2}\leq Tr\left(G_{mn}^{2}\right).$$

Using these, we can show that most eigenvalues approach the real axis:

Lemma 4.2
Assume the hypotheses of Theorem 1.2. Then
(a)

$$\frac{1}{n\sqrt{\rho_{m}}}\sum_{\lambda\in\Lambda\left(E_{mn}\right)}\left|\operatorname{Im}\lambda\right|^{2}=o\left(1\right). \tag{4.3}$$

(b)

$$\frac{1}{n\sqrt{\rho_{m}}}\sum_{\lambda\in\Lambda\left(E_{mn}\right)}\left|\operatorname{Re}\lambda\right|^{2}=O\left(1\right). \tag{4.4}$$

Proof
(a) In view of Schur's inequalities (4.2), it suffices to show that

$$\frac{1}{n\sqrt{\rho_{m}}}Tr\left(G_{mn}^{2}\right)=o\left(1\right). \tag{4.5}$$

Now

$$\begin{aligned}
Tr\left(G_{mn}^2\right) &= \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\left|e_{k-j}-\overline{e_{j-k}}\right|^2 \\
&\leq \frac{n}{2}\sum_{\ell=1}^{n-1}\left|e_\ell-\overline{e_{-\ell}}\right|^2 \\
&\leq \frac{n}{2}\left(\sum_{|\ell|\leq\sqrt{D\rho_m}}\left|e_\ell-\overline{e_{-\ell}}\right|^2+\sum_{|\ell|>\sqrt{D\rho_m}}\left|e_\ell-\overline{e_{-\ell}}\right|^2\right) \\
&\leq n\left(o\left(\sqrt{\rho_m}\right)+C_1\sqrt{\rho_m}e^{-C_2D}\right),
\end{aligned}$$

by (2.9) and (2.15). Then

$$\limsup_{n\to\infty}\frac{1}{n\sqrt{\rho_m}}Tr\left(G_{mn}^2\right)\leq C_1e^{-C_2D}$$

and since D may be arbitrarily large, (4.3) follows.
(c) By Schur's inequalities,

$$\begin{aligned}
\sum_{\lambda\in\Lambda(E_{mn})}(\operatorname{Re}\lambda)^2 &\leq Tr\left(F_{mn}^2\right) \\
&= \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(e_{k-j}+\overline{e_{j-k}}\right)^2 \\
&\leq Cn\sqrt{\rho_m},
\end{aligned}$$

by (2.16). ■

Next, we use the Bendixson-Hirsch inequalities to bound the real and imaginary parts of the eigenvalues. However, we first need

Lemma 4.3
Let $0\leq\chi<1, n\geq 1,$ *and let* $B=\left[\chi^{(j-k)^2/2}\right]_{1\leq j,k\leq n}$. *Then* B *is positive definite.*
Proof
If $X=[x_1\ x_2 ... x_n]^T$, then we see that

$$\begin{aligned}
X^TBX &= \sum_{j,k=1}^{n}x_j\chi^{((j-1)-(k-1))^2/2}x_k \\
&= \sum_{j,k=1}^{n}\left(x_j\chi^{(j-1)^2/2}\right)\chi^{-(j-1)(k-1)}\left(x_k\chi^{(k-1)^2/2}\right),
\end{aligned}$$

so it suffices to prove that the matrix

$$V_n = \left[\chi^{-(j-1)(k-1)}\right]_{1\le j,k\le n} = \left[\left(\chi^{-(j-1)}\right)^{(k-1)}\right]_{1\le j,k\le n}$$

is positive definite. Now V_n is symmetric, real, and a Vandermonde matrix, *so*

$$\det(V_n) = \prod_{1\le j<\ell\le n}\left[\chi^{-(\ell-1)} - \chi^{-(j-1)}\right] > 0.$$

But then all the leading submatrices $V_1, V_2, ..., V_{n-1}$ also have positive determinant, so V_n is positive definite. ■

