Large Sieve Inequalities via Subharmonic Methods and the Mahler Measure of the Fekete Polynomials

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Abstract

We investigate large sieve inequalities such as

$$\frac{1}{m}\sum_{j=1}^{m}\psi\left(\log\left|P\left(e^{i\tau_j}\right)\right|\right)\leq\frac{C}{2\pi}\int_0^{2\pi}\psi\left(\log\left[e\left|P\left(e^{i\tau}\right)\right|\right]\right)d\tau,$$

where ψ is convex and increasing, P is a polynomial or an exponential of a potential, and the constant C depends on the degree of P, and the distribution of the points $0\leq\tau_1<\tau_2<\cdots<\tau_m\leq 2\pi$. The method allows greater generality and is in some ways simpler than earlier ones. We apply our results to estimate the Mahler measure of Fekete polynomials.

1 [1]Results

The large sieve of number theory [14, p. 559] asserts that if

$$P(z)=\sum_{k=-n}^{n}a_kz^k$$

is a trigonometric polyonomial of degree $\leq n$, and

$$0\leq\tau_1<\tau_2<\cdots<\tau_m\leq 2\pi,$$

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and

$$\delta = \min\{\tau_2 - \tau_1, \tau_3 - \tau_2, \dots, \tau_m - \tau_{m-1}, 2\pi - (\tau_m - \tau_1)\},$$

then

$$\sum_{j=1}^{m} \left|P\left(e^{i\tau_j}\right)\right|^2 \leq \left(\frac{n}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} \left|P\left(e^{i\tau}\right)\right|^2 d\tau. \tag{1}$$

There are numerous extensions of this to L_p norms, or involving $\psi\left(\left|P\left(e^{i\tau}\right)\right|^p\right)$, where ψ is a convex function, and $p > 0$ [8], [12]. There are versions of this that estimate Riemann sums, for example,

$$\sum_{j=1}^{m} \left|P\left(e^{i\tau_j}\right)\right|^2 (\tau_j - \tau_{j-1}) \leq C\frac{1}{2\pi} \int_0^{2\pi} \left|P\left(e^{i\tau}\right)\right|^2 d\tau, \tag{2}$$

with C independent of n, P, $\{\tau_1, \tau_2, \dots, \tau_m\}$. These are often called forward Marcinkiewicz-Zygmund inequalities. Converse Marcinkiewicz-Zygmund Inequalities provide estimates for the integrals above in terms of the sums on the left-hand side [11], [13], [16].

A particularly interesting case is that of the L_0 norm. A result of the first author asserts that if $\{z_1, z_2, \dots, z_n\}$ are the nth roots of unity, and P is a polynomial of degree $\leq n$,

$$\prod_{j=1}^{n} |P(z_j)|^{1/n} \leq 2M_0(P), \tag{3}$$

where

$$M_0(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|P\left(e^{it}\right)\right| dt\right)$$

is the Mahler measure of P.

The focus of this paper is to show that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extend (3) to points other than the roots of unity. Given $c \geq 0$, $\kappa \in [0, \infty)$, and a positive measure ν of compact support and total mass at most

$\kappa \geq 0$ on the plane, we define the associated exponential of its potential by

$$P(z) = c \exp\left(\int \log|z-t|\, d\nu(t)\right).$$

We say that this is an *exponential of a potential of mass* $\leq \kappa$, and that its *degree* is $\leq \kappa$. The set of all such functions is denoted by $\mathbb{P}_\kappa$. Note that if P is a polynomial of degree $\leq n$, then

$$|P| \in \mathbb{P}_n.$$

More generally, the generalized polynomials studied by several authors [3], [7] also lie in $\mathbb{P}_\kappa$, for an appropriate κ. We prove:

Theorem 1.1 *Let* $\psi : \mathbb{R} \to [0,\infty)$ *be nondecreasing and convex. Let* $m \geq 1$, $\kappa > 0$, $\alpha > 0$, *and*