Lemma 4.4
Assume the hypotheses of Theorem 1.2. Then
(a)

$$\min_{\lambda\in\Lambda(E_{mn})} \operatorname{Re}(\lambda) \ge -o\left(\sqrt{\rho_m}\right). \tag{4.6}$$

(b)

$$\max_{\lambda\in\Lambda(E_{mn})} |\operatorname{Im}(\lambda)| = o\left(\sqrt{\rho_m}\right). \tag{4.7}$$

(c) All limit points of $\{\Lambda(E_{mn})/\sqrt{2\pi\rho_m}\}_{n\ge1}$ *lie in* $[0,1]$.
Proof
Let F_{mn} and G_{mn} be given by (4.1). The Bendixson-Hirsch inequalities [19, p. 195], [30, p. 490] assert that

$$\min_{\lambda\in\Lambda(E_{mn})} \operatorname{Re}(\lambda) \ge \lambda_{\min}(F_{mn}) \tag{4.8}$$

where $\lambda_{\min}(F_{mn})$ is the smallest eigenvalue of F_{mn} above. Moreover, for each $\lambda \in \Lambda(E_{mn})$, we have

$$\lambda_{\min}(G_{mn}) \le \operatorname{Im}\lambda \le \lambda_{\max}(G_{mn}) \tag{4.9}$$

Here $\lambda_{\min}(G_{mn})$, $\lambda_{\max}(G_{mn})$ denote the smallest and largest eigenvalues of G_{mn}. The advantage of using F_{mn} and G_{mn} is that they are Hermitian matrices, so we can use the variational formulation for their smallest and largest eigenvalues.
(a) Now

$$\begin{aligned}\lambda_{\min}(F_{mn}) &= \inf \sum_{j,k=1}^{n} \frac{1}{2}\left(e_{k-j} + \overline{e_{j-k}}\right) x_j x_k \\ &= \inf \sum_{\ell=-n+1}^{n-1} \frac{1}{2}\left(e_\ell + \overline{e_{-\ell}}\right) S_\ell, \end{aligned} \tag{4.10}$$

where

$$S_\ell = \sum_{1 \le j, j+\ell \le n} x_j x_{j+\ell} \tag{4.11}$$

and the inf is taken over all $\{x_j\}_{j=1}^n$ with $\sum_{j=1}^n x_j^2 = 1$, so that all $|S_\ell| \le 1$. Fix some large positive D. From (2.8),

$$\begin{aligned} &\lambda_{\min}(F_{mn}) \\ \ge\ & \inf \left\{ \sum_{|\ell| \le \sqrt{D\rho_m}} \chi_m^{\ell^2/2} (1 + o(1)) S_\ell + \sum_{|\ell| > \sqrt{D\rho_m}} \frac{1}{2} \left(e_\ell + \overline{e_{-\ell}} \right) S_j \right\} \\ \ge\ & \left(\inf \sum_{\ell=-n+1}^{n-1} \chi_m^{\ell^2/2} S_\ell \right) - \sum_{|\ell| > \sqrt{D\rho_m}} \chi_m^{\ell^2/2} - \sum_{|\ell| > \sqrt{D\rho_m}} \frac{1}{2} \left| e_\ell + \overline{e_{-\ell}} \right| - o(1) \sum_{|\ell| \le \sqrt{D\rho_m}} \chi_m^{\ell^2/2}. \end{aligned}$$

Here the first term in the last right-hand side is the smallest eigenvalue of matrix $\left(\chi_m^{(j-k)^2/2} \right)_{1 \le j,k \le n}$, which is positive definite by Lemma 4.3, so its smallest eigenvalue is positive. Also, using the bounds of Lemma 2.3(a),

$$\sum_{|\ell| > \sqrt{D\rho_m}} \left| e_\ell + \overline{e_{-\ell}} \right| \le C_1 \sqrt{\rho_m} e^{-C_2 D};$$

while a similar estimate holds for $\sum_{|\ell| > \sqrt{D\rho_m}} \chi_m^{\ell^2/2}$, and trivially,

$$o(1) \sum_{|\ell| \le \sqrt{D\rho_m}} \chi_m^{\ell^2/2} = o\left(\sqrt{\rho_m} \right).$$