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi.$$

Let $w_j \geq 0$, $1 \leq j \leq m$ *with*

$$\sum_{j=1}^{m} w_j = 1.$$

Let μ_m *denote the corresponding Riemann-Stieltjes measure, defined for* $\theta \in [0,2\pi]$ *by*

$$\mu_m([0,\theta]) := \sum_{j:\tau_j \leq \theta} w_j.$$

Let

$$\Delta := \sup\left\{\left|\mu_m([0,\theta]) - \frac{\theta}{2\pi}\right| : \theta \in [0,2\pi]\right\} \quad (4)$$

denote the discrepancy of μ_m. *Then for* $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} w_j \psi\left(\log P\left(e^{i\tau_j}\right)\right) \leq \left(1 + \frac{8}{\alpha}\kappa\Delta\right) \frac{1}{2\pi} \int_0^{2\pi} \psi\left(\log\left[e^{\alpha} P\left(e^{i\theta}\right)\right]\right) d\theta. \quad (5)$$

Example 1 Let us choose all equal weights,

$$w_j = \frac{1}{m}, \qquad 1 \le j \le m.$$

Then μ_m is counting measure,

$$\mu_m([0,\theta]) = \frac{1}{m}\#\{j : \tau_j \in [0,\theta]\}.$$

If we take $\psi(t) = \max\{0,t\}$, and $\alpha = 1$, and use the notation $\log^+ t = \max\{0, \log t\}$, we obtain

$$\frac{1}{m}\sum_{j=1}^{m} \log^+ P\left(e^{i\tau_j}\right) \le (1+8\kappa\Delta)\frac{1}{2\pi}\int_0^{2\pi} \log^+\left[eP\left(e^{i\theta}\right)\right] d\theta. \tag{6}$$

This result is new. Previous inequalities have been limited to sums involving $\psi\left(P\left(e^{i\tau_j}\right)^p\right)$, some $p > 0$. If we let $p > 0$, $\psi(t) = e^{pt}$, and $\alpha = \frac{1}{p}$, (5) becomes

$$\frac{1}{m}\sum_{j=1}^{m} P\left(e^{i\tau_j}\right)^p \le (1+8p\kappa\Delta)\frac{e}{2\pi}\int_0^{2\pi} P\left(e^{i\theta}\right)^p d\theta. \tag{7}$$

This choice of α is not optimal. The optimal choice is

$$\alpha = 4\kappa\Delta\left[-1+\sqrt{1+\frac{1}{2p\kappa\Delta}}\right]$$

but one needs further information on the size of $p\kappa\Delta$ to exploit this. For example, if $p\kappa\Delta \le 1$, the optimal choice is of order $\sqrt{\frac{\kappa\Delta}{p}}$, and choosing this α in (5), we obtain

$$\frac{1}{m}\sum_{j=1}^{m} P\left(e^{i\tau_j}\right)^p \le \left(1+C\sqrt{p\kappa\Delta}\right)\frac{1}{2\pi}\int_0^{2\pi} P\left(e^{i\theta}\right)^p d\theta, \tag{8}$$

where C is independent of p, κ, Δ, P.

For well distributed $\{\tau_1, \tau_2, \ldots, \tau_m\}$, Δ is of order $\frac{1}{m}$. In particular, when these points are equally spaced and include 2π, but not 0, so that

$$\tau_j = \frac{2j\pi}{m}, \qquad 1 \le j \le m,$$

we have

$$\Delta = \frac{2\pi}{m},$$

and (7) becomes

$$\frac{1}{m}\sum_{j=1}^{m} P\left(e^{i\tau_j}\right)^p \le \left(1+\frac{16\pi p\kappa}{m}\right)\frac{e}{2\pi}\int_0^{2\pi} P\left(e^{i\theta}\right)^p d\theta. \tag{9}$$