In summary,

$$\lambda_{\min}(F_{mn}) \ge -C \sqrt{\rho_m} e^{-CD} - o\left(\sqrt{\rho_m} \right).$$

Since D may be arbitrarily large, we obtain

$$\lambda_{\min}(F_{mn}) \ge -o\left(\sqrt{\rho_m} \right).$$

(b) Here we can use that the maximum absolute value of eigenvalues of G_{mn} is bounded by its largest row sum,

$$\begin{aligned} & \max_j \sum_{k=1}^{n} \frac{1}{2} \left| e_{k-j} - \overline{e_{j-k}} \right| \\ \leq & \sum_{\ell=0}^{n} \left| e_\ell - \overline{e_{-\ell}} \right| \\ \leq & \; o(1) \sum_{1 \leq \ell \leq \sqrt{D\rho_m}} \chi_m^{\ell^2/2} + \sum_{\ell > \sqrt{D\rho_m}} \left| e_\ell - \overline{e_{-\ell}} \right| \\ \leq & \; o\left(\sqrt{\rho_m}\right) + C_1 \sqrt{\rho_m} e^{-C_2 D}, \end{aligned}$$

by Lemma 2.2(b) and 2.3(a). Then as D is arbitrary, the maximum absolute value of eigenvalues of G_{mn} is $o\left(\sqrt{\rho_m}\right)$, and then (4.9) gives the result.
(c) This is an immediate consequence of (a), (b) and Lemma 4.1. ■

5. Proof of Theorem 1.2

Let us review what we have proved so far: in Lemma 4.1, we proved that all eigenvalues of E_{mn}, and hence of A_{mn}/a_m, have modulus $\leq \sqrt{2\pi\rho_m}\,(1 + o(1))$. Moreover, in Lemma 4.4, we proved that all limit points of $\{\Lambda(A_{mn}/a_m)/\sqrt{2\pi\rho_m}\}$ lie in $[0,1]$. We still need to prove:
(I) That the eigenvalue of largest absolute value of A_{mn}/a_m is $\geq \sqrt{2\pi\rho_m}\,(1 + o(1))$ and that the limit points of $\{\Lambda(A_{mn}/a_m)/\sqrt{2\pi\rho_m}\}_{n\geq 1}$ fill out $[0,1]$.
(II) That the measures μ_{mn} and $\mu_{mn}^{[2]}$ converge weakly in the sense described in Theorem 1. We begin with

Lemma 5.1
The assertion (III) of Theorem 1.2 is true.
Proof
Now μ_{mn} is a probability measure with support contained in a ball center 0 that is independent of n. Moreover, for $k \geq 1$,

$$\begin{aligned} \int t^k d\mu_{mn}(t) &= \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left(\lambda / \sqrt{2\pi\rho_m} \right)^k \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

by Theorem 3.1. Then for any polynomial P, we have

$$\lim_{n\to\infty} \int P(t)\, d\mu_{mn}(t) = P(0).$$

Next let f be continuous in an open subset of the complex plane containing $[0,1]$. Let $\varepsilon > 0$. For some small enough $\delta > 0$, the restriction of f to $[-\delta, 1+\delta]$ is continuous, so we can find a polynomial P such that

$$|f(x) - P(x)| < \frac{\varepsilon}{3} \text{ for } x \in [-\delta, 1+\delta].$$