Example 2 Another important choice of the weights w_j is

$$w_j = \frac{\tau_j - \tau_{j-1}}{2\pi}, \qquad 1 \le j \le m,$$

where now we assume $\tau_0 = 0$ and $\tau_m = 2\pi$. For this case (5) becomes an estimate for Riemann sums,

$$\begin{aligned}&\frac{1}{2\pi}\sum_{j=1}^{m}(\tau_j - \tau_{j-1})\,\psi\left(\log P\left(e^{i\tau_j}\right)\right)\\ &\qquad \le \left(1+\frac{8}{\alpha}\kappa\Delta\right)\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log\left[e^{\alpha}P\left(e^{i\theta}\right)\right]\right)d\theta.\end{aligned} \tag{10}$$

The discrepancy Δ in this case is

$$\Delta = \sup_j \frac{\tau_j - \tau_{j-1}}{2\pi}.$$

Remarks

(a) In many ways, the approach of this paper is simpler than that in [12] where Dirichlet kernels were used, or that of [8], where Carleson measures were used. The main idea is to use the Poisson integral inequality for subharmonic functions.

(b) We can reformulate (5) as

$$\begin{aligned}&\int_0^{2\pi}\psi\left(\log\left|P\left(e^{i\tau}\right)\right|\right)d\mu_m(\tau)\\ &\qquad \le \left(1+\frac{8}{\alpha}\kappa\Delta\right)\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log\left[e^{\alpha}P\left(e^{i\theta}\right)\right]\right)d\theta.\end{aligned}$$

In fact this estimate holds for any probability measure μ_m on $[0, 2\pi]$, not just the pure jump measures above.

(c) The one severe restriction above is that ψ is nonnegative. In particular, this excludes $\psi(x) = x$. For this case, we prove 2 different results:

Theorem 1.2 *Assume that* m, κ, $\{\tau_1, \tau_2, \ldots, \tau_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *are as in* Theorem 1.1. *Let*

$$Q(z) = \prod_{j=1}^{m} \left|z - e^{i\tau_j}\right|^{w_j}. \tag{11}$$

Then for $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} w_j \log P\left(e^{i\tau_j}\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \log P\left(e^{i\theta}\right) d\theta + \kappa \log \|Q\|_{L_\infty(|z|=1)}. \tag{12}$$

Remarks

If we choose all $w_j = \frac{1}{m}$, this yields

$$\prod_{j=1}^{m} P\left(e^{i\tau_j}\right)^{1/m} \leq \|Q\|^{\kappa}_{L_\infty(|z|=1)} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log P\left(e^{i\theta}\right) d\theta\right). \tag{13}$$

If we take $\left\{e^{i\tau_1}, e^{i\tau_2}, \ldots, e^{i\tau_m}\right\}$ to be the mth roots of unity, then

$$Q(z) = |z^m - 1|^{1/m}$$

and (13) becomes

$$\prod_{j=1}^{m} P\left(e^{i\tau_j}\right)^{1/m} \leq 2^{\kappa/m} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log P\left(e^{i\theta}\right) d\theta\right). \tag{14}$$

In the case $\kappa = m = n$, this gives the first author's inequality (3). In general however, it is not easy to bound $\|Q\|_{L_\infty(|z|=1)}$. Using an alternative method, we can avoid the term involving Q, when the spacing between successive τ_j is $O\left(\kappa^{-1}\right)$:

Theorem 1.3 *Assume that* m, κ *and* $\{\tau_1, \tau_2, \ldots, \tau_m\}$ *are as in* Theorem 1.1. *Let* $\tau_0 := \tau_m - 2\pi$ *and* $\tau_{m+1} := \tau_1 + 2\pi$. *Let*

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_m - \tau_{m-1}\}.$$

Let $A > 0$. *There exists* $B > 0$ *such that whenever* $\kappa \geq 1$ *and*

$$\delta \leq A\kappa^{-1},$$

then for all $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log P\left(e^{i\tau_j}\right) \leq \int_0^{2\pi} \log P\left(e^{i\theta}\right) d\theta + B. \tag{15}$$

One application of Theorem 1.2 is to estimation of Mahler measure. Recall that for a bounded measurable function Q on $[0, 2\pi]$, its Mahler measure is

$$M_0(Q) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|Q\left(e^{i\theta}\right)\right| d\theta\right).$$

It is well known that

$$M_0(Q) = \lim_{p \to 0+} M_p(Q),$$

where for $p > 0$,

$$M_p(Q) := \|Q\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|Q\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p}.$$

It is a simple consequence of Jensen's formula that if

$$Q(z) = c\prod_{k=1}^{n}(z - z_k)$$

is a polynomial, then

$$M_0(Q) = |c|\prod_{k=1}^{n} \max\{1, |z_k|\}.$$

The construction of polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The Littlewood polynomials,

$$L_n := \left\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k,\ \alpha_k \in \{-1, 1\} \right\},$$

which have coefficients ± 1, and the unimodular polynomials,

$$K_n := \left\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k,\ |\alpha_k| = 1 \right\}$$

are two of the most important classes considered. Beller and Newman [1] constructed unimodular polynomials of degree n whose Mahler measure is at least $\sqrt{n} - c/\log n$. Here we show that for Littlewood polynomials, we can achieve almost $\frac{1}{2}\sqrt{n}$, by considering the Fekete polynomials.

For a prime number p, the pth Fekete polynomial is

$$f_p(z) = \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) z^k,$$

where

$$\left(\frac{k}{p} \right) = \begin{cases} 1, & \text{if } x^2 \equiv k \pmod{p} \text{ has a non-zero solution } x \\ 0, & \text{if } p \text{ divides } k \\ -1, & \text{otherwise.} \end{cases}$$

Since f_p has constant coefficient 0 it is not a Littlewood polynomial, but

$$g_p(z) = f_p(z)/z$$

is a Littlewood polynomial, and has the same Mahler measure as f_p. Fekete polynomials are examined in detail in [2, pp. 37–42].

Theorem 1.4 *Let $\varepsilon > 0$. For large enough prime p, we have*

$$M_0(f_p) = M_0(g_p) \geq \left(\frac{1}{2} - \varepsilon \right) \sqrt{p}. \tag{16}$$

Remarks

From Jensen's inequality,

$$M_0\left(f_p\right) \leq \|f_p\|_2 = \sqrt{p-1}\,.$$

However $\frac{1}{2} - \varepsilon$ in Theorem 1.4 cannot be replaced by $1 - \varepsilon$. Indeed if p is prime, and we write $p = 4m+1$, then g_p is self-reciprocal, that is,

$$z^{p-1} g_p\left(\frac{1}{z}\right) = g_p\left(z\right),$$

and hence

$$g_p\left(e^{2it}\right) = e^{i(p-2)t} \sum_{k=0}^{(p-3)/2} a_k \cos\left(\left(2k+1\right)t\right), \qquad a_k \in \{-2, 2\}\,.$$

A result of Littlewood [10, Theorem 2] implies that

$$M_0\left(f_p\right) = M_0\left(g_p\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \left|g_p\left(e^{2it}\right)\right| dt \leq \left(1 - \varepsilon_0\right) \sqrt{p-1},$$

for some absolute constant $\varepsilon_0 > 0$. It is an interesting question whether there is a sequence of Littlewood polynomials (f_n) such that for an arbitrary $\varepsilon > 0$, and n large enough,

$$M_0\left(f_n\right) \geq \left(1 - \varepsilon\right) \sqrt{n}.$$

The results are proved in the next section.

2 Proofs

We assume the notation of Theorem 1.1. We let

$$r = 1 + \frac{\alpha}{\kappa}, \tag{17}$$

and define the Poisson kernel for the ball $|z| \leq r$ (cf. [15, p. 8]),

$$\mathcal{P}_r\left(se^{i\theta}, re^{it}\right) = \frac{r^2 - s^2}{r^2 - 2rs\cos\left(t - \theta\right) + s^2},$$

where $0 \leq s < r$ and $t, \theta \in \mathbb{R}$.