In as much as f and P are continuous, we can find $\eta \in (0,\delta)$ such that if $S = \{z : dist\,(z,[0,1]) < \eta\}$, then

$$|f(z) - f(\operatorname{Re} z)| < \frac{\varepsilon}{3} \text{ and } |P(z) - P(\operatorname{Re} z)| < \frac{\varepsilon}{3}$$

so that for all $z \in S$,

$$|f(z) - P(z)| < \varepsilon.$$

Then for n so large that the support of μ_{mn} is contained in S,

$$\begin{aligned}
& \left| \int f(t)\, d\mu_{mn}(t) - f(0) \right| \\
\leq & \left| \int [f(t) - P(t)]\, d\mu_{mn}(t) \right| + \left| \int P(t)\, d\mu_{mn}(t) - P(0) \right| + |P(0) - f(0)| \\
\leq & \varepsilon + \left| \int P(t)\, d\mu_{mn}(t) - P(0) \right| + \varepsilon \\
\rightarrow & 2\varepsilon,
\end{aligned}$$

as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, (1.17) follows. ■

Lemma 5.2
The assertion (IV) of Theorem 1.2 is true.
Proof
First note that

$$\begin{aligned}
\left|\mu_{mn}^{[2]}\right|(\mathbb{C}) &= \frac{1}{n\sqrt{\pi\rho_m}} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda|^2 \\
&= \frac{1}{n\sqrt{\pi\rho_m}} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left[|\operatorname{Re}\lambda|^2 + |\operatorname{Im}\lambda|^2\right] \\
&\leq C, \qquad (5.1)
\end{aligned}$$

independently of n, by Lemma 4.2(a), (b). Next, Theorem 3.1 shows that for each fixed, j,

$$\begin{aligned} & \lim_{n\to\infty}\int t^j d\mu_{mn}^{[2]}(t) \\ = & \lim_{n\to\infty}\frac{1}{n\sqrt{\pi\rho_m}}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}\lambda^2\left(\lambda/\sqrt{2\pi\rho_m}\right)^j \\ = & \lim_{n\to\infty}\frac{\sqrt{2}}{n\left(\sqrt{2\pi\rho_m}\right)^{j+1}}Tr\left(\left(\frac{A_{mn}}{a_m}\right)^{j+2}\right)=\sqrt{\frac{2}{j+2}}. \end{aligned}$$

(5.2)

In view of Lemma 4.4, the limit points of the supports of $\mu_{mn}^{[2]}$ lie in $[0,1]$. Because of (5.1), every subsequence of $\left\{\mu_{mn}^{[2]}\right\}$ contains another subsequence converging weakly to some measure ω with support in $[0,1]$ and finite total mass. As $\mu_{mn}^{[2]}$ is complex, this requires some clarification. Decompose, in Hahn-Jordan fashion,

$$\mu_{mn}^{[2]}=\left(\operatorname{Re}\mu_{mn}^{[2]+}-\operatorname{Re}\mu_{mn}^{[2]-}\right)+i\left(\operatorname{Im}\mu_{mn}^{[2]+}-\operatorname{Im}\mu_{mn}^{[2]-}\right)$$

where all four measures on the right-hand side are non-negative. By Lemma 4.4 and (5.1), the sequence $\left\{\operatorname{Re}\mu_{mn}^{[2]+}\right\}$ all have support contained in a fixed neighborhood of $[0,1]$, and

$$\sup_{n\geq 1}\operatorname{Re}\mu_{mn}^{[2]+}(\mathbb{C})<\infty.$$

Hence we can choose a weakly convergent subsequence in the usual sense: for f continuous in an open set containing $[0,1]$,

$$\lim\int f\ d\operatorname{Re}\mu_{mn}^{[2]+}=\int f\ d\sigma_+$$

where σ_+ is a positive measure supported on $[0,1]$ and we take limits through the subsequence. Similar assertions apply to $\left\{\operatorname{Re}\mu_{mn}^{[2]-}\right\}$ and $\left\{\operatorname{Im}\mu_{mn}^{[2]\pm}\right\}$. So let ω be a weak limit of a subsequence of $\left\{\mu_{mn}^{[2]}\right\}$ in this sense. By (5.2),

$$\int_0^1 t^j d\omega(t)=\sqrt{\frac{2}{j+2}}\text{ for } j\geq 0.$$

In particular then $\omega([0,1]) = 1$. To identify ω, make, the substitution $t = e^{-s/(j+2)}$ in

$$\sqrt{\frac{2}{\pi}} \int_0^1 t^{j+1} |\log t|^{-1/2} \, dt = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{j+2}} \int_0^\infty e^{-s} s^{-1/2} ds = \sqrt{\frac{2}{j+2}}.$$