Proof of Theorem 1.1

Step 1 The Basic Inequality

Let $P \in \mathbb{P}_\kappa \backslash \{0\}$, so that for some $c > 0$ and some measure ν with total mass $\leq \kappa$ and compact support,

$$\log P(z) = \log c + \int \log |z - t| \, d\nu(t).$$

As $\log P$ is subharmonic, and as ψ is convex and increasing, $\psi(\log P)$ is subharmonic [15, Theorem 2.6.3, p. 43]. Then we have for $|z| < r$, the inequality [15, Theorem 2.4.1, p. 35]

$$\psi(\log P(z)) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi\left(\log P\left(re^{it}\right)\right) \mathcal{P}_r\left(z, re^{it}\right) dt.$$

Choosing $z = e^{i\tau_j}$, multiplying by w_j, and adding over j gives

$$\sum_{j=1}^{m} w_j \psi\left(\log P\left(e^{i\tau_j}\right)\right) - \frac{1}{2\pi} \int_0^{2\pi} \psi\left(\log P\left(re^{it}\right)\right) dt$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} \psi\left(\log P\left(re^{it}\right)\right) \mathcal{H}(t) \, dt \qquad (18)$$

where

$$\mathcal{H}(t) := \sum_{j=1}^{m} w_j \mathcal{P}_r\left(e^{i\tau_j}, re^{it}\right) - 1$$
$$= \int_0^{2\pi} \mathcal{P}_r\left(e^{i\tau}, re^{it}\right) d\left(\mu_m(\tau) - \frac{\tau}{2\pi}\right).$$

Here we have used the elementary property of the Poisson kernel, that it integrates to 1 over any circle center 0 inside its ball of definition.

Step 2 Estimating $\mathcal{H}$

We integrate this relation by parts, and note that both $\mu_m[0,0] = 0$ and

$\mu_m[0,2\pi] = 1$. This gives

$$\mathcal{H}(t) = -\int_0^{2\pi} \left(\frac{\partial}{\partial \tau}\mathcal{P}_r\left(e^{i\tau}, re^{it}\right)\right)\left(\mu_m([0,\tau]) - \frac{\tau}{2\pi}\right) d\tau$$

and hence

$$|\mathcal{H}(t)| \leq \Delta \int_0^{2\pi} \left|\frac{\partial}{\partial \tau}\mathcal{P}_r\left(e^{i\tau}, re^{it}\right)\right| d\tau. \tag{19}$$

Now

$$\frac{\partial}{\partial \tau}\mathcal{P}_r\left(e^{i\tau}, re^{it}\right) = \frac{\left(r^2-1\right)2r\sin(t-\tau)}{\left(r^2-2r\cos(t-\tau)+1\right)^2}$$

so a substitution $s = t-\tau$ and $2\pi-$periodicity give

$$\begin{aligned}\int_0^{2\pi} \left|\frac{\partial}{\partial \tau}\mathcal{P}_r\left(e^{i\tau}, re^{it}\right)\right| d\tau &= \int_{-\pi}^{\pi} \left|\frac{\partial}{\partial s}\mathcal{P}_r\left(e^{is}, r\right)\right| ds \\ &= -2\int_0^{\pi} \frac{\partial}{\partial s}\mathcal{P}_r\left(e^{is}, r\right) ds \\ &= -2\left[\mathcal{P}_r\left(e^{i\pi}, r\right) - \mathcal{P}_r(1,r)\right] = \frac{8r}{r^2-1}.\end{aligned} \tag{20}$$

Combining (18)–(20), gives

$$\sum_{j=1}^{m} w_j\psi\left(\log P\left(e^{i\tau_j}\right)\right) \leq \left(1+\Delta\frac{8r}{r^2-1}\right)\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log P\left(re^{it}\right)\right) dt. \tag{21}$$

Step 3 Return to the unit circle

Next, we estimate the integral on the right-hand side in terms of an integral over the unit circle. Let us assume that ν has total mass $\lambda(\leq \kappa)$. Let

$$S(z) = |z|^{\lambda} P\left(\frac{r}{z}\right)$$

so that

$$\log S(z) = \log c + \int \log|r - tz|\, d\nu(t),$$

a function subharmonic in $\mathbb{C}$. Then the same is true of $\psi(\log S)$, so its integrals over circles centre 0 increase with the radius [15, Theorem 2.6.8, p. 46]. In particular

$$\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log S\left(e^{i\theta}\right)\right) d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log S\left(re^{i\theta}\right)\right) d\theta$$

and a substitution $\theta \to -\theta$ gives

$$\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log P\left(re^{i\theta}\right)\right)d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\psi\left(\lambda\log r+\log P\left(e^{i\theta}\right)\right)d\theta$$
$$\le \frac{1}{2\pi}\int_0^{2\pi}\psi\left(\kappa\log r+\log P\left(e^{i\theta}\right)\right)d\theta$$
$$\le \frac{1}{2\pi}\int_0^{2\pi}\psi\left(\alpha+\log P\left(e^{i\theta}\right)\right)d\theta,$$

recall our choice (17) of r. Then (21) becomes

$$\sum_{j=1}^{m} w_j\psi\left(\log P\left(e^{i\tau_j}\right)\right)$$
$$\le \left(1+\Delta\frac{8r}{r^2-1}\right)\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log\left[e^{\alpha}P\left(e^{i\theta}\right)\right]\right)d\theta$$
$$\le \left(1+8\Delta\frac{\kappa}{\alpha}\right)\frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log\left[e^{\alpha}P\left(e^{i\theta}\right)\right]\right)d\theta.$$

□

Proof of Theorem 1.2

Write

$$\log P(z) = \log c + \int \log|z-t|\,d\nu(t)$$

so

$$\sum_{j=1}^{m} w_j \log P\left(e^{i\tau_j}\right) = \log c + \int\left(\sum_{j=1}^{m} w_j\log\left|e^{i\tau_j}-t\right|\right)d\nu(t)$$
$$= \log c + \int \log Q(t)\,d\nu(t), \tag{22}$$

recall (11). Now as all zeros of Q are on the unit circle,

$$g(u) := \log Q(u) - \log\|Q\|_{L_\infty(|z|=1)} - \log|u|$$

is harmonic in the exterior $\{u : |u| > 1\}$ of the unit ball, with limit 0 at ∞, and with $g(u) \le 0$ for $|u| = 1$. By the maximum principle for subharmonic functions,

$$g(u) \le 0, \qquad |u| > 1.$$

We deduce that for $|u| > 1$,

$$\log Q(u) \leq \log \|Q\|_{L_\infty(|z|=1)} + \log^+ |u|.$$

Moreover, inside the unit ball, we can regard Q as the absolute value of a function analytic there (with any choice of branches). So the last inequality holds for all $u \in \mathbb{C}$. Then assuming (as above) that ν has total mass $\lambda \leq \kappa$,

$$\begin{aligned}\int \log Q(t)\, d\nu(t) &\leq \lambda \log \|Q\|_{L_\infty(|z|=1)} + \int \log^+ |t|\, d\nu(t) \\ &= \lambda \log \|Q\|_{L_\infty(|z|=1)} + \int \left(\frac{1}{2\pi} \int_0^{2\pi} \log \left| e^{i\theta} - t \right| d\theta \right) d\nu(t) \\ &\leq \kappa \log \|Q\|_{L_\infty(|z|=1)} + \frac{1}{2\pi} \int_0^{2\pi} \left(\int \log \left| e^{i\theta} - t \right| d\nu(t) \right) d\theta. \end{aligned} \tag{23}$$