So

$$\int_0^1 t^j d\omega(t) = \sqrt{\frac{2}{\pi}} \int_0^1 t^{j+1} |\log t|^{-1/2} \, dt \text{ for } j \geq 0. \tag{5.3}$$

We next use the fact that the Hausdorff moment problem has a unique solution [27], but taking account of the fact that ω is a possibly complex measure. Write

$$\omega = (\sigma_+ - \sigma_-) + i(\tau_+ - \tau_-)$$

where all the measures on the right are finite nonnegative measures. Taking imaginary parts in (5.3) and then rearranging gives

$$\int_0^1 t^j d\tau_-(t) = \int_0^1 t^j d\tau_+(t), \; j \geq 0,$$

so as the Hausdorff moment problem has a unique solution [27], $\tau_+ = \tau_-$. So ω is a purely real measure. We then obtain

$$\int_0^1 t^j d\sigma_+(t) = \sqrt{\frac{2}{\pi}} \int_0^1 t^{j+1} |\log t|^{-1/2} \, dt \; + \int_0^1 t^j d\sigma_-(t), \text{ for } j \geq 0.$$

Again, this gives

$$d\sigma_+ = \sqrt{\frac{2}{\pi}} t |\log t|^{-1/2} \; dt \; + d\sigma_-$$

and hence

$$d\omega(t) = \sqrt{\frac{2}{\pi}} t |\log t|^{-1/2} \; dt.$$

Since this is independent of the subsequence, $\left\{\mu_{mn}^{[2]}\right\}$ converges weakly to $\sqrt{\frac{2}{\pi}} |\log t|^{-1/2} t \; dt$. The proof can be completed as in Lemma 5.1, using also (5.1). ■.

Proof of Theorem 1.2

(I) The requisite asymptotic upper bound for the eigenvalue of largest modulus follows from Lemma 4.1. The matching asymptotic lower bound follows from the lemma above, which shows that as $\sqrt{\frac{2}{\pi}} |\log t|^{-1/2} t \; dt$ contains 1 in its support, so necessarily there are eigenvalues λ of A_{mn}/a_m with $\lambda/\sqrt{2\pi\rho_m}$ arbitrarily close to 1.

(II) In Lemma 4.4, we showed that all limit points of $\{\Lambda(A_{mn}/a_m)/\sqrt{2\pi\rho_m}\}_{n\geq 1}$

are contained in $[0,1]$. That they fill out $[0,1]$ follows from Lemma 5.2. (III), (IV) were proved in the two lemmas above. ■

6. Proof of Theorem 1.4

We begin by obtaining finer estimates on the location of the eigenvalues:

Lemma 6.1
Assume the hypotheses of Theorem 1.4. Then
(a)

$$\min_{\lambda\in\Lambda(E_{mn})} \operatorname{Re}(\lambda) \geq -o(1). \tag{6.1}$$

(b) If in addition (1.36) holds, then

$$\max_{\lambda\in\Lambda(E_{mn})} |\operatorname{Im}(\lambda)| = O(1). \tag{6.2}$$