In the second last line we used a well known identity [15, Exercise 2.2, p. 29], and in the last line we used the fact that the sup norm of Q on the unit circle is larger than 1. This is true because

$$\frac{1}{2\pi} \int_0^{2\pi} \log Q\left(e^{i\theta}\right) d\theta = \sum_{j=1}^{m} w_j \frac{1}{2\pi} \int_0^{2\pi} \log \left| e^{i\tau_j} - e^{i\theta} \right| d\theta = 0,$$

while $\log Q < 0$ in a neighborhood of each τ_j, so that $\log Q\left(e^{i\theta}\right) > 0$ on a set of θ of positive measure. Substituting (23) into (22) gives

$$\sum_{j=1}^{m} w_j \log P\left(e^{i\tau_j}\right) \leq \kappa \log \|Q\|_{L_\infty(|z|=1)} + \frac{1}{2\pi} \int_0^{2\pi} \log \left| P\left(e^{i\theta}\right) \right| d\theta. \qquad \square$$

Proof of Theorem 1.3

Note first that our choice of τ_0, τ_{m+1} give

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} = 2\pi.$$

It suffices to prove that for every $a \in \mathbb{C}$,

$$\begin{aligned}\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log \left| e^{i\tau_j} - a \right| &\leq \int_0^{2\pi} \log \left| e^{it} - a \right| dt + B\kappa^{-1} \\ &= 2\pi \log^+ |a| + B\kappa^{-1}. \end{aligned} \tag{24}$$

For, we can integrate this against the measure $d\nu(a)$ that appears in the representation of $P \in \mathbb{P}_\kappa$. Since

$$\log\left|e^{i\tau} - a\right| = \log\left|e^{i\tau} - \overline{a}^{-1}\right| + \log|a|$$

for $\tau \in \mathbb{R}$ and $|a| < 1$, we can assume that $|a| \geq 1$. Moreover it is sufficient to prove (24) in the case $|a| \geq 1 + \kappa^{-1}$. Indeed the case $|a| \in \left[1, 1+\kappa^{-1}\right]$ follows easily from the case $|a| = 1 + \kappa^{-1}$, and the fact that the left-hand and right-hand sides in (24) increase as we increase $|a|$, while keeping $\arg(a)$ fixed. We may also assume that $a \in [1+\kappa^{-1}, \infty)$, simply rotate the unit circle. To prove (24), we use the integral form of the error for the trapezoidal rule [6, p. 288, (4.3.16)]: if f'' exists and is integrable in $[\alpha, \beta]$,

$$\int_\alpha^\beta f(t)\,dt - \frac{\beta-\alpha}{2}\left(f(\alpha) + f(\beta)\right) = \frac{1}{2}\int_\alpha^\beta f''(t)(\alpha - t)(\beta - t)\,dt.$$

From this we deduce that if f'' does not change sign on $[\alpha, \beta]$,

$$\left|\int_\alpha^\beta f(t)\,dt - \frac{\beta-\alpha}{2}\left(f(\alpha) + f(\beta)\right)\right| \leq \frac{(\beta-\alpha)^2}{2}\left|f'(\beta) - f'(\alpha)\right|. \tag{25}$$

Moreover, if f'' changes sign at most twice, then

$$\left|\int_\alpha^\beta f(t)\,dt - \frac{\beta-\alpha}{2}\left(f(\alpha) + f(\beta)\right)\right| \leq 3(\beta-\alpha)^2 \max_{t\in[\alpha,\beta]}\left|f'(t)\right|. \tag{26}$$

Now let

$$f(t) := \log\left|e^{it} - a\right|.$$

Then

$$f'(t) = \frac{a\sin t}{1 + a^2 - 2a\cos t} \text{ and } f''(t) = \frac{-2a^2 + \left(1 + a^2\right)a\cos t}{\left(1 + a^2 - 2a\cos t\right)^2}.$$