Proof
We again use (4.10), (4.11). Fix some large positive D. From Lemma 2.4(a),

$$\begin{aligned}
&\lambda_{\min}(F_{mn}) \\
\geq\ & \inf\left\{\sum_{|\ell|\leq\sqrt{D\rho_m\log\rho_m}} \frac{1}{2}\left(e_\ell+\overline{e_{-\ell}}\right)S_\ell + \sum_{|\ell|>\sqrt{D\rho_m\log\rho_m}} \frac{1}{2}\left(e_\ell+\overline{e_{-\ell}}\right)S_\ell\right\} \\
\geq\ & \left(\inf\sum_{\ell=-n+1}^{n-1}\chi_m^{\ell^2/2}S_\ell\right) - \sum_{|\ell|>\sqrt{D\rho_m\log\rho_m}}\chi_m^{\ell^2/2} - \sum_{|\ell|>\sqrt{D\rho_m\log\rho_m}}\frac{1}{2}\left|e_\ell+\overline{e_{-\ell}}\right| \\
&-C\sum_{|\ell|\leq\sqrt{D\rho_m\log\rho_m}}\chi_m^{\ell^2/2}\left\{o\left(\frac{\ell^3}{\rho_m^2}\right)+o\left(\frac{\ell^2}{\rho_m^{3/2}}\right)+o\left(\frac{\ell^6}{\rho_m^{7/2}}\right)\right\}.
\end{aligned}$$

Here as before, the first term is the smallest eigenvalue of a positive definite matrix, so is nonnegative. Also if D is large enough, Lemma 2.3(a) shows that for some C_1, C_2 independent of D,

$$\begin{aligned}
&\sum_{|\ell|>\sqrt{D\rho_m\log\rho_m}}\chi_m^{\ell^2/2} + \sum_{|\ell|>\sqrt{D\rho_m\log\rho_m}}\left|e_\ell+\overline{e_{-\ell}}\right| \\
\leq\ & C_3\sqrt{\rho_m}\exp\left(-C_4 D\log\rho_m\right) \\
=\ & C_3\rho_m^{1/2-C_4D} = o(1), \qquad (6.3)
\end{aligned}$$

if D is large enough. Next,

$$\sum_{|\ell|\leq\sqrt{D\rho_m\log\rho_m}}\chi_m^{\ell^2/2}\left\{O\left(\frac{\ell^3}{\rho_m^2}\right)+o\left(\frac{\ell^2}{\rho_m^{3/2}}\right)+o\left(\frac{\ell^6}{\rho_m^{7/2}}\right)\right\}=o\left(1\right),$$

by Lemma 2.3(b). So

$$\lambda_{\min}\left(F_{mn}\right)\geq-o\left(1\right)$$

Then (4.8) gives the result.
(b) As in the proof of Lemma 4.4(b), we bound the largest absolute value of any eigenvalue of G_{mn} by

$$\begin{aligned}&\sum_{\ell=1}^{n}\left|e_\ell-\overline{e_{-\ell}}\right|\\ \leq&\sum_{1\leq\ell\leq\sqrt{D\rho_m\log\rho_m}}\left|e_\ell-\overline{e_{-\ell}}\right|+C_1\sqrt{\rho_m}e^{-C_4D\log\rho_m},\end{aligned}$$

by Lemma 2.3(a). Here using Lemma 2.4(b),

$$\begin{aligned}&\sum_{1\leq\ell\leq\sqrt{D\rho_m\log\rho_m}}\left|e_\ell-\overline{e_{-\ell}}\right|\\ \leq\ &C\sum_{1\leq\ell\leq\sqrt{D\rho_m\log\rho_m}}\chi_m^{\ell^2/2}\left\{O\left(\frac{\ell^3}{\rho_m^2}\right)+o\left(\frac{\ell^2}{\rho_m^{3/2}}\right)+o\left(\frac{\ell^6}{\rho_m^{7/2}}\right)\right\}\leq C.\end{aligned}$$

by Lemma 2.3(b). So the eigenvalue of maximum modulus of G_{mn} is $O\left(1\right)$, and then (4.9) gives the result. ■

We note that when (1.36) holds, Lemma 6.1(b) shows that the conclusions (II), (III) of Theorem 1.4 hold when we replace $\mu_{mn}^{[1]}$ by the measures in (1.37). Next, we need:

Lemma 6.2
The assertions (II) and (III) of Theorem 1.4 are true.
Proof
First note that $\mu_{mn}^{[1]}$ is a (possibly signed) measure with

$$\mu_{mn}^{[1]}\left(\mathbb{C}\right)=\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}\operatorname{Re}\lambda=\operatorname{Re}\frac{1}{n}Tr\left(A_{mn}/a_m\right)=1. \tag{6.4}$$

Moreover, the limit points of the supports of $\left\{\mu_{mn}^{[1]}\right\}$ lie in $[0,1]$, and for each fixed $j \geq 0$,

$$\begin{aligned}\int t^j d\mu_{mn}^{[1]}(t) &= \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} \operatorname{Re}\lambda\left(\lambda/\sqrt{2\pi\rho_m}\right)^j \\ &= \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} \lambda\left(\lambda/\sqrt{2\pi\rho_m}\right)^j - \frac{i}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} (\operatorname{Im}\lambda)\left(\lambda/\sqrt{2\pi\rho_m}\right)^j.\end{aligned}$$

(6.5)

Here

$$\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} \lambda\left(\lambda/\sqrt{2\pi\rho_m}\right)^j = \frac{1}{n\left(\sqrt{2\pi\rho_m}\right)^j} Tr\left(\left[A_{mn}/a_m\right]^{j+1}\right) = \frac{1+o(1)}{\sqrt{j+1}}$$

by Theorem 3.1. Moreover, if $j=0$, the second sum in the right-hand side of (6.5) is 0, while if $j \geq 1$, Cauchy-Schwarz gives

$$\begin{aligned}&\left|\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} (\operatorname{Im}\lambda)\left(\lambda/\sqrt{2\pi\rho_m}\right)^j\right| \\ \leq\ & \sqrt{\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\operatorname{Im}\lambda|^2}\sqrt{\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} \left|\lambda/\sqrt{2\pi\rho_m}\right|^{2j}} \\ \leq\ & C\sqrt{\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\operatorname{Im}\lambda|^2}\sqrt{\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} \left|\lambda/\sqrt{2\pi\rho_m}\right|^{2}} \\ =\ & C\sqrt{\frac{1}{n\sqrt{2\pi\rho_m}}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\operatorname{Im}\lambda|^2}\sqrt{\frac{1}{n\sqrt{2\pi\rho_m}}\sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\lambda|^2} = o(1),\end{aligned}$$

by Lemma 4.2. Thus for $j \geq 0$,

$$\int t^j d\mu_{mn}^{[1]}(t) = \frac{1+o(1)}{\sqrt{j+1}}.$$

Next,

$$\begin{aligned}\left|\mu_{mn}^{[1]}\right|(\mathbb{C}) &= \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}|\operatorname{Re}\lambda| \\ &= \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m),\operatorname{Re}\lambda<0}|\operatorname{Re}\lambda| + \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m),\operatorname{Re}\lambda\geq 0}\operatorname{Re}\lambda \\ &= o(1) + \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m),\operatorname{Re}\lambda\geq 0}\operatorname{Re}\lambda \\ &= \frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}\operatorname{Re}\lambda + o(1) \\ &= 1 + o(1).\end{aligned}$$

Here we have used Lemma 6.1 and (6.4). Next the substitution $t = e^{-s/(j+1)}$ shows that for each $j \geq 0$,

$$\sqrt{\frac{1}{\pi}}\int_0^1 t^j \left|\log t\right|^{-1/2}\ dt = \sqrt{\frac{1}{j+1}},$$

so for $j \geq 0$,

$$\lim_{n\to\infty}\int t^j d\mu_{mn}^{[1]}(t) = \sqrt{\frac{1}{\pi}}\int_0^1 t^j \left|\log t\right|^{-1/2}\ dt.$$

We can now complete the proof as in Lemma 5.2. ■

Proof of Theorem 1.4

We proved (I) in Lemma 6.1, and (II), (III) in Lemma 6.2. ■

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