Elementary calculus shows that $|f'|$ achieves its maximum on $[0, 2\pi]$ when $\cos t = \frac{2a}{1+a^2}$. Then $|\sin t| = \frac{a^2-1}{a^2+1}$. Hence, as $a \geq 1 + \kappa^{-1}$, and $\kappa \geq 1$,

$$\left|f'(t)\right| \leq \left(a - a^{-1}\right)^{-1} \leq \kappa \text{ for all } t \in \mathbb{R}. \tag{27}$$

Also, since f'' has at most two zeros in the period, the total variation $V_0^{2\pi} f'$ on $[0, 2\pi]$ satisfies

$$V_0^{2\pi} f' \leq 6 \max_{[0,2\pi]} |f'| \leq 6\kappa. \tag{28}$$

Now we apply (25) to (28) to the interval $[\alpha, \beta] = [\tau_{j-1}, \tau_j]$ and add over j. We also use our conventions on τ_{m+1} and τ_m. Then

$$\begin{aligned}
&\left| \int_0^{2\pi} f(t)\, dt - \sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} f(\tau_j) \right| \\
&= \left| \sum_{j=1}^{m} \left(\int_{\tau_{j-1}}^{\tau_j} f(t)\, dt - \frac{\tau_j - \tau_{j-1}}{2} \left[f(\tau_{j-1}) + f(\tau_j) \right] \right) \right| \\
&\leq \frac{1}{2} \delta^2 V_0^{2\pi} f' + 6\delta^2 \kappa \leq 9A^2 \kappa^{-1}.
\end{aligned}$$

So we have (24) with $B = 9A^2$. □

Proof of Theorem 1.4

We begin by recalling two facts about zeros of Littlewood and unimodular polynomials:

(I) $\exists\ c > 0$ such that every unimodular polynomial of degree $\leq n$ has at most $c\sqrt{n}$ real zeros [4].

(II) $\exists\ c > 0$ such that every Littlewood polynomial of degree $\leq n$ has at most $c \log^2 n / \log\log n$ zeros at 1 [5].

Now suppose that 1 is a zero of f_p with multiplicity $m = m(p)$. By (I) or (II), $m = O\left(p^{1/2}\right)$. Let

$$h_m(z) = (z-1)^m$$

and

$$F_p(z) = f_p(z) / h_m(z).$$

Note that all coefficients of F_p are integers (as $1/h_m(z)$ has Maclaurin series with integer coefficients), so $F_p(1)$ is a non-zero integer. Also h_m is monic and has all zeros on the unit circle, so its Mahler measure is 1. Then as Mahler measure is multiplicative,

$$M_0(f_p) = M_0(F_p)\, M_0(h_m) = M_0(F_p).$$

Let $z_p = \exp\left(\frac{2\pi i}{p}\right)$. The special case (3) of Theorem 1.2 gives

$$\begin{aligned} M_0(f_p) &\geq \frac{1}{2}\left(|F_p(1)| \prod_{k=1}^{p-1} \left|F_p\left(z_p^k\right)\right|\right)^{1/p} \\ &\geq \frac{1}{2}\left(1 \cdot \prod_{k=1}^{p-1} \left|\frac{f_p\left(z_p^k\right)}{\left(z_p^k - 1\right)^m}\right|\right)^{1/p}. \end{aligned}$$

It is known [2, Section 5] that for $1 \leq k \leq p-1$,

$$f_p\left(z_p^k\right) = \sqrt{\left(\frac{-1}{p}\right) p}.$$

Then

$$M_0(f_p) \geq \frac{1}{2}\left(\frac{\sqrt{p}^{p-1}}{p^m}\right)^{1/p} = \frac{1}{2}\sqrt{p}\, p^{-\left(\frac{1}{2}+m\right)/p}.$$

Since $m = O\left(p^{1/2}\right)$, the bound (16) follows for large p. □

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